Algebra and Geometry
Lecture 2
Matrices and systems of linear equations

A matrix is a rectangular array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The values a_{11}, a_{12}, … are called the elements or entries of the matrix.
A matrix with m rows and n columns is called an $m \times n$ matrix, or m-by-n matrix, m and n are called its *dimensions*.

The 1×1 matrix is identified with its single entry.

$$
\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{Vmatrix}
$$

$$
\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}
$$

A matrix with the same number of rows and columns, say n, is called a square matrix of *order* n.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \hdotsfor{4} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Diagonal: $a_{11},\ a_{22},\ a_{33},\ \dots,\ a_{nn}$

Secondary diagonal: $a_{1n},\ a_{2,n-1},\ a_{3,n-2},\ \dots,\ a_{n1}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$A = (a_{ij})$$

Matrices with a single row are called *row vectors*, and those with a single column are called *column vectors*.

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{pmatrix} \quad \text{size } m \times 1$$

$$b = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} \quad \text{size } 1 \times n$$

Two matrices $A = (a_{ij})$ and $B = (a_{ij})$ of the same size are called equal if $a_{ij} = b_{ij}$ for all i and j.

Addition

The sum of the matrices of the same sizes $A = (a_{ij})$ and $B = (b_{ij})$ is defined by the formula $A + B = (a_{ij} + b_{ij})$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ 2 & 3 & -6 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 8 & 0 \end{pmatrix}$$

Zero matrix

$$0_{mn} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$0_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad 0_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Addition, properties

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$-A = \begin{pmatrix} -a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & -a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{m1} & -a_{m2} & \dots & -a_{mn} \end{pmatrix}$$

$$A + (-A) = 0$$

Multiplication by a scalar

$$A = (a_{ij}),\ \ \alpha A = (\alpha a_{ij})$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{pmatrix}$$

$$(\alpha + \beta)A = \alpha A + \beta A$$

$$\alpha(A + B) = \alpha A + \alpha B$$

Matrix multiplication

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$ Size $m \times n$

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$$ Size $n \times p$

$C = AB$, size $m \times p$

$C = (c_{ij})$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ size } 3 \times 3$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \text{ size } 3 \times 2$$

$C = AB$, size 3×2

$\boldsymbol{a_{11}}$	$\boldsymbol{a_{12}}$	$\boldsymbol{a_{13}}$		$\boldsymbol{b_{11}}$	b_{12}
a_{21}	a_{22}	a_{23}		$\boldsymbol{b_{21}}$	b_{22}
a_{31}	a_{31}	a_{33}		$\boldsymbol{b_{31}}$	b_{32}

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

$$
\begin{matrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{matrix}
\qquad\qquad
\begin{matrix} b_{11} & \boldsymbol{b_{12}} \\ b_{21} & \boldsymbol{b_{22}} \\ b_{31} & \boldsymbol{b_{32}} \end{matrix}
$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

$$
\begin{matrix} a_{11} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ a_{31} & a_{31} & a_{33} \end{matrix}
\qquad\qquad
\begin{matrix} \boldsymbol{b_{11}} & b_{12} \\ \boldsymbol{b_{21}} & b_{22} \\ \boldsymbol{b_{31}} & b_{32} \end{matrix}
$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

Properties of multiplication

$$(AB)C = A(BC)$$

$$(A + B)C = AB + AC$$

Multiplication of a matrix and a vector

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix}$$

$$AB \neq BA$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \dots & \lambda_m \end{pmatrix}$$

Multiplication by a diagonal matrix from the left

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \dots & \lambda_m \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \dots & \lambda_1 a_{1n} \\ \lambda_2 a_{21} & \lambda_2 a_{22} & \dots & \lambda_2 a_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ \lambda_m a_{m1} & \lambda_m a_{m2} & \dots & \lambda_m a_{mn} \end{pmatrix}$$

Rule. When we multiply by the diagonal matrix from the left, the *rows* of the matrix are multiplied by the corresponding diagonal entries.

Multiplication by a diagonal matrix from the right

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 a_{11} & \lambda_2 a_{12} & \dots & \lambda_n a_{1n} \\ \lambda_1 a_{21} & \lambda_2 a_{22} & \dots & \lambda_n a_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda_1 a_{m1} & \lambda_2 a_{m2} & \dots & \lambda_n a_{mn} \end{pmatrix}$$

Rule. When we multiply by the diagonal matrix from the right, the *columns* of the matrix are multiplied by the corresponding diagonal entries.

The identity matrix is defined by the relation

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

This designation is shortened to I. In Russian math books this matrix is designated by E_n or simply E.
Main property: for the $m \times n$ matrix A there hold the relations

$$I_m A = A,\ AI_n = A,$$

or shortly $IA = A$, $AI = A$.

Transposed matrix

For the $m \times n$ matrix A its transposed matrix A^T is $n \times m$ matrix defined as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

Properties of this operation:

$$(A+B)^T = A^T + B^T, \ (\lambda A)^{\mathrm{T}} = \lambda A^{\mathrm{T}}, \ (AB)^T = B^T A^T$$

Powers of a *square* matrix

$A^0 = E$, $A^1 = A$, $A^2 = AA$, $A^3 = A^2 A$, …

Example:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

$$A^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix}, \quad A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad AB = ? \quad BA = ?$$

Upper- and lower-triangular matrices.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 0 & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix} \quad (1,2,3,4,5) \quad \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

Find the products of the row vector and column vector.

Matrix polynomial

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$$

$$f(t) = 2t - 3$$

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$$

$$f(A)=2A-3E=2\begin{pmatrix}2 & -3\\ 4 & 1\end{pmatrix}-3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}4 & -6\\ 8 & 2\end{pmatrix}-\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}=\begin{pmatrix}1 & -6\\ 8 & -1\end{pmatrix}.$$

$$f(t)=3t^2-5t+2$$

$$A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$$

$$f(A)=3A^2-5A+2E=3\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix}^2-5\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix}+2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=3\begin{pmatrix}1&4&12\\0&1&4\\0&0&1\end{pmatrix}-$$

$$-\begin{pmatrix}5&10&15\\0&5&10\\0&0&5\end{pmatrix}+\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}=\begin{pmatrix}0&2&21\\0&0&2\\0&0&0\end{pmatrix}.$$

Invertible matrix. $AB=BA=I$, A^{-1}

$$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$$B=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$$

$$AB=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}=\begin{pmatrix}ad-bc&0\\0&ad-bc\end{pmatrix}=(ad-bc)I$$

$$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$$

Systems of linear algebraic equation

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Solution.

Three possible cases.

1. The system has infinitely many solutions.
2. The system has a single solution.
3. The system has no solution.

General solution, consistent, inconsistent, determinate, indeterminate.

Homogeneous and non-homogeneous systems

Elementary transformations.

Type 1: Swap two equations.

Type 2: Multiply an equation by a nonzero scalar.

Type 3: Add to one equation another one multiplied by an arbitrary number.