22. Taylor's formula

Taylor's formula for polynomials and for arbitrary differentiable functions

Taylor's formula for polynomials

19A/31:02 (09:58)

Consider the polynomial $P_n(x)$ of degree $n \in \mathbb{N}$:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Let us select some point $x_0 \in \mathbb{R}$. It is known from the course of algebra that any polynomial can be expanded in powers of $(x - x_0)$; the degree of the polynomial will not change:

$$P_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + \underline{c_n}(x - x_0)^n. \qquad (1)$$

We want to obtain formulas for the coefficients c_k, $k = 0, \ldots, n$, using the differentiation operation.

To find the coefficient c_0, it is enough to calculate the value of the polynomial at the point x_0:

$((x-x_0)^3)' =$

$$c_0 = \underline{P_n(x_0)}. \qquad 0! = 1$$

$= 3(x-x_0)^2$

$0!$

To find the coefficient c_1, we firstly differentiate the polynomial:

$$P_n'(x) = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \cdots + nc_n(x - x_0)^{n-1}.$$

The coefficient c_1 is a free term of the derivative $P_n'(x)$ and, therefore, to find it, it suffices to calculate the value of the derivative at the point x_0:

$$c_1 = \underline{P_n'(x_0)}. \qquad 1! = 1$$

$1!$

Let us find the second derivative of the polynomial:

$$P_n''(x) = 2c_2 + \boxed{2 \cdot 3c}_3(x - x_0) + \cdots + (n-1)nc_n(x - x_0)^{n-2}.$$

Substituting the value $x = x_0$ into this derivative, we obtain the formula for the coefficient c_2:

$$c_2 = \frac{P_n''(x_0)}{2}. \qquad 2! = 1 \cdot 2 = 2$$

In this case, the formula contains not only the value of the derivative at the point x_0, but also the factor $\frac{1}{2}$.

$3! = \cancel{1}\cdot 2\cdot 3$

A factor of $\frac{1}{6}$ will appear in the formula for the coefficient c_3. This factor is more convenient to represent as $\frac{1}{3!}$ using the factorial function $n! = 1\cdot2\cdot3\cdots n$:

$$c_3 = \frac{P_n'''(x_0)}{3!}.$$

Continuing the process of differentiation, we finally obtain a derivative of order n, which contains a single term:

$$P_n^{(n)}(x) = 2\cdot3\cdots(n-1)nc_n.$$

$P_n^{(m)}(x) = 0$

Thus, for the coefficient c_n, we obtain the formula

$m = n+1, n+2, \ldots$

$$c_n = \frac{P_n^{(n)}(x_0)}{n!}.$$

Taking into account that $0! = 1$, all the formulas obtained for the coefficients c_k can be written in the following general form:

$$\boxed{c_k = \frac{P_n^{(k)}(x_0)}{k!}, \quad k = 0, 1, \ldots, n.}$$

$x - 0 = x$

Substituting the representations for the coefficients c_k into formula (1), we obtain *Taylor's formula for the polynomial* $P_n(x)$:

$$P_n(x) = \sum_{k=0}^{n} \frac{P_n^{(k)}(x_0)}{k!}(x-x_0)^k. \tag{2}$$

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{P_n^{(k)}(0)}{k!} x^k}$$

This formula allows us to obtain the expansion of the polynomial $P_n(x)$ in powers of $(x-x_0)$ using the values of the derivatives of the polynomial of order 0 up to n at the point x_0. Note that all derivatives of the polynomial $P_n(x)$ of higher orders $(n+1,\ n+2,\ \ldots)$ vanish.

The version of Taylor's formula (2) when $x_0 = 0$ is also called *Maclaurin's formula*.

Deriving the binomial formula using Taylor's formula for polynomials

`19A/41:00 (06:16)`

Let $n \in \mathbb{N}$, $b \in \mathbb{R}$. Consider a polynomial of the form $P_n(x) = (x+b)^n$ and expand it in powers of x. To do this, we use Taylor's formula (2) with $x_0 = 0$.

