Chapter 3

FUNDAMENTALS OF DISCRETIZATION

3.1 Local and Global Numbering

In solving an engineering problem with the finite element method (FEM), the domain is discretized by employing elements. The characteristics of the problem dictate the dimensionality of the problem, i.e., one, two, or three dimensional. A brief summary of the common element types utilized in a finite element analysis (FEA) is presented in Fig. 3.1. Once the domain of the problem is discretized by elements, a unique element number identifies each element and a unique node number identifies each node in the domain. As illustrated in Fig. 3.2, nodes are also numbered within each element, and are called local node numbers. The unique node numbering within the entire domain is called global node numbering. This is part of the computational procedure in FEA.

3.2 Approximation Functions

The variation of the field variable, $\phi^{(e)}$, over an element is approximated by an appropriate choice of functions, as illustrated in Fig. 3.3. The selection of these functions is the core of the finite element method. The approximation functions should be reliable in the sense that as the mesh becomes more refined, the approximate solution should converge to the exact solution monotonically. Oscillatory convergence is unreliable because it is possible to observe an increase in error with the refined mesh. Oscillatory and monotonic convergences are demonstrated in Fig. 3.4. Common approximation functions are usually polynomials since their differentiation and integration are rather straightforward compared to other functions.

In order to achieve a monotonically convergent solution, the polynomials chosen as approximation functions must satisfy four requirements:

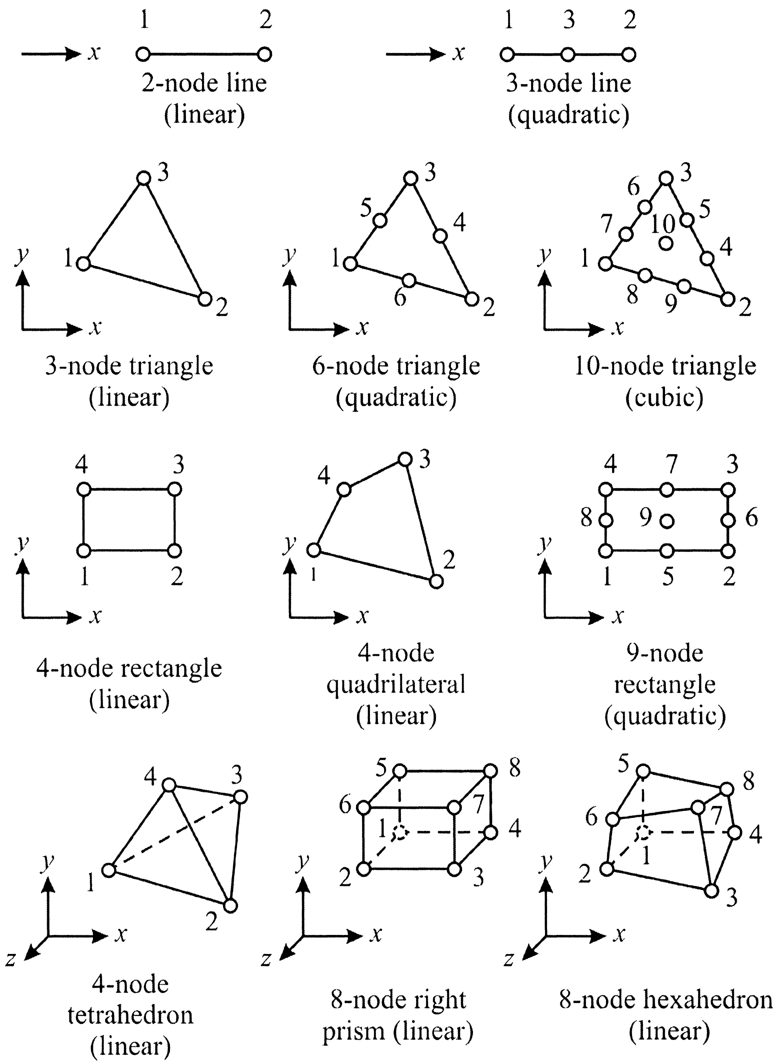

Fig. 3.1 Commonly used one-, two-, and three-dimensional finite elements.

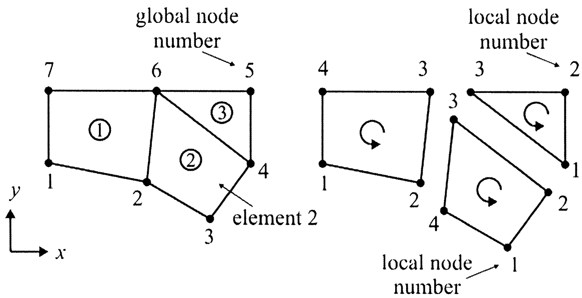

Fig. 3.2 Element numbers, global node numbers, and local node numbers.

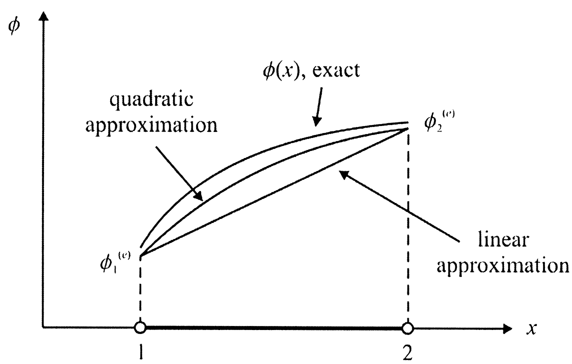

Fig. 3.3 Element approximation functions.

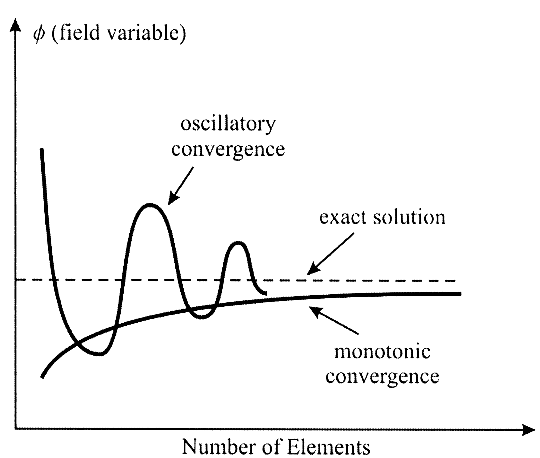

Fig. 3.4 Oscillatory and monotonic convergence of approximate solution.

Requirement 1. *Continuous behavior of the approximation function within the element—no kinks or jumps.*

Requirement 2. *Compatibility along the common nodes, boundaries or surfaces between adjacent elements—no gaps between elements*

The elements satisfying the continuity and compatibility requirements are called conformal elements (Fig. 3.5).

Requirement 3. *Completeness, permitting rigid body motion of the element and ensuring (constant) variation of* ϕ *and its derivatives within the element.*

The reason for this requirement is best illustrated by considering a cantilever beam under a concentrated load in the middle (Fig. 3.6). As a result of this loading, deformation occurs only to the left of the load. The section of the beam to the right of the load experiences only rigid-body translations and rotations (constant displacements and zero strain), i.e., no stresses and strains occur. Therefore, the element approximation functions must permit such behavior. Complete polynomials satisfy these requirements.

A complete polynomial of order n in one dimension can be written in compact form as

$$P_n(x) = \sum_{k=1}^{n+1} \alpha_k x^{k-1} \tag{3.1}$$

leading to complete polynomials of order 0, 1, and 2 (constant, linear, and quadratic) as

$$\begin{aligned} P_0(x) &= \alpha_1 \\ P_1(x) &= \alpha_1 + \alpha_2 x \\ P_2(x) &= \alpha_1 + \alpha_2 x + \alpha_3 x^2 \end{aligned} \tag{3.2}$$

In two dimensions, the compact form for a complete polynomial of order n can be written as

$$P_n(x) = \sum_{k=1}^{\frac{(n+1)(n+2)}{2}} \alpha_k x^i y^j \qquad i + j \le n \tag{3.3}$$

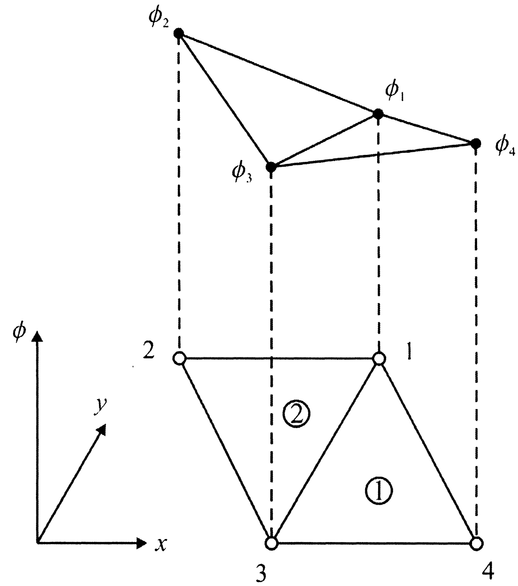

Fig. 3.5 Compatibility of approximation functions.

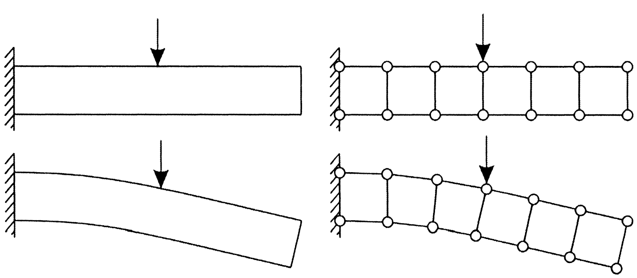

Fig. 3.6 A cantilever beam loaded at the middle and its FEA model.

Constant, linear, and quadratic complete polynomials in two dimensions can be written as

$$\begin{aligned} &P_0(x,y) = \alpha_1 \\ &P_1(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \\ &P_2(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \end{aligned} \tag{3.4}$$

The Pascal triangle shown in Fig. 3.7 is useful for including the appropriate terms to obtain complete approximating functions in any order.

$$\begin{array}{ccccccccccl}
 & & & & \alpha_1 & & & & & & \text{zeroth order} \\
 & & & \alpha_2 x & & \alpha_3 y & & & & & \text{first order} \\
 & & \alpha_4 x^2 & & \alpha_5 xy & & \alpha_6 y^2 & & & & \text{second order} \\
 & \alpha_7 x^3 & & \alpha_8 x^2 y & & \alpha_9 xy^2 & & \alpha_{10} y^3 & & & \text{third order} \\
\alpha_{11} x^4 & & \alpha_{12} x^3 y & & \alpha_{13} x^2 y^2 & & \alpha_{14} xy^3 & & \alpha_{15} y^4 & & \text{fourth order}
\end{array}$$

Fig. 3.7 Pascal's triangle for complete polynomials.