Let us calculate the derivatives of the polynomial $P_n(x)$ of order 0, 1, 2 at the point 0:

$$P_n^{(0)}(x) = (x+b)^n, \quad P_n^{(0)}(0) = b^n;$$

$$P_n'(x) = n(x+b)^{n-1}, \quad P_n'(0) = nb^{n-1};$$

$$P_n''(x) = n(n-1)(x+b)^{n-2}, \quad P_n''(0) = n(n-1)b^{n-2}.$$

It is easy to see that the formula for the derivative of the polynomial $P_n(x)$ of order k at the point 0 can be written in the general form:

$$P_n^{(k)}(0) = n(n-1)\cdots(n-k+1)b^{n-k}, \quad k = 0, \ldots, n.$$

Let us transform the resulting formula by multiplying and dividing it by the product $(n-k)\cdots 3\cdot 2 = (n-k)!$:

$$P_n^{(k)}(0) = \frac{n(n-1)\cdots(n-k+1)(n-k)\cdots 3\cdot 2}{(n-k)\cdots 3\cdot 2}b^{n-k} =$$
$$= \frac{n!}{(n-k)!}b^{n-k}.$$

Substitute the found values of the derivatives into formula (2) for $x_0 = 0$:

$$(x+b)^n = \sum_{k=0}^{n} \frac{P_n^{(k)}(0)}{k!}x^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}b^{n-k}x^k.$$

The expression found can be simplified by using the formula for the number of combinations $C_n^k = \frac{n!}{(n-k)!k!}$:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}$$

$$(x+b)^n = \sum_{k=0}^{n} C_n^k b^{n-k}x^k.$$

Replacing x by the value $a \in \mathbb{R}$, we obtain the *binomial formula*:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}.$$

$$k + n - k = n$$

Taylor's formula
for arbitrary differentiable functions 19B/00:00 (07:55)

Taylor's formula found earlier for polynomials (2) is an exact equality: both left-hand and right-hand sides in this equality contain polynomials of degree n. This equality holds for all $x \in \mathbb{R}$.

Suppose that, instead of the polynomial $P_n(x)$, we consider an arbitrary function $f(x)$ defined on some interval containing the point x_0. Also suppose that the function f is differentiable at the point x_0 up to the order n.

Now we cannot write relation (2) in the form of equality, but we can introduce into consideration the quantity $r_n(x_0, x)$, by which the function $f(x)$ differs from the sum given on the right-hand side of (2):

$$r_n(x_0, x) \overset{\text{def}}{=} f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

The value $r_n(x_0, x)$ is called the *remainder term of Taylor's formula* for the function f.

Using the remainder term, we can write the *Taylor's formula for an arbitrary differentiable function* f as follows:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + r_n(x_0, x).$$

? is small?

If for some $n \in \mathbb{N}$, $x_0 \in \mathbb{R}$, $x \in \mathbb{R}$, the value $r_n(x_0, x)$ is small, then this means that the function f can be approximated in the point x by a polynomial of degree n according to Taylor's formula, that is, we can obtain an approximation for the function f in the form of a simpler function (a polynomial).

$n = 1$

We have reason to expect that the remainder term $r_n(x_0, x)$ will be small, at least in a situation where the point x is close to the point x_0. Indeed, taking $n = 1$, we obtain

$f^{(0)}(x_0) = f(x_0)$

$0! = 1$

$1! = 1$

$$r_n(x_0, x) = f(x) - \sum_{k=0}^{1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k =$$
$$= f(x) - \big(f(x_0) + f'(x_0)(x - x_0)\big). = \bar{o}(x - x_0)$$

$f(x) = f(x_0) + f'(x_0)(x - x_0) + \bar{o}(x - x_0)$

$x \to x_0$

Since $n = 1$ and, therefore, the function f is differentiable at the point x_0, we obtain, by the definition of differentiability, that the right-hand side of the last equality is $o(x - x_0)$ as $x \to x_0$. This means that the remainder term in this case approaches 0, as $x \to x_0$, faster than the linear function $(x - x_0)$, that is, it is small enough for x close to x_0.

We can also expect that while increasing n, that is, in a situation where the function is differentiable more times, the rate of approach to 0 of the remainder term, as $x \to x_0$, will be higher. However, in order to prove this, we need to study the properties of the remainder term.

Various representations of the remainder term in Taylor's formula

General formula for the remainder term in Taylor's formula

19B/07:55 (17:45)

Now we derive a remainder term formula, which will include some arbitrary function φ. Choosing various specific functions as this arbitrary function, we can obtain different representations of the remainder term.

In deriving the general formula, we will assume that for the function f, in addition to differentiability n times at the point x_0, the following conditions are satisfied. We select the point x, assuming for definiteness that $x > x_0$, and require that the function f is differentiable $(n+1)$ times on the segment $[x_0, x]$. Note that this implies the continuity of the first n derivatives of the function f on this segment. We do not require continuity of the derivative $f^{(n+1)}$.

Introduce the following auxiliary function F:

$$\begin{aligned}F(t) &= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k = \\ &= f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2}(x-t)^2 - \\ &\quad - \frac{f'''(t)}{3!}(x-t)^3 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n.\end{aligned}$$

It is easy to see that the function F coincides with the remainder term $r_n(x_0, x)$ at x_0, and it is equal to zero at x:

$$F(x_0) = r_n(x_0, x), \qquad F(x) = 0. \tag{3}$$

Let us calculate the derivative of the function F. To do this, we firstly find the derivatives for the individual summands (recall that differentiation is performed with respect to the variable t):

$$\begin{aligned}&\big(f(x)\big)' = 0, \\ &\big(-f(t)\big)' = -f'(t), \\ &\big(-f'(t)(x-t)\big)' = f'(t) - f''(t)(x-t), \\ &\Big(-\frac{f''(t)}{2}(x-t)^2\Big)' = f''(t)(x-t) - \frac{f'''(t)}{2}(x-t)^2, \\ &\Big(-\frac{f'''(t)}{3!}(x-t)^3\Big)' = \frac{f'''(t)}{2}(x-t)^2 - \frac{f^{(3)}(t)}{3!}(x-t)^3, \\ &\ldots \\ &\Big(-\frac{f^{(n)}(t)}{n!}(x-t)^n\Big)' = \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n.\end{aligned}$$

We see that as a result of differentiation of each summand (starting from the third one), a term appears that is opposite to one of the terms of the previous summand. Thus, after summing and collecting terms, we get the following formula:

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n. \tag{4}$$

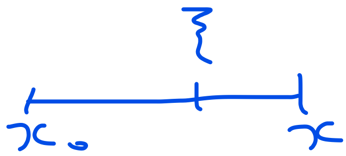
Now we apply the Cauchy's mean value theorem to the function $F(t)$ and an arbitrary function $\varphi(t)$ on the segment $[x_0, x]$. The function F satisfies all the conditions of this theorem. For the function φ, it is necessary to require that it be continuous on the segment $[x_0, x]$, differentiable on the interval (x_0, x), and that $\varphi'(t) \neq 0$ for $t \in (x_0, x)$.

By virtue of the Cauchy's mean value theorem, there exists a point $\xi \in (x_0, x)$ such that the following relation holds for the functions F and φ:

$$\frac{F(x) - F(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{F'(\xi)}{\varphi'(\xi)}.$$

Substitute the values of $F(x)$, $F(x_0)$ into the resulting relation (see (3)) and use formula (4) for $F'(t)$:

$$\frac{0 - r_n(x_0, x)}{\varphi(x) - \varphi(x_0)} = \frac{-f^{(n+1)}(\xi)(x-\xi)^n}{n!\varphi'(\xi)}.$$



As a result, we obtain the following *formula for the remainder term containing an arbitrary function* $\varphi(t)$:

$$r_n(x_0, x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n\big(\varphi(x) - \varphi(x_0)\big)}{n!\varphi'(\xi)}. \tag{5}$$

Representation of the remainder term in the form of Cauchy and in the form of Lagrange

`19B/25:40 (12:51)`

Based on formula (5), we can obtain various representations of the remainder term by choosing specific functions as the function φ.