The order of the polynomial as an approximation function is dictated by the total number of nodes in an element, i.e., the number of coefficients, α_i, in the approximation function must be the same as the number of nodes in the element.

Requirement 4. *Geometric isotropy for the same behavior in each direction.*

Using complete polynomials satisfies this requirement of translation and rotation of the coordinate system. If the required degree of completeness does not provide a number of terms equal to the number of nodes, then this requirement can be satisfied by disregarding the non-symmetrical terms. In the case of a 4-noded rectangular element, the first-order complete polynomial has 3 coefficients, one less than the number of nodes. In order to circumvent this deficiency, the order of the polynomial can be increased to "complete" in the second degree, having 6 coefficients, two more than the number of nodes. As a result, two of the additional higher-order terms, which are $\alpha_4 x^2$, $\alpha_5 xy$, and $\alpha_6 y^2$, must be removed from the approximation function.

In order to satisfy the condition of geometric isotropy, only the term $\alpha_5 xy$ is retained in the approximation function, leading to

$$P_2(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \tag{3.5}$$

Approximation functions satisfying these four requirements ensure monotonic convergence of the solution as the element sizes decrease.

The element is referred to as C^0 continuous when only the field variable (none of its derivatives) maintains continuity along its boundary. If the field variable and its r^{th} derivative maintain continuity, the element is C^r continuous. A more extensive discussion is given by Huebner et al. (2001).

3.3 Coordinate Systems

3.3.1 Generalized Coordinates

The coefficients of the approximation functions, α_i, are referred to as the generalized coordinates. They are not identified with particular nodes. The generalized coordinates are independent parameters that specify the magnitude of the prescribed distribution of the field variable. They have no direct physical interpretation, but rather are linear combinations of the physical nodal degrees of freedom.

3.3.2 Global Coordinates

Global coordinates are convenient for specifying the location of each node, the orientation of each element, and the boundary conditions and loads for the entire domain. Also, the solution to the field variable is generally represented with respect to the global coordinates. However, approximation functions described in terms of the global coordinates are not convenient to use in the evaluation of integrals necessary for the construction of the element matrix.

3.3.3 Local Coordinates

A local coordinate system whose origin is located within the element is introduced in order to simplify the algebraic manipulations in the derivation of the element matrix. The use of natural coordinates in expressing the approximation functions is particularly advantageous because special integration formulas can often be employed to evaluate the integrals in the element matrix. Natural coordinates also play a crucial role in the development of elements with curved boundaries (discussed under isoparametric elements, Sec. 6.2.2.5).

3.3.4 Natural Coordinates

A local coordinate system that permits the specification of a point within the element by a dimensionless parameter whose absolute magnitude never exceeds unity is referred to as a natural coordinate system. Natural coordinates are dimensionless. They are defined with respect to the element rather than with reference to the global coordinates. Also, the natural coordinates are functions of the global coordinates in which the element is defined. As illustrated in Fig. 3.8, the basic purpose of the natural coordinate system is to describe the location of a point inside an element in terms of coordinates associated with the nodes of the element.

3.3.4.1 Natural Coordinates in One Dimension

As shown in Fig. 3.9, within a one-dimensional element (line segment), defined by two nodes (one at each end), the location of a point P denoted by x (global coordinate) on the element can be expressed in terms of length or centroidal coordinates.

3.3.4.1.1 Length Coordinates

The location of point P, x, is expressed as a linear combination of the global nodal coordinates, x_1 and x_2, and the length coordinates, ξ_1 and ξ_2, as

$$x = \xi_1 x_1 + \xi_2 x_2 \tag{3.6}$$

As shown in Fig. 3.9, ξ_1 and ξ_2 are defined as the ratios of lengths $\xi_1 = L_1/L$ and $\xi_2 = L_2/L$, with L representing the length of the line segment, $L = x_2 - x_1$. Since $L = L_1 + L_2$, ξ_1, and ξ_2 are not independent of each other and must satisfy the constraint relation

$$\xi_1 + \xi_2 = 1 \tag{3.7}$$

Solving for ξ_1 and ξ_2 via these equations written in matrix form as

$$\begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} \tag{3.8}$$

results in

$$\xi_1 = \frac{x_2 - x}{x_2 - x_1} \quad \text{and} \quad \xi_2 = \frac{x - x_1}{x_2 - x_1} \tag{3.9}$$

Such coordinates, whose behavior is shown in Fig. 3.10, have the property that one particular coordinate has a unit value at one node of the element and a zero value at the other node(s), i.e., $\xi_1(x_1) = 1$ and $\xi_1(x_2) = 0$, and $\xi_2(x_1) = 0$ and $\xi_2(x_2) = 1$.

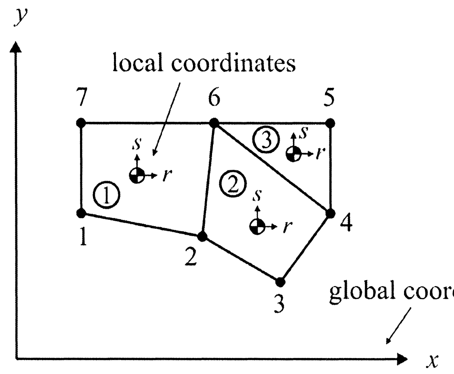

Fig. 3.8 Local and global coordinates in two dimensions.

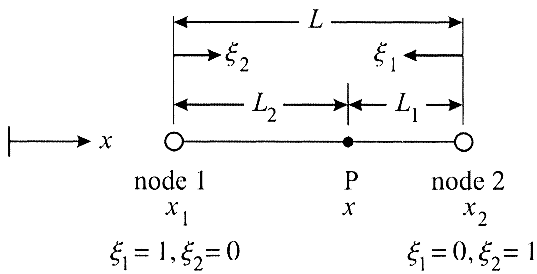

Fig. 3.9 Length coordinates in one dimension.

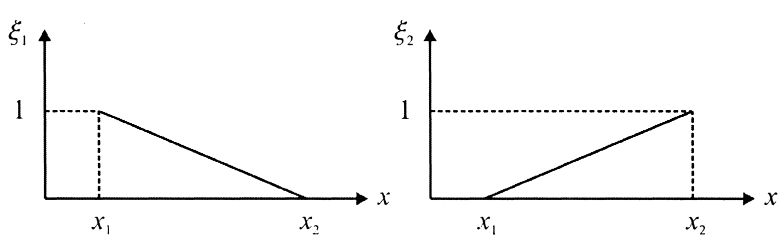

Fig. 3.10 Variation of length coordinates within the element.

3.3.4.1.2 Centroidal Coordinates

As shown in Fig. 3.11, x (the location of point P) with respect to a local coordinate system, r, located at the centroid of the line element becomes

$$x = r + x_1 + \frac{L}{2} \tag{3.10}$$

The local coordinate r is normalized in the form $\xi = r/(L/2)$ in order to achieve a dimensionless coordinate, ξ, and to ensure that its range never exceeds unity. Thus, the location of the point P becomes

$$x = \frac{L}{2}\xi + x_1 + \frac{L}{2} \tag{3.11}$$

Substituting for L ($L = x_2 - x_1$) and rearranging terms leads to

$$x = \frac{1}{2}(1-\xi)x_1 + \frac{1}{2}(1+\xi)x_2 \tag{3.12}$$

or

$$x = \sum_{i=1}^{2} N_i x_i \tag{3.13}$$

with $N_1 = (1-\xi)/2$ and $N_2 = (1+\xi)/2$. As shown in Fig. 3.12, $N_1(-1) = 1$ and $N_1(1) = 0$, and $N_2(-1) = 0$ and $N_2(1) = 1$.

3.3.4.2 Natural Coordinates in Two Dimensions

3.3.4.2.1 Area Coordinates

As shown in Fig. 3.13, within a two-dimensional element (triangular area) defined by three nodes, one at each apex, the location of a point P, denoted by (x, y) (global coordinates), on the element can be expressed as linear combinations of the global nodal coordinates, (x_1, y_1), (x_2, y_2), and (x_3, y_3), and the area coordinates, ξ_1, ξ_2, and ξ_3, as

$$\begin{aligned} x &= \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \\ y &= \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 \end{aligned} \tag{3.14}$$

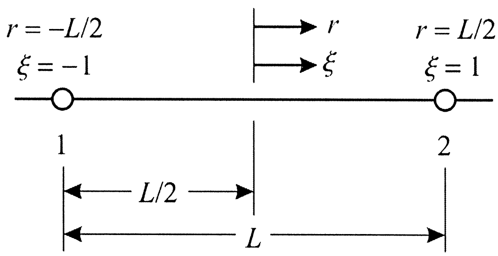

Fig. 3.11 Centroidal coordinates in one dimension.

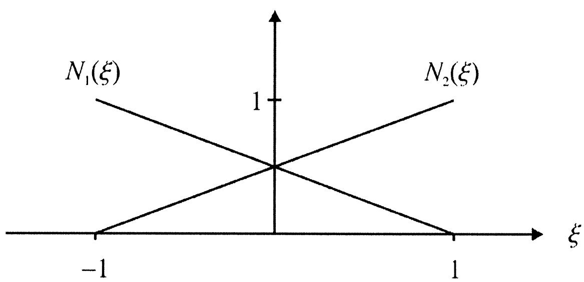

Fig. 3.12 Variation of centroidal coordinates within the element.

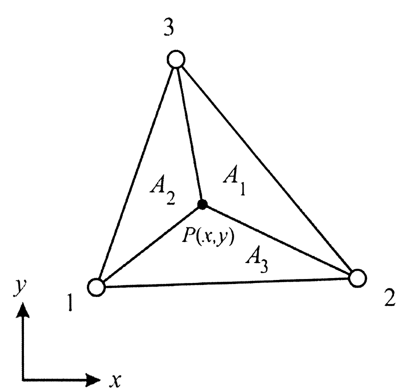

Fig. 3.13 Definition of area coordinates in a triangular element.