First, put $\varphi(t) = \varphi_1(t) = x - t$. This function satisfies all the necessary conditions: it is continuous and differentiable on the segment $[x_0, x]$, and, moreover, its derivative $\varphi_1'(t)$ is equal to -1, that is, it does not vanish on interval (x_0, x).

Substituting in (5) the values $\varphi_1(x) = 0$, $\varphi_1(x_0) = x - x_0$, and $\varphi_1'(\xi) = -1$, we obtain:

$$r_n(x_0, x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n(x - x_0)}{n!}. \tag{6}$$

Let us additionally transform the resulting formula by representing the value ξ in the form $\xi = x_0 + \theta(x - x_0)$. Since $\xi \in (x_0, x)$, we obtain that $\theta \in (0, 1)$. Note that this expression equals x_0 when $\theta = 0$, and it equals x when $\theta = 1$.

Replacing the value ξ in formula (6) with the expression $x_0 + \theta(x - x_0)$, we obtain:

$$r_n(x_0, x) =$$

$$= \frac{f^{(n+1)}\big(x_0+\theta(x-x_0)\big)\Big(x-\big(x_0+\theta(x-x_0)\big)\Big)^n (x-x_0)}{n!} =$$

$$= \frac{f^{(n+1)}\big(x_0+\theta(x-x_0)\big)(1-\theta)^n}{n!}(x-x_0)^{n+1}.$$

The resulting representation of the remainder term is called the *Cauchy form of the remainder term.*

Now we choose the following power function $\varphi_2(t) = (x-t)^{n+1}$ as the function $\varphi(t)$. This function also satisfies all the necessary conditions: it is continuous and differentiable on the segment $[x_0, x]$, and, moreover, its derivative $\varphi_2'(t)$ is equal to $-(n+1)(x-t)^n$ and, therefore, does not vanish on the interval (x_0, x).

Substituting in (5) the values $\varphi_2(x) = 0$, $\varphi_2(x_0) = (x-x_0)^{n+1}$, and $\varphi_2'(\xi) = -(n+1)(x-\xi)^n$, we obtain:

$$r_n(x_0,x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n\big(0-(x-x_0)^{n+1}\big)}{n!\big(-(n+1)(x-\xi)^n\big)} =$$

$$= \underline{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}.$$

The resulting representation of the remainder term is called the *Lagrange form of the remainder term.* This representation is interesting in that it is similar to the term in Taylor's formula corresponding to $k = n+1$, except that the derivative is found not at the point x_0, but at some point ξ from the interval (x_0, x).

$n+1$

Let us write Taylor's formula with the remainder term in the Lagrange form:

$\xi = 1$ $\quad n=20 \quad 20!$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}. \tag{7}$$

$x_0 \quad \xi \quad x$

$(x-x_0)^{20} = \left(\frac{1}{10}\right)^{20} = 0.\underbrace{00\ldots0}_{20}1$

Representation of the remainder term in the Peano form 20A/00:00 (20:31)

We noted earlier that for $n = 1$ the remainder term of Taylor's formula has the form $o(x-x_0)$, $x \to x_0$. It turns out that similar representations for the remainder term in the form of little-o can also be obtained for other values of n.

Theorem (on Taylor's formula with the remainder term in the Peano form).

Let the function f be n times continuously differentiable on the segment $[x_0, x]$. Then the following expansion of the function f by Taylor's formula takes place:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o\big((x-x_0)^n\big), \quad x \to x_0. \tag{8}$$

The representation of the remainder term $r_n(x_0, x) = o\big((x-x_0)^n\big)$, $x \to x_0$, used in this formula is called the *remainder term in the Peano form*. Thus, when expanding the function f by Taylor's formula up to the derivative of order n, the remainder term decreases, as $x \to x_0$, faster than the function $(x-x_0)^n$.

Remark.

The assertion of the theorem remains valid in the case when the derivative of order n is not continuous. However, the continuity condition for this derivative allows us to simplify the proof.

Proof.

The conditions of the theorem allow us to expand the function f by Taylor's formula, taking the $n-1$ term in it and representing the remainder term in the Lagrange form (see (7)):

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n)}(\xi)}{n!}(x-x_0)^n. \tag{9}$$

The value of ξ, which is an argument of the function $f^{(n)}$ in the remainder term, depends on x_0 and x. The point x_0 does not change, but we can change the point x, moving it closer to x_0. When the point x changes, the point ξ will change in some way, so we can consider ξ as a function of x: $\xi = \xi(x)$. We do not know how exactly $\xi(x)$ will change when x changes, however the following double estimate will always be valid: $x_0 < \xi(x) < x$. From this double estimate, by virtue of the second theorem on passing to the limit in inequalities for functions, it follows that

$$\lim_{x \to x_0} \xi(x) = x_0. \tag{10}$$

Thus, although the properties of the function $\xi(x)$ are unknown to us, it can be stated that its limit, as $x \to x_0$, exists and is equal to x_0.

Now turn to the function $f^{(n)}(\xi)$. It can be considered as a superposition of the form $f^{(n)}\big(\xi(x)\big) = \big(f^{(n)} \circ \xi\big)(x)$, where the external function is $f^{(n)}(t)$

and the internal function is $\xi(x)$. Since, by the condition of the theorem, the function $f^{(n)}(t)$ is continuous in a neighborhood of x_0, we can calculate the limit of superposition $f^{(n)}\big(\xi(x)\big)$, as $x \to x_0$, using the theorem on the limit of superposition in the case when the external function is continuous. By virtue of this theorem, we can move the limit sign under the sign of the external function, and then use the limit relation (10):

$$\lim_{x\to x_0} f^{(n)}\big(\xi(x)\big) = f^{(n)}\left(\lim_{x\to x_0} \xi(x)\right) = f^{(n)}(x_0). \tag{11}$$

Denote $\alpha(x) = f^{(n)}\big(\xi(x)\big) - f^{(n)}(x_0)$. Then it follows from the limit relation (11) that

$$\lim_{x\to x_0} \alpha(x) = \lim_{x\to x_0}\Big(f^{(n)}\big(\xi(x)\big) - f^{(n)}(x_0)\Big) = f^{(n)}(x_0) - f^{(n)}(x_0) = 0.$$

So, we have proved that the function $f^{(n)}\big(\xi(x)\big)$ can be represented as

$$f^{(n)}\big(\xi(x)\big) = f^{(n)}(x_0) + \alpha(x), \text{ where } \alpha(x) \to 0 \text{ as } x \to x_0.$$

Substitute the obtained representation of the function $f^{(n)}\big(\xi(x)\big)$ into the remainder term from the right-hand side of relation (9):

$$\frac{f^{(n)}(\xi)}{n!}(x-x_0)^n = \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{\alpha(x)}{n!}(x-x_0)^n. \tag{12}$$

The first term on the right-hand side of equality (12) can be added to the sum of Taylor's formula (9) as the term corresponding to $k = n$. The second term can be represented as $\tilde{\alpha}(x)(x-x_0)^n$, where $\tilde{\alpha}(x) = \frac{\alpha(x)}{n!} \to 0$ as $x \to x_0$. Thus, this second term is $o\big((x-x_0)^n\big)$, $x \to x_0$.

After the indicated transformations on the right-hand side of relation (9) are performed, this relation takes the form (8). □

$x_0 = 0$

Expansions of elementary functions by Taylor's formula in a neighborhood of zero

$(e^{x_0})^{(k)} = e^{x_0} = e^{0} = \underline{\underline{1}}$

Expansions of functions e^x, $\sin x$, $\cos x$, ~~$\sinh x$, $\cosh x$~~

`20A/20:31 (19:50)`

FUNCTION e^x.