As illustrated in Fig. 3.13, ξ_1, ξ_2, and ξ_3 are defined as the ratios of areas $\xi_1 = A_1/A$, $\xi_2 = A_2/A$, and $\xi_3 = A_3/A$, with A representing the area of the triangle. Since $A_1 + A_2 + A_3 = 1$, ξ_1, ξ_2, and ξ_3 are not independent of each other and must satisfy the constraint relation

$$\xi_1 + \xi_2 + \xi_3 = 1 \tag{3.15}$$

Solving for ξ_1, ξ_2, and ξ_3 via Eq. (3.14) and (3.15) written in matrix form as

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} \tag{3.16}$$

results in

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & y_{23} & x_{32} \\ (x_3 y_1 - x_1 y_3) & y_{31} & x_{13} \\ (x_1 y_2 - x_2 y_1) & y_{12} & x_{21} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \tag{3.17}$$

where $x_{mn} = x_m - x_n$, $y_{mn} = y_m - y_n$, and

$$2A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{3.18}$$

As shown in Fig. 3.14, one particular area coordinate has a unit value at one node of the element and a zero value at the other node(s); $\xi_i(x_j) = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

The exact evaluation of the area integrals over a triangle can be obtained by employing the expression

$$I = \int_A \xi_1^m \xi_2^n \xi_3^\ell \, dxdy = \frac{m!n!\ell!}{(m+n+\ell+2)!} 2A \tag{3.19}$$

3.3.4.2.2 Centroidal Coordinates

In the case of a two-dimensional element with a quadrilateral shape defined by four nodes, one at each corner, the location of a point P, denoted by (x, y), on the element can be expressed with respect to the centroidal coor-

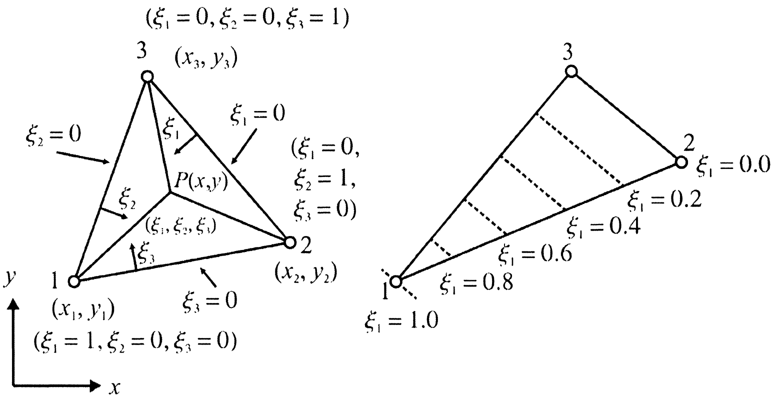

Fig. 3.14 Area coordinates within a triangular element.

dinate system (ξ,η) whose origin coincides with the centroid of the quadrilateral area, as shown in Fig. 3.15. The relationship between (x, y) and (ξ,η) can be expressed as

$$
\begin{aligned}
x &= a_x + b_x\xi + c_x\eta + d_x\xi\eta \\
y &= a_y + b_y\xi + c_y\eta + d_y\xi\eta
\end{aligned}
\tag{3.20}
$$

Also, these relations map a quadrilateral shape in global coordinates to a unit square in natural (centroidal) coordinates. Evaluation of these equations along $\eta = -1$ leads to

$$
\begin{aligned}
x &= a_x + b_x\xi - c_x - d_x\xi \\
y &= a_y + b_y\xi - c_y - d_y\xi
\end{aligned}
\tag{3.21}
$$

Eliminating the coordinate ξ from the resulting equations yields the linear relationship between the global coordinates

$$y = A + Bx \tag{3.22}$$

in which A and B are known explicitly. Considering the remaining sides of the square in the centroidal coordinates defined by the lines $\eta = 1$, $\xi = 1$, and $\xi = -1$ results in a straight-sided quadrilateral.

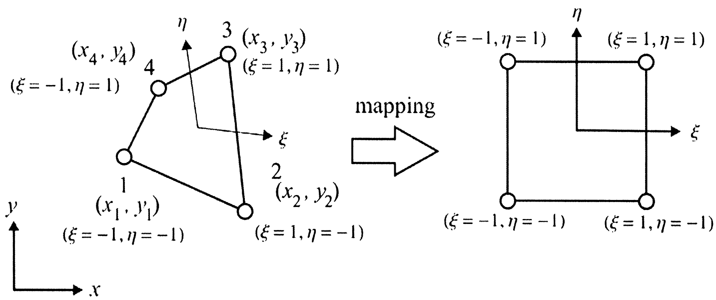

Fig. 3.15 Centroidal coordinates within a quadrilateral element.

Evaluation of x at $\xi = \pm 1$ and $\eta = \pm 1$ (four corners) leads to

$$\begin{aligned}
x_1 &= a_x - b_x - c_x + d_x \\
x_2 &= a_x + b_x - c_x - d_x \\
x_3 &= a_x + b_x + c_x + d_x \\
x_4 &= a_x - b_x + c_x - d_x
\end{aligned} \tag{3.23}$$

Solving for the coefficients a_x, b_x, c_x, and d_x, substituting back into Eq. (3.20), and collecting the terms multiplying x_i gives

$$\begin{aligned}
x = &\frac{1}{4}(1-\xi)(1-\eta)x_1 + \frac{1}{4}(1+\xi)(1-\eta)x_2 \\
&+ \frac{1}{4}(1+\xi)(1+\eta)x_3 + \frac{1}{4}(1-\xi)(1+\eta)x_4
\end{aligned} \tag{3.24}$$

A similar operation performed on y in Eq. (3.20) yields

$$\begin{aligned}
y = &\frac{1}{4}(1-\xi)(1-\eta)y_1 + \frac{1}{4}(1+\xi)(1-\eta)y_2 \\
&+ \frac{1}{4}(1+\xi)(1+\eta)y_3 + \frac{1}{4}(1-\xi)(1+\eta)y_4
\end{aligned} \tag{3.25}$$

Defining

$$\begin{aligned}
N_1 &= \frac{1}{4}(1-\xi)(1-\eta) \qquad & N_2 &= \frac{1}{4}(1+\xi)(1-\eta) \\
N_3 &= \frac{1}{4}(1+\xi)(1+\eta) \qquad & N_4 &= \frac{1}{4}(1-\xi)(1+\eta)
\end{aligned} \tag{3.26}$$

allows Eq. (3.24) and (3.25) to be rewritten as

$$x=\sum_{i=1}^{4} N_i(\xi,\eta)x_i \quad \text{and} \quad y=\sum_{i=1}^{4} N_i(\xi,\eta)y_i \tag{3.27}$$

Note that N_i can be written in compact form as

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{3.28}$$

with ξ_i and η_i representing the coordinates of the corner nodes in the natural coordinate system. It is worth noting that $N_i(\xi_j,\eta_j)=\delta_{ij}$, where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i \neq j$. The variations of N_i within a quadrilateral element are given schematically in Fig. 3.16.

3.4 Shape Functions

Shape functions constitute the subset of element approximation functions. They cannot be chosen arbitrarily. As discussed in the previous section, the element approximation functions are chosen to be *complete polynomials* with unknown generalized coordinates. For a one-dimensional element with m nodes as shown in Fig. 3.17, the element approximation function for the field variable, $\phi(x)$, is assumed as a polynomial of order $(m-1)$

$$\phi^{(e)}(x)=\alpha_1+\alpha_2 x+\alpha_3 x^2+\alpha_4 x^3+\cdots+\alpha_{m-1}x^{m-2}+\alpha_m x^{m-1} \tag{3.29}$$

or

$$\phi^{(e)}(x)=\mathbf{g}^T \boldsymbol{\alpha} \tag{3.30}$$

where

$$\mathbf{g}^T=\left\{1 \quad x \quad x^2 \quad \cdots \quad x^{m-1}\right\} \tag{3.31}$$

and

$$\boldsymbol{\alpha}^T=\left\{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \cdots \quad \alpha_m\right\} \tag{3.32}$$

Note that the number of generalized coordinates (α_i, $i=1,2,\ldots,m$) is equal to the number of nodes within the element.

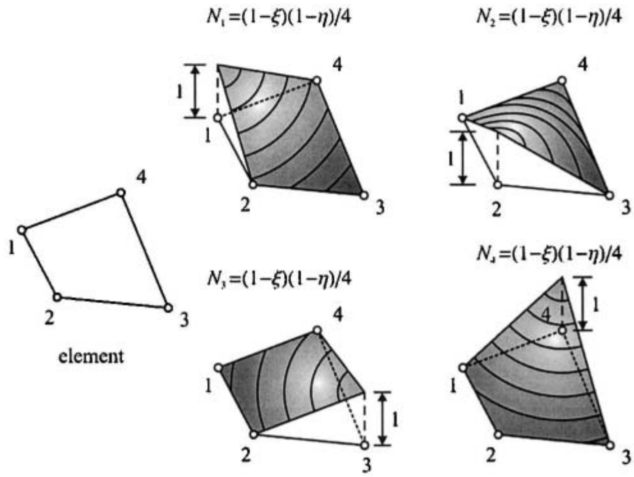

Fig. 3.16 Variation of N_i within a quadrilateral.

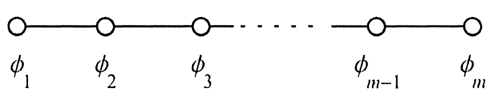

Fig. 3.17 A one-dimensional element with *m* nodes.

The field variable, $\phi^{(e)}(x)$, can also be expressed within the element through the use of its nodal values, $\phi_i \;\; (i = 1, m)$, in the form

$$\begin{aligned} \phi^{(e)}(x) = N_1\,\phi_1 + N_2\,\phi_2 + N_3\,\phi_3 + N_4\,\phi_4 + \cdots \\ + N_{m-1}\,\phi_{m-1} + N_m\,\phi_m \end{aligned} \tag{3.33}$$

or

$$\phi^{(e)}(x) = \mathbf{N}^T\,\boldsymbol{\varphi} \tag{3.34}$$

where

$$\mathbf{N}^T = \{N_1 \quad N_2 \quad N_3 \quad \cdots \quad N_m\} \tag{3.35}$$

and

$$\boldsymbol{\varphi}^T = \{\phi_1 \quad \phi_2 \quad \phi_3 \quad \cdots \quad \phi_m\} \tag{3.36}$$

in which N_i $(i=1, m)$ are referred to as shape functions. These functions are associated with node i and must have a unit value at node i and a zero value at all other nodes. Furthermore, they must have the same degree of polynomial variation as in the element approximation function.

The explicit form of the shape functions can be determined by solving for the generalized coordinates, α_i, in terms of the nodal coordinates, x_i, and nodal values, ϕ_i $(i=1,2,\ldots,m)$, through Eq. (3.29), and rearranging the resulting expressions in the form of Eq. (3.34). At each node, the field variable $\phi^{(e)}(x)$ is evaluated as

$$\begin{aligned}
\phi_1 &= \alpha_1 + \alpha_2 x_1 + \alpha_3 x_1^2 + \alpha_4 x_1^3 + \cdots + \alpha_{m-1} x_1^{m-2} + \alpha_m x_1^{m-1} \\
\phi_2 &= \alpha_1 + \alpha_2 x_2 + \alpha_3 x_2^2 + \alpha_4 x_2^3 + \cdots + \alpha_{m-1} x_2^{m-2} + \alpha_m x_2^{m-1} \\
&\vdots \\
\phi_m &= \alpha_1 + \alpha_2 x_m + \alpha_3 x_m^2 + \alpha_4 x_m^3 + \cdots + \alpha_{m-1} x_m^{m-2} + \alpha_m x_m^{m-1}
\end{aligned} \tag{3.37}$$

or in matrix form

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_m \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{m-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{m-1} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{Bmatrix} \quad \text{or} \quad \boldsymbol{\varphi} = \mathbf{A}\boldsymbol{\alpha} \tag{3.38}$$

Solving for the generalized coordinates in terms of nodal coordinates and nodal values of the field variable yields

$$\boldsymbol{\alpha} = \mathbf{A}^{-1}\boldsymbol{\varphi} \tag{3.39}$$

Substituting for the generalized coordinates in Eq. (3.30) results in

$$\phi^{(e)}(x) = \mathbf{g}^T \mathbf{A}^{-1}\boldsymbol{\varphi} \tag{3.40}$$

Comparison of Eq. (3.40) and (3.34) leads to the explicit form of the shape functions N_i as

$$\mathbf{N}^T = \mathbf{g}^T \mathbf{A}^{-1} \tag{3.41}$$

This formulation illustrates the determination of the shape functions for a one-dimensional element; its extension to two dimensions is straightforward. The properties of shape functions are:

1. $N_i = 1$ at node i and $N_i = 0$ at all other nodes.