Since $(e^x)^{(n)} = e^x$, $n = 0, 1, 2, \ldots$, we obtain that the derivatives of this function of any order are equal to 1 at the point 0. Therefore, the expansion of the function e^x by Taylor's formula at the point $x_0 = 0$ with the remainder term in the Peano form will be as follows:

$$e^x = \sum_{k=0}^{n} \frac{(e^x)^{(k)}\big|_{x=0}}{k!} x^k + \bar{\bar{o}}(x^n) = \sum_{k=0}^{n} \frac{x^k}{k!} + \bar{\bar{o}}(x^n)$$

$x \to 0$

$n=1$ $\quad e^x = 1 + x + \bar{\bar{o}}(x)$ $\quad e^x - 1 \sim x$

$k=0$, $k=1$

$e^x - 1 = x + \bar{\bar{o}}(x)$ $\quad x \to 0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n) =$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n), \quad x \to 0.$$

$1 = \lim\limits_{x\to 0} \frac{e^x - 1}{x} \left(\frac{0}{0}\right) = \lim\limits_{x\to 0} \frac{x + \bar{\bar{o}}(x)}{x} = \lim\limits_{x\to 0} \frac{x}{x} + \lim\limits_{x\to 0} \frac{\bar{\bar{o}}(x)}{x} = 1$, $\quad \frac{\bar{\bar{o}}(x)}{x} \to 0$

$\frac{\bar{\bar{o}}(1)}{1} \to 0$

From the obtained formula, the previously proved equivalence $e^x \sim 1 + x$, $x \to 0$, follows, since for $n = 1$ the expansion takes the form $e^x = 1 + x + o(x)$, $x \to 0$.

Remark.

This and all subsequent expansions of elementary functions are valid for both positive and negative values of x belonging to the domain of definition of the function.

$\underset{0}{\overset{0}{\sin x}}, \ \underset{1}{\overset{1}{\cos x}}, \ \underset{2}{\overset{0}{-\sin x}}, \ \underset{3}{\overset{-1}{-\cos x}}, \ \underset{4}{\overset{0}{\sin x}}, \ \underset{5}{\overset{1}{\cos x}}, \dots$

Function $\sin x$.

Let us sequentially find the derivatives of the function $\sin x$ at the point 0. The function $\sin x$ itself vanishes at the point 0. Its first derivative is $\cos x$, so it equals 1 at the point 0. The second derivative is $(-\sin x)$, it vanishes at the point 0. Finally, the third derivative is $(-\cos x)$, it equals -1 at the point 0. The fourth derivative coincides with the original function $\sin x$, therefore, starting from it, the set of values at the point 0 will be repeated: 0, 1, 0, -1, ...

So, we obtain that even-order derivatives vanish at the point 0, and odd-order derivatives take alternating values of 1 and -1, starting from 1. Therefore, the expansion of the function $\sin x$ by Taylor's formula at the point $x_0 = 0$ with the remainder term in the Peano form will be as follows:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) =$$

$$= \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + o(x^{2n+2}), \quad x \to 0.$$

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

The remainder term has the form $o(x^{2n+2})$, since we can take into account one more (zero-valued) term corresponding to $k = 2n + 2$ in the sum.

It should be noted that the expansion of the function $\sin x$ contains only odd powers of x, starting with x in the first power, and their signs alternate.

From the obtained formula, the previously proved equivalence $\sin x \sim x$, $x \to 0$, follows, since for $n = 0$ the expansion takes the form $\sin x = x + o(x^2)$, $x \to 0$.

$\lim \frac{x + \bar{\bar{o}}(x^2)}{x} = \lim \frac{x}{x} + \lim \frac{\bar{\bar{o}}(x^2)}{x} = 1$

FUNCTION $\cos x$.

With successive differentiation of the function $\cos x$, we will obtain the following functions (starting with the zero derivative): $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, $\cos x$, $-\sin x, \dots$ At the point 0, these functions take the following values: $1, 0, -1, 0, 1, 0, \dots$ In this case, the derivatives of odd order vanish at the point 0, and the derivatives of even order take alternating values of 1 and -1, starting from 1. Therefore, the expansion of the function $\cos x$ by Taylor's formula at the point $x_0 = 0$ with the remainder term in the Peano form will be as follows:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1}) =$$
$$= \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + o(x^{2n+1}), \quad x \to 0.$$

The remainder term has the form $o(x^{2n+1})$, since we can take into account one more (zero-valued) term corresponding to $k = 2n + 1$ in the sum.

It should be noted that the expansion of the function $\cos x$ contains only even powers of x, starting with $x^0 = 1$, and their signs alternate.

From this formula, the previously proved equivalence $\cos x \sim 1 - \frac{x^2}{2}$, $x \to 0$, follows, since for $n = 1$ the expansion takes the form $\cos x = 1 - \frac{x^2}{2} + o(x^3)$, $x \to 0$.

FUNCTIONS $\sinh x$ AND $\cosh x$.

Since $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$ and, in addition, $\sinh 0 = 0$ and $\cosh 0 = 1$, we obtain that the successive differentiation of the hyperbolic sine and cosine at the point 0 gives alternating values of 1 and 0. Moreover, for the function $\sinh x$, as for the function $\sin x$, nonzero values correspond to derivatives of odd order and for the function $\cosh x$, as for the function $\cos x$, nonzero values correspond to derivatives of even order (starting from order 0). The difference between the expansions of hyperbolic functions and the expansions of the corresponding trigonometric ones is only that the signs do not alternate in these expansions:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) =$$
$$= \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}), \quad x \to 0;$$
$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \dots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}) =$$

$$= \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+1}), \quad x \to 0.$$

Expansions of the functions $\ln(1+x)$ **and** $(1+x)^\alpha$

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$f(x) = f(0) + \frac{f'(0)}{1} \cdot x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots$

Function $\ln(1+x)$.

In this case, we find the expansion of the logarithm function in a neighborhood of point 1; moreover, the estimate $x > -1$ must be satisfied for x.

Let us calculate several initial derivatives of the function $\ln(1+x)$ at the point 0 and substitute them in the corresponding terms of Taylor's formula:

$$\left(\ln(1+x)\right)^{(0)}\Big|_{x=0} = \ln 1 = 0, \quad f(0) = 0;$$

$$\left(\ln(1+x)\right)'\Big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1, \quad f'(0)x = x;$$

$$\left(\ln(1+x)\right)''\Big|_{x=0} = -\frac{1}{(1+x)^2}\Big|_{x=0} = -1, \quad \frac{f''(0)x^2}{2} = -\frac{x^2}{2};$$

$$\left(\ln(1+x)\right)'''\Big|_{x=0} = \frac{2}{(1+x)^3}\Big|_{x=0} = 2, \quad \frac{f'''(0)x^3}{3!} = \frac{x^3}{3};$$

$$\left(\ln(1+x)\right)^{(4)}\Big|_{x=0} = -\frac{2 \cdot 3}{(1+x)^4}\Big|_{x=0} = -3!, \quad \frac{f^{(4)}(0)x^4}{4!} = -\frac{x^4}{4}.$$

$\left((1+x)^{-1}\right)' = -1 \cdot (1+x)^{-2}$

$\left(-(1+x)^{-2}\right)' = 2(1+x)^{-3}$

$\frac{2}{3!} = \frac{2}{2 \cdot 3}$

$\frac{-3!}{4!} = \frac{-2 \cdot 3}{2 \cdot 3 \cdot 4}$

Thus, the terms have alternating signs in this expansion. In addition, in the denominator, instead of the factorial, only one factor remains, since all other factors are cancelled out with the coefficients of the corresponding derivatives:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + o(x^n) =$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k} + o(x^n), \quad x \to 0.$$

The resulting expansion of the function $\ln(1+x)$ contains all powers of x, starting from the first power, and their signs alternate. Moreover, the denominator does not have factorials.