2. $\sum_{i=1}^{m} N_i = 1$.

3.4.1 Linear Line Element with Two Nodes

3.4.1.1 Global Coordinate

For a line element with two nodes, the field variable, $\phi^{(e)}$, is approximated by a linear function (refer to Fig. 3.18) in terms of the global coordinate, x, as

$$\phi^{(e)}(x) = \alpha_1 + \alpha_2 x \tag{3.42}$$

This element approximation function ensures the inter-element continuity of only the field variable. The nodal values of the function are identified by ϕ_1 and ϕ_2.

Evaluation of the function at each node with coordinates x_1 and x_2 leads to

$$\phi_1 = \alpha_1 + \alpha_2 x_1 \quad \text{and} \quad \phi_2 = \alpha_1 + \alpha_2 x_2 \tag{3.43}$$

Solving for α_1 and α_2 and substituting for them in the element approximation function results in

$$\phi^{(e)}(x) = N_1(x)\phi_1 + N_2(x)\phi_2 \tag{3.44}$$

where $N_1 = (x_2 - x)/(x_2 - x_1)$ and $N_2 = (x - x_1)/(x_2 - x_1)$. These functions, referred to as interpolation or shape functions, are the same as the length coordinates, ξ_1 and ξ_2, and they also vary linearly with x (Fig. 3.19), as does the element approximation function. Because $N_i(x_j) = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$,

$$1 = \sum_{i=1}^{2} N_i \tag{3.45}$$

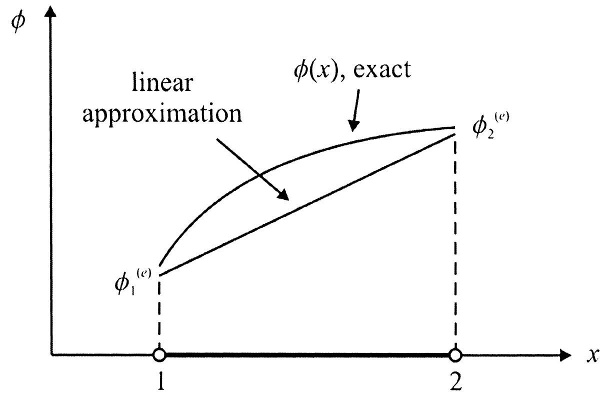

Fig. 3.18 Linear approximation for the field variable ϕ within a line element.

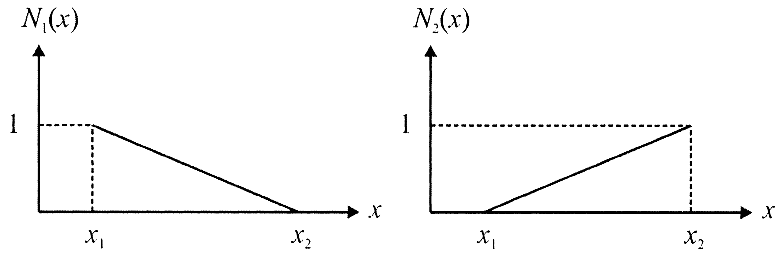

Fig. 3.19 Variation of linear shape functions within a 1-D line element.

3.4.1.2 Centroidal Coordinate

For a line element with two nodes, the field variable, $\phi^{(e)}$, is approximated by a linear function in terms of the natural (centroidal) coordinate, ξ, as

$$\phi^{(e)}(\xi) = \alpha_1 + \alpha_2 \xi \tag{3.46}$$

This element approximation function ensures the inter-element continuity of the field variable. The nodal values of the function are identified by ϕ_1 and ϕ_2. Evaluation of the function at each node with coordinates $\xi = -1$ and $\xi = 1$ leads to

$$\phi_1 = \alpha_1 - \alpha_2 \quad \text{and} \quad \phi_2 = \alpha_1 + \alpha_2 \tag{3.47}$$

Solving for α_1 and α_2 and substituting for them in the element approximation function results in

$$\phi^{(e)}(\xi) = N_1(\xi)\phi_1 + N_2(\xi)\phi_2 \tag{3.48}$$

where $N_1(\xi) = (1-\xi)/2$ and $N_2(\xi) = (1+\xi)/2$. These functions, referred to as interpolation or shape functions, vary linearly with ξ (Fig. 3.20), as in the case of the element approximation function.

Also, they have the property

$$1 = \sum_{i=1}^{2} N_i \tag{3.49}$$

because $N_i(\xi_j) = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

3.4.2 Quadratic Line Element with Three Nodes: Centroidal Coordinate

For a line element with three nodes, the field variable, $\phi^{(e)}$, is approximated by a quadratic function (schematic given in Fig. 3.21) in terms of the natural (centroidal) coordinate, ξ, as

$$\phi^{(e)}(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 \tag{3.50}$$

in order to ensure the inter-element continuity of the field variable. The element nodes are identified as 1, 2, and 3, with their nodal values as ϕ_1, ϕ_2, and ϕ_3. The middle node is located at the center of the line element. Evaluation of the function at each node with coordinates $\xi = -1$, $\xi = 0$, and $\xi = 1$ leads to

$$\phi_1 = \alpha_1 - \alpha_2 + \alpha_3, \quad \phi_3 = \alpha_1, \quad \phi_2 = \alpha_1 + \alpha_2 + \alpha_3 \tag{3.51}$$

Solving for α_1, α_2, and α_3 and substituting for them in the element approximation function results in

$$\phi^{(e)}(\xi) = N_1(\xi)\phi_1 + N_2(\xi)\phi_2 + N_3(\xi)\phi_3 \tag{3.52}$$

where $N_1(\xi) = \xi/[2(\xi-1)]$, $N_2(\xi) = \xi/[2(\xi+1)]$, and $N_3(\xi) = -(\xi+1)(\xi-1)$. These functions, referred to as interpolation or shape functions, vary quadratically with ξ (Fig. 3.22), as in the case of element approximation function.

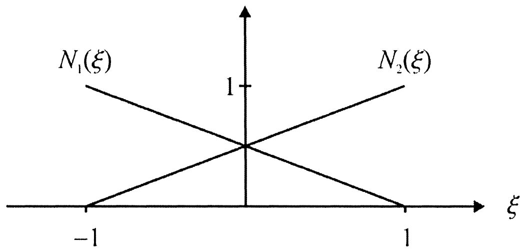

Fig. 3.20 Variation of linear shape functions within a 1-D line element.

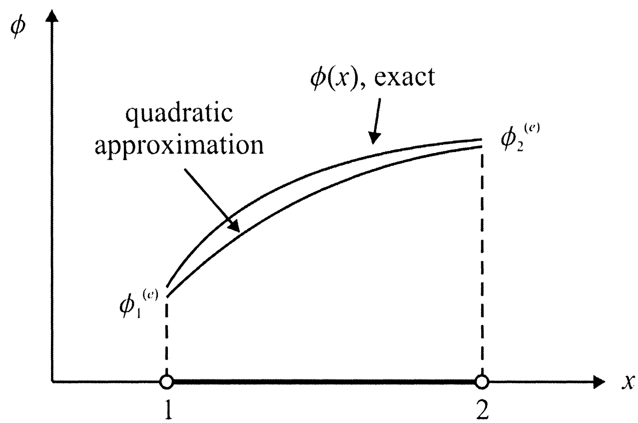

Fig. 3.21 Quadratic approximation for the field variable ϕ within a line element.

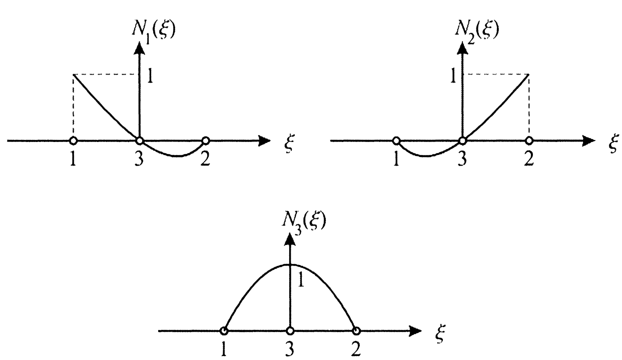

Fig. 3.22 Variation of quadratic shape functions within a 1-D line element.

Also, they have the property

$$1 = \sum_{i=1}^{3} N_i \tag{3.53}$$

because $N_i(\xi_j) = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

3.4.3 Linear Triangular Element with Three Nodes: Global Coordinate

Within a two-dimensional element (triangular area) defined by three nodes, one at each apex, the variation of the field variable, $\phi^{(e)}(x, y)$, can be approximated by a linear function (as illustrated in Fig. 3.23) of the form

$$\phi^{(e)}(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{3.54}$$

This function ensures the inter-element continuity of the field variable $\phi^{(e)}(x, y)$.

The element nodes are identified as 1, 2, and 3 in a counterclockwise orientation, with their nodal values as ϕ_1, ϕ_2, and ϕ_3. The nodal coordinates are specified by (x_1, y_1), (x_2, y_2), and (x_3, y_3).

The nodal values of the field variable must be satisfied as

$$\begin{aligned} \phi_1 &= \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \\ \phi_2 &= \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \\ \phi_3 &= \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \end{aligned} \tag{3.55}$$

leading to the determination of the generalized coefficients in the form

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \tag{3.56}$$

where

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{3.57}$$

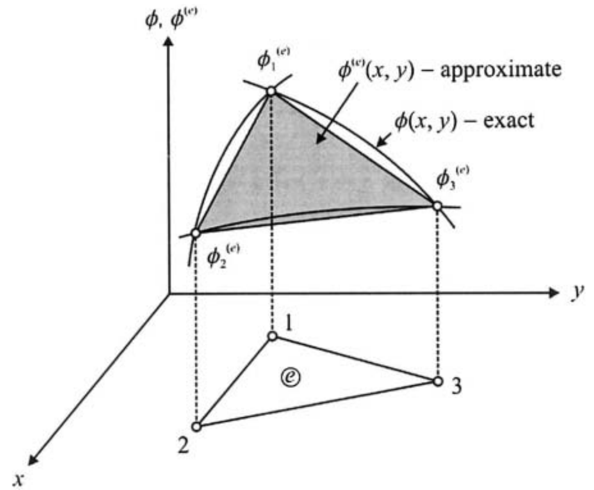

Fig. 3.23 Linear approximation for the field variable ϕ within a triangular element.

Substitution of α_1, α_2, and α_3 into the expression for the element approximation function results in

$$\phi^{(e)}(x,y) = N_1(x,y)\phi_1 + N_2(x,y)\phi_2 + N_3(x,y)\phi_3 \tag{3.58}$$

where the shape functions $N_1 = \xi_1$, $N_2 = \xi_2$, and $N_3 = \xi_3$ are the same as the area coordinates with properties $\xi_i(x_j, y_j) = \delta_{ij}$ and $\sum_{i=1}^{3} \xi_i = 1$. Their variation within the element is given in Fig. 3.24.

3.4.4 Quadratic Triangular Element with Six Nodes

The field variable can be approximated by a complete quadratic function within a triangular element in the form

$$\phi^{(e)}(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \tag{3.59a}$$

or

$$\phi^{(e)}(x,y) = \mathbf{g}^T \boldsymbol{\alpha} \tag{3.59b}$$

where the vectors $\mathbf{g}$ and $\boldsymbol{\alpha}$ are defined by

$$\mathbf{g}^T = \left\{1 \quad x \quad y \quad x^2 \quad xy \quad y^2\right\} \tag{3.60}$$

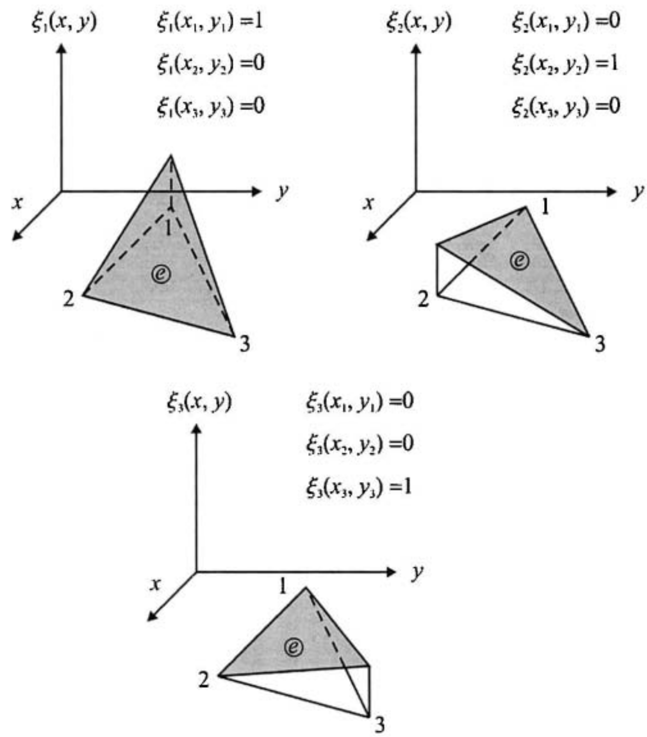

Fig. 3.24 Variation of linear shape functions within a triangular element.

and

$$\boldsymbol{\alpha}^T = \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6\} \tag{3.61}$$

However, this representation requires a triangular element with six nodes, as shown in Fig. 3.25, in order to determine its six unknown coefficients, α_i.

At each node, the field variable, $\phi^{(e)}(x_i, y_i)$, is evaluated as

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4 y_4 & y_4^2 \\ 1 & x_5 & y_5 & x_5^2 & x_5 y_5 & y_5^2 \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \quad \text{or} \quad \boldsymbol{\varphi} = \mathbf{A}\boldsymbol{\alpha} \tag{3.62}$$

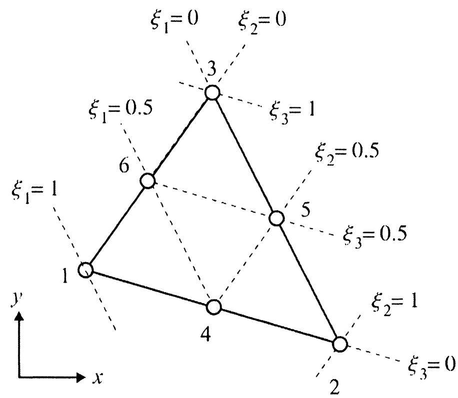

Fig. 3.25 Variation of linear shape functions within a triangular element.

Solving for the generalized coordinates in terms of nodal coordinates and nodal values of the field variable yields

$$\boldsymbol{\alpha} = \mathbf{A}^{-1} \boldsymbol{\varphi} \tag{3.63}$$

Substituting for the generalized coordinates in Eq. (3.59) results in

$$\phi^{(e)}(x, y) = \mathbf{g}^T \mathbf{A}^{-1} \boldsymbol{\varphi} \tag{3.64}$$

However, $\phi^{(e)}(x, y)$ can also be expressed within the element through the use of its nodal values ϕ_i as

$$\phi^{(e)}(x, y) = \sum_{i=1}^{6} N_i(x, y)\phi_i \quad \text{or} \quad \phi^{(e)}(x, y) = \mathbf{N}^T \boldsymbol{\varphi} \tag{3.65}$$

where $\mathbf{N}$ is the vector of shape functions, N_i $(i = 1, 6)$. Comparison of the last two equations results in the explicit form of the shape functions N_i as

$$\mathbf{N}^T = \mathbf{g}^T \mathbf{A}^{-1} \tag{3.66}$$

In providing the explicit forms of the shape functions, lengthy expressions are avoided by utilizing the expressions for the area coordinates of ξ_1, ξ_2, and ξ_3, as derived in Eq. (3.17), thus leading to

$$\mathbf{N}^T = \{(2\xi_1 - 1)\xi_1 \quad (2\xi_2 - 1)\xi_2 \quad (2\xi_3 - 1)\xi_3 \quad 4\xi_1\xi_2 \quad 4\xi_2\xi_3 \quad 4\xi_3\xi_1\} \tag{3.67}$$

or

$$\begin{aligned} N_1 &= (2\xi_1 - 1)\xi_1 , & N_2 &= (2\xi_2 - 1)\xi_2 , & N_3 &= (2\xi_3 - 1)\xi_3 \\ N_4 &= 4\xi_1\xi_2 , & N_5 &= 4\xi_2\xi_3 , & N_6 &= 4\xi_3\xi_1 \end{aligned} \tag{3.68}$$

Variation of these shape functions within the element is shown in Fig. 3.26.

3.4.5 Linear Quadrilateral Element with Four Nodes: Centroidal Coordinate

For a quadrilateral element with four nodes, the field variable, $\phi^{(e)}(x, y)$, is approximated by a linear function (refer to Fig. 3.27) in terms of the natural (centroidal) coordinates, $-1 \le \xi \le 1$ and $-1 \le \eta \le 1$, as

$$\phi^{(e)}(\xi,\eta) = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta \tag{3.69}$$

This element approximation function ensures the inter-element continuity of only the field variable. The nodal values of the function are identified by ϕ_1, ϕ_2, ϕ_3, and ϕ_4. Evaluation of the function at each node with coordinates $(\xi_1 = -1, \eta_1 = -1)$, $(\xi_2 = 1, \eta_2 = -1)$, $(\xi_3 = 1, \eta_3 = 1)$, and $(\xi_4 = -1, \eta_4 = 1)$ leads to

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \tag{3.70}$$

Solving for α_1, α_2, α_3, and α_4 results in

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{3.71}$$

and their substitution in the element approximation function yields

$$\phi^{(e)}(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\phi_i \tag{3.72}$$

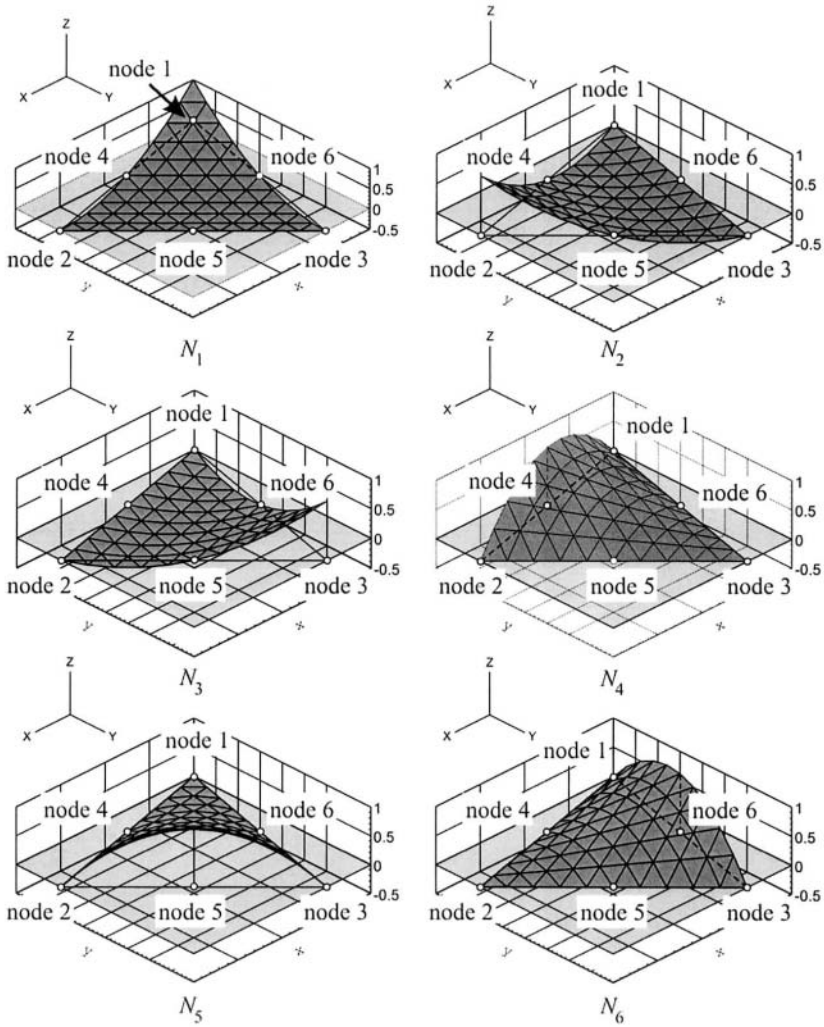

Fig. 3.26 Variation of quadratic shape functions within a triangular element.