From this formula, the previously proved equivalence $\ln(1+x) \sim x$, $x \to 0$, follows, since for $n = 1$ the expansion takes the form $\ln(1+x) = x + o(x)$, $x \to 0$.

$((1+x)^\alpha)' = \alpha(1+x)^{\alpha-1}$

$((1+x)^\alpha)'' = \alpha(\alpha-1)(1+x)^{\alpha-2} =$

FUNCTION $(1+x)^\alpha$, $\alpha \neq 0$.

In this case, we also find the expansion of the function in a neighborhood of the point 1; moreover, any real number, except 0, can be taken as α.

Apply the formula for the derivative of a power function of order n:

$$((1+x)^\alpha)^{(n)} = \underbrace{\alpha(\alpha-1)\cdots(\alpha-n+1)}\cancel{(1+x)^{\alpha-n}}, \quad n = 0, 1, 2, \ldots$$

$(1+x)^3$

For derivatives at the point 0, we will sequentially obtain the values 1, α, $\alpha(\alpha-1)$, $\alpha(\alpha-1)(\alpha-2)$, … Therefore, the expansion of the function $(1+x)^\alpha$ by Taylor's formula at the point $x_0 = 0$ with the remainder term in the Peano form will be as follows:

$\alpha \in \mathbb{N}$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots +$$
$$+ \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n) =$$
$$= \sum_{k=0}^{n} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k + o(x^n), \quad x \to 0.$$

$\alpha = 3$

$\alpha = \frac{1}{2}$ $\alpha = -2$ $\alpha = \frac{1}{3}$ …

Note that if $\alpha \in \mathbb{N}$, then, starting from some order, all derivatives vanish, and we obtain the version of Taylor's formula for polynomials in which the remainder term equals 0.

From this formula, the previously proved equivalence $(1+x)^\alpha \sim 1 + \alpha x$, $x \to 0$, follows, since for $n = 1$ the expansion takes the form $(1+x)^\alpha = 1 + \alpha x + o(x)$, $x \to 0$.

Example of using expansions to calculate limits

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Consider the following limit: $\lim_{x\to 0} \frac{x - \sin x}{x^3}$. To calculate this limit, we cannot use the equivalence $\sin x \sim x$, $x \to 0$, since equivalences can be used only in products and quotients. Instead, we apply the expansion of the function $\sin x$ by Taylor's formula with the remainder term $o(x^3)$:

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3!} + o(x^3)\right)}{x^3}.$$

$\alpha(x) \to 0$

When removing the brackets, we can omit the minus sign in front of the term $o(x^3)$, since this term simply denotes some function that decreases faster than x^3 as $x \to 0$:

$= \alpha(x)\cdot \cancel{x^3}$ $\quad \frac{\bar{o}(g)}{g} \to 0$

$$\lim_{x\to 0} \frac{\cancel{x} - \cancel{x} + \frac{x^3}{3!} + o(x^3)}{x^3} = \lim_{x\to 0} \frac{\cancel{x^3}}{3!\cancel{x^3}} + \lim_{x\to 0} \frac{o(x^3)}{\cancel{x^3}}.$$

$\to 0$

$\to \frac{1}{3!} = \frac{1}{6}$

In the first limit of the right-hand side, we can reduce the factors x^3. As a result, we obtain the limit equal to $\frac{1}{6}$.

The second limit is 0, because, by the definition of the "little-o", expression $o(x^3)$ can be represented as $\alpha(x)x^3$, where $\alpha(x) \to 0$ as $x \to 0$.

Thus, the initial limit is $\frac{1}{6}$.

Note that when using the equivalence $\sin x \sim x$ we would obtain an incorrect answer equal to 0.