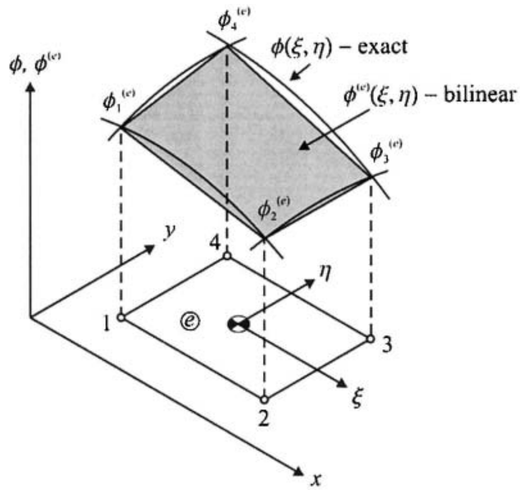

Fig. 3.27 Bi-linear approximation for the field variable ϕ within a quadrilateral element.

in which

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{3.73}$$

with ξ_i and η_i representing the coordinates of the corner nodes in the natural coordinate system. The shape functions have the property $N_i(\xi_j, \eta_j) = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. They are graphically illustrated in Fig. 3.28.

3.5 Isoparametric Elements: Curved Boundaries

The modeling of domains involving curved boundaries by using straight-sided elements may not provide satisfactory results. However, the family of elements known as "isoparametric elements" is suitable for such boundaries. The shape (or geometry) and the field variable of these elements are described by the same interpolation functions of the same order. The representation of geometry (element shape) in terms of linear (or nonlinear) shape functions can be considered as a mapping procedure that transforms a square in local coordinates to a regular quadrilateral (or distorted shape) in global coordinates (Fig. 3.29) (Ergatoudis et al. 1968).

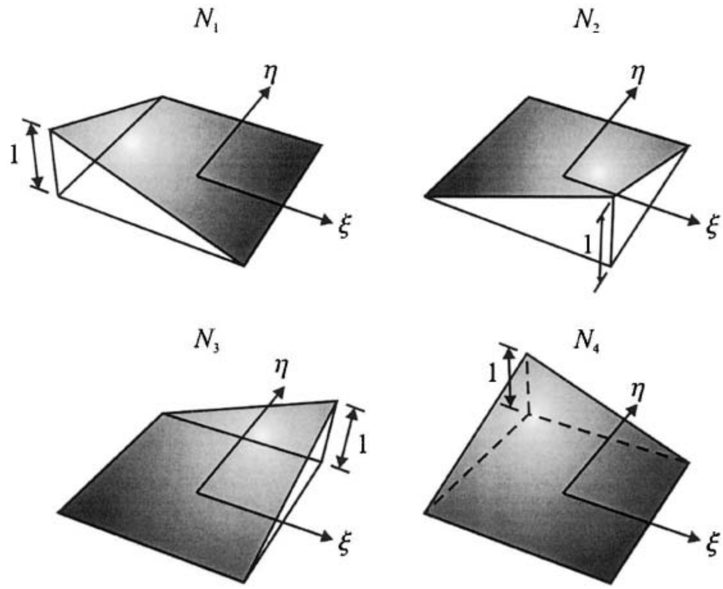

Fig. 3.28 Variation of bi-linear shape functions within a quadrilateral element.

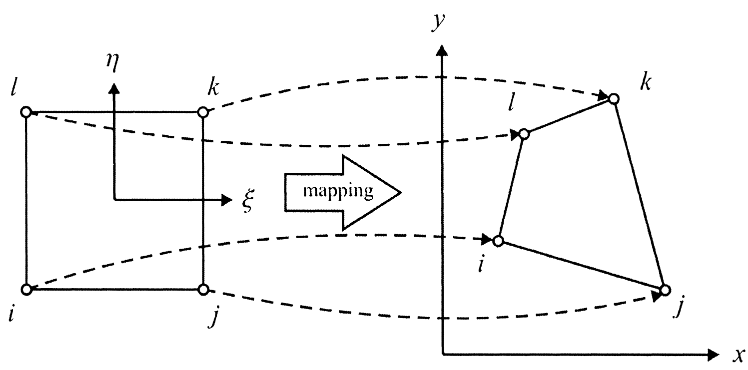

Fig. 3.29 Mapping from a unit square to an arbitrary straight-sided quadrilateral.

The most widely used elements are triangular or quadrilateral because of their ability to approximate complex geometries. An arbitrary straight-sided quadrilateral in global coordinates, (x, y), can be obtained by a point mapping from the "standard square" defined in natural coordinates, (ξ, η). The mapping shown in Fig. 3.29 can be achieved by

$$\begin{aligned} x &= \frac{1}{4}(1-\xi)(1-\eta)x_1 + \frac{1}{4}(1+\xi)(1-\eta)x_2 \\ &\quad + \frac{1}{4}(1+\xi)(1+\eta)x_3 + \frac{1}{4}(1-\xi)(1+\eta)x_4 \\ y &= \frac{1}{4}(1-\xi)(1-\eta)y_1 + \frac{1}{4}(1+\xi)(1-\eta)y_2 \\ &\quad + \frac{1}{4}(1+\xi)(1+\eta)y_3 + \frac{1}{4}(1-\xi)(1+\eta)y_4 \end{aligned} \tag{3.74}$$

or

$$x = \sum_{i=1}^{4} N_i(\xi, \eta)x_i \quad \text{and} \quad y = \sum_{i=1}^{4} N_i(\xi, \eta)y_i \tag{3.75}$$

in which

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{3.76}$$

with $(\xi_1 = -1,\ \eta_1 = -1)$, $(\xi_2 = 1,\ \eta_2 = -1)$, $(\xi_3 = 1,\ \eta_3 = 1)$, and $(\xi_4 = -1,\ \eta_4 = 1)$.

In the case of an element with curved boundaries in global coordinates, quadratic shape functions can be used to map it on to a unit square in local coordinates, as shown in Fig. 3.30. The mapping can be achieved by

$$x = \sum_{i=1}^{8} N_i(\xi, \eta)x_i \quad \text{and} \quad y = \sum_{i=1}^{8} N_i(\xi, \eta)y_i \tag{3.77}$$

in which

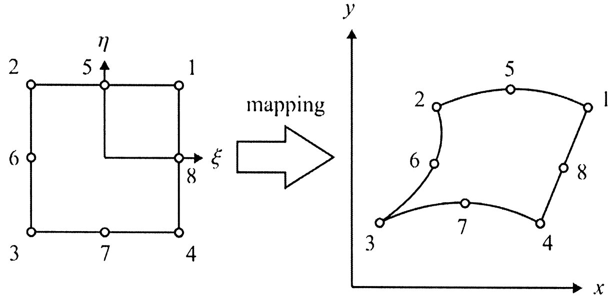

Fig. 3.30 Mapping from a unit square to a quadrilateral with curved sides.

$$
\begin{aligned}
N_1 &= \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{2}(N_5 + N_8) \\
N_2 &= \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{2}(N_5 + N_6) \\
N_3 &= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{2}(N_6 + N_7) \\
N_4 &= \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{2}(N_7 + N_8) \\
N_5 &= \frac{1}{2}(1-\xi^2)(1+\eta) \\
N_6 &= \frac{1}{2}(1-\xi)(1-\eta^2) \\
N_7 &= \frac{1}{2}(1-\xi^2)(1-\eta) \\
N_8 &= \frac{1}{2}(1+\xi)(1-\eta^2)
\end{aligned}
\tag{3.78}
$$

When the elements have curved boundaries, or arbitrary nodal locations (such as the quadrilaterals), the integrals appearing in the expression for the element matrix are most easily evaluated by using a natural coordinate system. Since it is more advantageous to use natural coordinates, the variables of integration are changed so that the integrals can be evaluated using natural coordinates. In two dimensions, the integral over an arbitrary quadrilateral region of $dx\,dy$ becomes an integral over a square area of $d\xi\,d\eta$ in a natural coordinate system in the form

$$\int_A f(x,y)dxdy = \int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)|\mathbf{J}|d\xi d\eta \tag{3.79}$$

where $|\mathbf{J}|$ is the determinant of the Jacobian matrix relating the term $dxdy$ to $d\xi d\eta$ from advanced calculus as

$$dxdy = |\mathbf{J}|d\xi d\eta \tag{3.80}$$

The Jacobian matrix, $\mathbf{J}$, is given by

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{3.81}$$

whose determinant is always positive, $|\mathbf{J}| > 0$, for a one-to-one mapping.

It is not necessary to use interpolation or shape functions of the same order for describing both the geometry and field variable of an element. If the geometry is described by a lower-order model (in comparison to that for the field variable), the element is called a "subparametric element." On the other hand, if the geometry is described by a higher-order interpolation function, then the element is termed a "superparametric" element.

3.6 Numerical Evaluation of Integrals

The evaluation of line or area integrals appearing in the finite element equations can be performed numerically by employing the Gaussian integration method (Stroud and Secrest 1966). This method locates sampling points (also called Gaussian points) to achieve the greatest accuracy.

3.6.1 Line Integrals

The line integrals encountered commonly are of the form

$$I = \int_a^b f(x)dx \tag{3.82}$$

The limits of this integral can be changed by introducing a new variable as

$$x = \frac{1}{2}\left[(b-a)\xi + (b+a)\right] \tag{3.83}$$

Thus, the integral given by Eq. (3.82) can be rewritten as

$$I = \int_{-1}^{1} f(\xi) J d\xi \tag{3.84}$$

in which the variables ξ and J are given by

$$\xi = \frac{2}{b-a}\left[x - \frac{(b+a)}{2}\right] \tag{3.85}$$

and

$$J = \frac{dx}{d\xi} = \frac{b-a}{2} \tag{3.86}$$

Integrals expressed in the form of Eq. (3.84) are almost always evaluated numerically. The most commonly used Gaussian integration technique approximates the integral in the form

$$I = \int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i) \tag{3.87}$$

The weights of the numerical integration are denoted by w_i, and the number of evaluation points, ξ_i (referred to as the Gaussian points), depends on the order of the polynomial approximation of the integrand.

In general, the integrand $f(\xi)$ in Eq. (3.87) can be approximated as

$$f(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3 + \ldots + \alpha_{2n}\xi^{2n-1} \tag{3.88}$$

resulting in

$$I = \int_{-1}^{1} f(\xi) d\xi = 2\alpha_1 + \frac{2}{3}\alpha_3 + \ldots + \frac{2}{2n-1}\alpha_{2n-1} \tag{3.89}$$

and

$$\begin{aligned} I = \sum_{i=1}^{n} w_i f(\xi_i) = \alpha_1 \sum_{i=1}^{n} w_i + \alpha_2 \sum_{i=1}^{n} w_i \xi_i + \alpha_3 \sum_{i=1}^{n} w_i \xi_i^2 + \ldots \\ + \alpha_{2n} \sum_{i=1}^{n} w_i \xi_i^{2n-1} \end{aligned} \tag{3.90}$$

Equating the coefficients of the α_i's in Eq. (3.89) and (3.90) leads to

$$\sum_{i=1}^{n} w_i = 2 , \quad \sum_{i=1}^{n} w_i \xi_i = 0$$
$$\sum_{i=1}^{n} w_i \xi_i^2 = \frac{2}{3} , \quad \sum_{i=1}^{n} w_i \xi_i^{2n-2} = \frac{2}{2n-1} \tag{3.91}$$
$$\sum_{i=1}^{n} w_i \xi_i^{2n-1} = 0$$

providing $2n$ equations in n unknowns for positions ξ_i and n unknowns for weights w_i. Hence, for a polynomial of degree $p = 2n-1$, it is sufficient to use n sampling points for exact integration, i.e., the exact integration is obtained if $n \geq (p+1)/2$. This means that for "n" sampling points, a polynomial of degree $(2n-1)$ can be integrated exactly.

Rewriting Eq. (3.84) in its final form as

$$I = \int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left[\frac{b-a}{2}\xi + \frac{b+a}{2}\right]d\xi \tag{3.92}$$

and assuming a third-order polynomial (p = 3) approximation for $f(\xi)$ in Eq. (3.92), this integral is approximated with two sampling points (n = 2) as

$$I \approx w_1 f(\xi_1) + w_2 f(\xi_2) \tag{3.93}$$

where $-1 \leq \xi_1$, $\xi_2 \leq 1$, and w_1, w_2 (Gaussian weights), ξ_1, and ξ_2 are to be determined. For each coefficient of the cubic representation of $f(\xi)$, Eq. (3.91) yields

$$\int_{-1}^{1} \xi^3 d\xi = 0 = w_1\xi_1^3 + w_2\xi_2^3 \tag{3.94a}$$

$$\int_{-1}^{1} \xi^2 d\xi = \frac{2}{3} = w_1\xi_1^2 + w_2\xi_2^2 \tag{3.94b}$$

$$\int_{-1}^{1} \xi d\xi = 0 = w_1\xi_1 + w_2\xi_2 \tag{3.94c}$$

$$\int_{-1}^{1} d\xi = 2 = w_1 + w_2 \tag{3.94d}$$

Multiplying Eq. (3.94c) by ξ_1^2 and subtracting it from Eq. (3.94a) gives

$$w_2\xi_2(\xi_2^2 - \xi_1^2) = w_2\xi_2(\xi_2 - \xi_1)(\xi_2 + \xi_1) = 0 \tag{3.95}$$

For this equality to be valid, the possibilities are:

1. $w_2 = 0 \quad \rightarrow$ one-term formula—reject.
2. $\xi_2 = 0 \quad \rightarrow \quad w_1 = 0$ one-term formula—reject.
3. $\xi_1 = \xi_2 \quad \rightarrow \quad w_1 = 0$ one-term formula—reject.
4. $\xi_2 = -\xi_1 \quad \rightarrow$ ACCEPTED.

Thus, substituting for $\xi_2 = -\xi_1$ in Eq. (3.94) leads to

$$w_1 = w_2 \tag{3.96a}$$

$$2\xi_1^2 = \frac{2}{\sqrt{3}} \rightarrow \xi_1 = \frac{1}{\sqrt{3}}, \xi_2 = -\frac{1}{\sqrt{3}} \tag{3.96b}$$

$$w_1 = w_2 = 1 \tag{3.96c}$$

The numerical integration, Eq. (3.93) becomes

$$I \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \tag{3.97}$$

The Gaussian points and weights for polynomials of order up to 5 are summarized in Table 3.1. The Gaussian points for higher order polynomial approximation are given by Abramowitz and Stegun (1972).

An example is considered that evaluates the line integral given by

$$I = \int_{-0.25}^{0.25} e^x dx \tag{3.98}$$

Table 3.1 Positions and weights for Gauss integration.

Gauss Points	ξ_i	w_i
n=1	0.00	2.00
n=2	$\pm\sqrt{1/3}$	1.00
n=3	0.00	8/9
	$\pm\sqrt{3/5}$	5/9
n=4	±0.339981	0.652145
	±0.861136	0.347854
n=5	0.00	0.568888
	±0.538469	0.478628
	±0.906179	0.236926

This integral can be rewritten as

$$I = \frac{1}{4}\int_{-1}^{1} e^{\xi/4} d\xi \tag{3.99}$$

Applying Gauss's formula with $n = 2$ integration points, this integral is approximated as

$$I \approx \frac{1}{4}\left[e^{-1/4\sqrt{3}} + e^{1/4\sqrt{3}}\right] = 0.505217 \tag{3.100}$$

The exact solution is $I = 2 \times \sinh(0.25) = 0.505224$.

3.6.2 Triangular Area Integrals

The area integrals over a triangular region given in the form

$$I = \int_A f(x, y)\, dA \tag{3.101}$$

can be rewritten as

$$I = \int_0^1 \int_0^{1-\xi_2} f(\xi_1, \xi_2)\left|\mathbf{J}\right| d\xi_1 d\xi_2 \tag{3.102}$$

in which $|\mathbf{J}|$ is the determinant of the *Jacobian* matrix expressed as

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi_1} & \dfrac{\partial y}{\partial \xi_1} \\ \dfrac{\partial x}{\partial \xi_2} & \dfrac{\partial y}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} (x_1 - x_3) & (y_1 - y_3) \\ (x_2 - x_3) & (y_2 - y_3) \end{bmatrix} = 2A \qquad (3.103)$$

relating the area coordinates (discussed in Sec. 3.3.4.2.1) to Cartesian coordinates

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi_1} \\ \dfrac{\partial}{\partial \xi_2} \end{Bmatrix} = [\mathbf{J}] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \qquad (3.104)$$

The extent of the triangular area of integration is defined by the coordinates (x_i, y_i) (with $i = 1,2,3$) of the vertices. The Gaussian approximation to the integration is expressed as

$$I = \int_0^1 \int_0^{1-\xi_2} f(\xi_1, \xi_2) |\mathbf{J}| \, d\xi_1 d\xi_2 \approx 2A \sum_{i=1}^{n} w_i f(\xi_{1i}, \xi_{2i}) \qquad (3.105)$$

in which the weights of the numerical integration are denoted by w_i. The number of evaluation points, ξ_{1i} and ξ_{2i}, are referred to as the Gaussian integration points and they depend on the order of the polynomial approximation of the integrand. Depending on the degree of approximation, the weights and the evaluation points are given by Huebner et al. (2001).

An example is considered that evaluates the area integral given by

$$I = \int_A xy \, dA \qquad (3.106)$$

in which the area A is defined by a triangle whose vertices are (1,1), (3,2), and (2,3), as shown in Fig. 3.31. This integral can also be evaluated exactly by using Eq. (3.19).

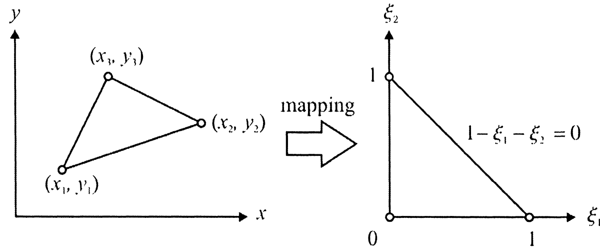

Fig. 3.31 A triangular element and its mapping.

The coordinates (x, y) of a point within a triangular area can be expressed as linear combinations of the nodal coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3) and the area coordinates ξ_1, ξ_2, and ξ_3 as

$$x = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = \xi_1 + 3\xi_2 + 2\xi_3 = -\xi_1 + \xi_2 + 2 \quad (3.107a)$$

$$y = \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 = \xi_1 + 2\xi_2 + 3\xi_3 = -2\xi_1 - \xi_2 + 3 \quad (3.107b)$$

with

$$\xi_1 + \xi_2 + \xi_3 = 1 \quad (3.107c)$$

Substituting for x and y in the integrand of Eq. (3.106) results in

$$I = 2A \int_0^1 \int_0^{1-\xi_2} (2\xi_1^2 - \xi_2^2 - \xi_1\xi_2 - 7\xi_1 + \xi_2 + 6) d\xi_1 d\xi_2 \quad (3.108)$$

Utilizing $n = 3$ Gaussian points as shown in Fig. 3.32, approximation to the integration by Eq. (3.105) becomes

$$I \approx 2A[w_1 f(\xi_{11}, \xi_{21}) + w_2 f(\xi_{12}, \xi_{22}) + w_3 f(\xi_{13}, \xi_{23})] \quad (3.109)$$

in which $w_1 = w_2 = w_3 = 1/6$, $\xi_{11} = 1/2$, $\xi_{21} = 0$, $\xi_{12} = 1/2$, $\xi_{22} = 1/2$, $\xi_{13} = 0$, and $\xi_{23} = 1/2$. The area of the triangle is obtained from Eq. (3.18) as $2A = 3$. Thus, the Gaussian approximation leads to

$$\begin{aligned} I \approx 2A[&w_1(2\xi_{11}^2 - \xi_{21}^2 - \xi_{11}\xi_{21} - 7\xi_{11} + \xi_{21} + 6) \\ &+ w_2(2\xi_{12}^2 - \xi_{22}^2 - \xi_{12}\xi_{22} - 7\xi_{12} + \xi_{22} + 6) \\ &+ w_3(2\xi_{13}^2 - \xi_{23}^2 - \xi_{13}\xi_{23} - 7\xi_{13} + \xi_{23} + 6)] \end{aligned}$$

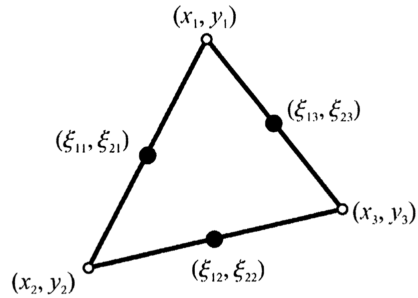

Fig. 3.32 Three Gaussian points, located at mid-sides, for approximate integration.

and

$$I \approx 3\frac{1}{6}\left[\left(2\frac{1}{4}-7\frac{1}{2}+6\right)+\left(2\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-7\frac{1}{2}+\frac{1}{2}+6\right)+\left(-\frac{1}{4}+\frac{1}{2}+6\right)\right]$$

and

$$I \approx \frac{1}{2}\left[6+\frac{25}{4}\right]=6.125 \tag{3.110}$$

For the exact evaluation, substituting for x and y in the integrand results in

$$I = \int_A (\xi_1^2 + 6\xi_2^2 + 6\xi_3^2 + 5\xi_1\,\xi_2 + 13\xi_2\xi_3 + 5\xi_1\,\xi_3)dA \tag{3.111}$$

Utilizing the formula of Eq. (3.19) for exact integration results in

$$\begin{aligned} I &= \frac{2A}{4!}(2!+6\times 2!+6\times 2!+5\times 1!\times 1!+13\times 1!\times 1!+5\times 1!\times 1!) \\ &= \frac{3}{24}[13\times 2+23]=\frac{49}{8}=6.125 \end{aligned} \tag{3.112}$$

3.6.3 Quadrilateral Area Integrals

The quadrilateral area integrals appearing in the form

$$I = \int_a^b \int_c^d f(x,y)\,dx\,dy \tag{3.113}$$

can be rewritten as

$$I = \int_{-1}^{1} \int_{-1}^{1} f(\xi,\eta)|\mathbf{J}|\,d\xi\,d\eta \tag{3.114}$$

in which $|\mathbf{J}|$ is the determinant of the Jacobian matrix expressed as

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \quad \text{relating} \quad \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = [\mathbf{J}] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \tag{3.115}$$

These integrals can be evaluated first with respect to one variable and then with respect to the other leading to

$$I = \int_{-1}^{1} \int_{-1}^{1} f(\xi,\eta)|\mathbf{J}|\,d\xi\,d\eta \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f(\xi_i,\eta_j)\left|\mathbf{J}(\xi_i,\eta_j)\right| \tag{3.116}$$

in which w_i represent the weights of the numerical integration, and ξ_i and η_i are the Gaussian integration points. They are given by Abramowitz and Stegun (1972) and depend on the order of the polynomial approximation of the integrand.

An example is considered that evaluates the area integral given by

$$I = \int_A xy\,dA \tag{3.117}$$

in which the area A is defined by a quadrilateral whose vertices are (1,1), (3,2), (4,4), and (2,3) as shown in Fig. 3.33.

The coordinates (x, y) of a point within a quadrilateral area can be expressed as linear combinations of the nodal coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) and the natural coordinates ξ and η as

$$\begin{aligned} x = \frac{1}{4}\big[&(1-\xi)(1-\eta) + 3(1+\xi)(1-\eta) \\ &+ 4(1+\xi)(1+\eta) + 2(1-\xi)(1+\eta)\big] \end{aligned} \tag{3.118}$$

$$\begin{aligned} y = \frac{1}{4}\big[&(1-\xi)(1-\eta) + 2(1+\xi)(1-\eta) \\ &+ 4(1+\xi)(1+\eta) + 3(1-\xi)(1+\eta)\big] \end{aligned} \tag{3.119}$$

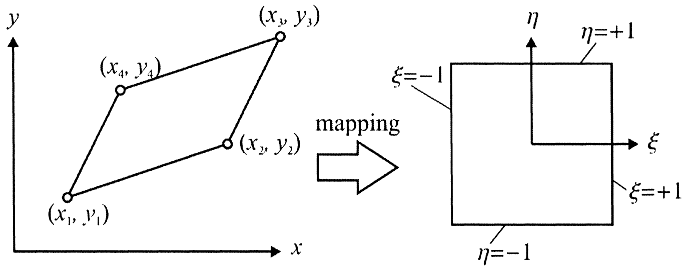

Fig. 3.33 A four-noded quadrilateral element and its mapping.

The Jacobian matrix is obtained as

$$\mathbf{J} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

with its determinant $|\mathbf{J}| = 3/4$.

Utilizing two Gaussian points as shown in Fig. 3.34, the approximation to the integration becomes

$$\begin{aligned} I \approx \frac{3}{4}\big[& w_1 w_1 f(\xi_1,\eta_1) + w_1 w_2 f(\xi_1,\eta_2) \\ & + w_2 w_1 f(\xi_2,\eta_1) + w_2 w_2 f(\xi_2,\eta_2)\big] \end{aligned} \tag{3.120}$$

in which $w_1 = w_2 = 1$, $\xi_1 = -1/\sqrt{3}$, $\xi_2 = 1/\sqrt{3}$, $\eta_1 = -1/\sqrt{3}$, and $\eta_2 = 1/\sqrt{3}$.

The function $f(\xi,\eta)$ is expressed as

$$\begin{aligned} f(\xi,\eta) = \frac{1}{16}\big[& (1-\xi)(1-\eta) + 3(1+\xi)(1-\eta) \\ & + 4(1+\xi)(1+\eta) + 2(1-\xi)(1+\eta)\big] \\ \times & \big[(1-\xi)(1-\eta) + 2(1+\xi)(1-\eta) \\ & + 4(1+\xi)(1+\eta) + 3(1-\xi)(1+\eta)0\big] \end{aligned} \tag{3.121}$$

Evaluation of the function $f(\xi,\eta)$ at Gaussian integration points results in their numerical evaluations as

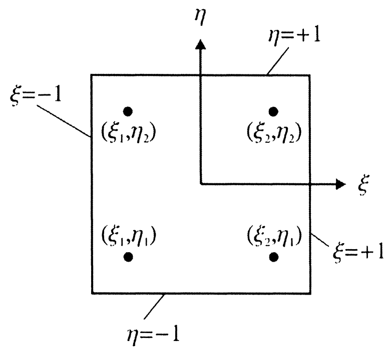

Fig. 3.34 Two Gaussian points, in each direction, for approximate integration.

$$f(\xi_1,\eta_1)=11.33012702$$
$$f(\xi_1,\eta_2)=6.16666667$$
$$f(\xi_2,\eta_1)=6.16666667$$
$$f(\xi_2,\eta_2)=2.66987298$$

Finally, the approximation to the integral from Eq. (3. 120) is determined to be 19.75.

By using Eq. (3.19), the exact evaluation of this integral can be obtained by integration over two triangular regions defined by the vertices (1,1), (3,2), and (2,3) and (3,2), (4,4), and (2,3). The exact integration over these two regions are obtained as 6.125 and 12.125. Their summation provides the exact integration over a quadrilateral defined by vertices (1,1), (3,2), (4,4), and (2,3). Thus, the exact integration becomes 19.75.

3.7 Problems

3.1. The *completeness criterion* for convergence of finite element solutions requires that the interpolating function must be able to reproduce exactly (that is, interpolate to the exact value at every point in the element). In particular, the approximation function $\phi(x,y)$ is specified as

$$\phi(x,y)=a+bx+cy=\sum N_i\phi_i$$

where a, b, and c are arbitrary constants, ϕ_i are the nodal values, and $N_i(x,y)$ are the interpolating functions.

(a) Derive a set of three equations that the interpolating functions $N_i(x, y)$ must satisfy for completeness.

(b) Show that the standard and quadratic linear interpolation functions for a triangular domain satisfy these requirements.

3.2. Using the coordinate transformation equations given in Sec. 3.5 for an 8-noded quadrilateral element, determine the isoparametric element shape whose nodal locations are

Node No.	x	y
1	6.0	3.0
2	–4.0	3.0
3	–5.0	–3.0
4	4.0	–3.0
5	1.0	4.0
6	–3.0	0.5
7	0.0	–2.0
8	5.0	0.0

3.3. The isoparametric formulation is useful for triangular, as well as for quadrilateral, elements. Also, the area coordinates (ξ_1, ξ_2, ξ_3) are commonly employed for triangular elements instead of using the local coordinates (r, s). However, because only two of these are independent coordinates, one of them, say ξ_3, can be eliminated in favor of ξ_1 and ξ_2. Thus, for a 3-noded triangle, the interpolation functions are $N_i = \xi_i$ $(i = 1,2,3)$ and the coordinate transformations, using $\xi_3 = 1 - \xi_1 - \xi_2$, are

$$x = \xi_1 x_1 + \xi_2 x_2 + (1 - \xi_1 - \xi_2) x_3$$
$$y = \xi_1 y_1 + \xi_2 y_2 + (1 - \xi_1 - \xi_2) y_3$$

As illustrated in Fig. 3.35, this clearly maps a triangle with vertices (1,0), (0,1), and (0,0) in the ξ_1 - ξ_2 plane into a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) in the x-y plane. Also, the integrals in the x-y plane may be related to integrals in the ξ_1 - ξ_2 plane by

$$\iint (\quad) dx dy = \int_0^1 \int_0^{1-\xi_1} (\quad) |J| d\xi_2 d\xi_1$$

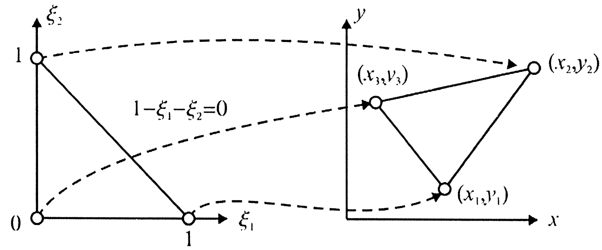

Fig. 3.35 A triangular element and its mapping.

Explicitly determine the coordinate transformations and the Jacobian matrix for the 6-noded triangle having the side nodes located at the midpoint of each side. Explain how it is possible to obtain a triangular element in the x-y plane with one or more curved sides. What is the form of the curve?

3.4. For a 4-noded element shown in Fig. 3.36, the mapping is achieved by

$$x=\sum_{i=1}^{4} N_i(\xi,\eta)x_i \quad \text{and} \quad y=\sum_{i=1}^{4} N_i(\xi,\eta)y_i$$

where

$$N_1=\frac{1}{4}(1-\xi)(1-\eta), \quad N_2=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3=\frac{1}{4}(1+\xi)(1+\eta), \quad N_4=\frac{1}{4}(1-\xi)(1+\eta)$$

(a) For this element explicitly determine the Jacobian determinant, and show that it is strictly linear in the local coordinates ξ and η and that the term proportional to the product $\xi\eta$ vanishes.

(b) Show that the Jacobian determinant becomes

$$|\mathbf{J}|=\frac{1}{4}\left[(x_4-x_3)(y_2-y_3)-(x_2-x_3)(y_4-y_3)\right]$$

for $\xi=\eta=1$.

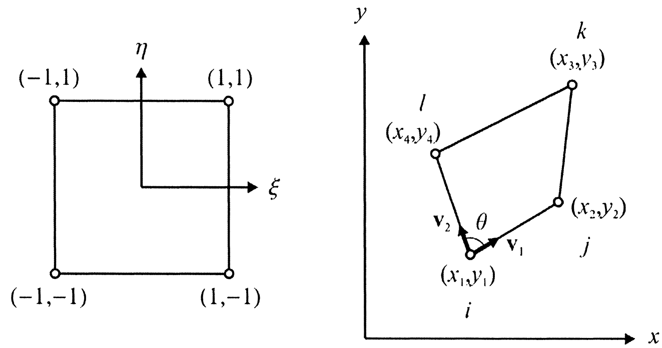

Fig. 3.36 A four-noded quadrilateral element and its mapping.

(c) Using the definition of the cross-product of the vectors v_1 and v_2 shown in Fig. 3.36, show that at $\xi = \eta = 1$

$$|\mathbf{J}| > 0 \quad \text{if} \quad 0 < \theta < \pi$$

(d) Based on the results of parts (a) and (c), provide a short argument to show that $|\mathbf{J}| > 0$ throughout the element and, hence, the coordinate transformation $(\xi, \eta) \to (x, y)$ is unique and invertible if the interior angles at all nodes are less than 180°.