1. Antiderivative and indefinite integral

$f \to f'(x)$

Definition of an antiderivative and indefinite integral

$f = F'(x)$?

`2.1A/00:00 (16:47)`

Definition.

Let the function f be defined on the interval (a, b). Let the function F be a differentiable function on this interval, with $F'(x) = f(x)$ for $x \in (a, b)$. Then the function F is called the *antiderivative* (or *primitive function*) of the function f on a given interval.

The process of finding an antiderivative is called *indefinite integration* (or *antidifferentiation*). If a function has an antiderivative on (a, b), then it is called *integrable* on (a, b).

Hereinafter we, as a rule, will not specify interval on which the function is integrable.

The question arises: how many different antiderivatives exist? Let F_1 be the antiderivative of the function f, that is, $F_1'(x) = f(x)$. Let $F_2(x) = F_1(x) + C$, where C is a constant. Then the function F_2 is also the antiderivative of the function f, since

$C' = 0$ $\quad F_1(x)$

$$F_2'(x) = \big(F_1(x) + C\big)' = F_1'(x) = f(x).$$

$F(x) = F_1(x) + C$ $\quad C \in \mathbb{R}$

Therefore, if we add a constant to some antiderivative, then we will also get a primitive function. So, there exists an infinite number of antiderivatives, differing from each other by a constant term.

There are no other antiderivatives: *all possible antiderivatives can be obtained by adding a constant to some selected antiderivative.* Let us formalize this fact as a theorem.

Theorem (on antiderivatives of a given function).

Let F_1 and F_2 be antiderivatives of f on (a, b). Then there exists a constant $C \in \mathbb{R}$ such that $F_2(x) = F_1(x) + C$.

Proof.

We introduce the auxiliary function $h(x) = F_2(x) - F_1(x)$. The function $h(x)$ is differentiable on (a, b) as the difference of differentiable functions. Let us find its derivative:

$$h'(x) = \big(F_2(x) - F_1(x)\big)' = F_2'(x) - F_1'(x) = f(x) - f(x) = 0.$$

Thus, $h'(x)$ is equal to 0 at any point $x \in (a, b)$. Then, by corollary 1 of Lagrange's theorem [1, Ch. 21], the function $h(x)$ is a constant on the interval (a, b):

$\int f(x)\,dx$

$$h(x) = C, \quad x \in (a, b).$$

Therefore, $F_2(x) - F_1(x) = C$, $F_2(x) = F_1(x) + C$. □

So, knowing one antiderivative, we can obtain all the other antiderivatives, since they all differ from the chosen antiderivative by a constant term.

DEFINITION.

The *indefinite integral* $\int f(x)\,dx$ of the function f is the set of all its antiderivatives: if F_1 is some antiderivative of the function f (that is, $F_1'(x) = f(x)$), then

$$\int f(x)\,dx \stackrel{\text{def}}{=} \{F_1(x) + C,\ \ C \in \mathbb{R}\}.$$

The symbol $\int$ is called the *integral sign*, the function $f(x)$ is called the *integrand*, and the expression $f(x)\,dx$ under the integral sign is called the *element of integration*.

As a rule, curly braces are not used and, moreover, it is not indicated that C is an arbitrary real constant:

$(x^\alpha)' = \alpha x^{\alpha-1}$

$$\int f(x)\,dx = F_1(x) + C. \Longleftarrow \boxed{F_1'(x) = f(x)} \quad C \in \mathbb{R}$$

Table of indefinite integrals

2.1A/16:47 (12:44)

$C' = 0$ $\qquad (Ax + C)' = A$

$$\int 0\,dx = C.$$

$\alpha \neq -1$

$$\int A\,dx = Ax + C, \quad A \in \mathbb{R}.$$

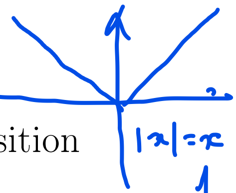
$$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad x > 0, \quad \alpha \in \mathbb{R} \setminus \{-1\}.$$

$\left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \frac{\cancel{(\alpha+1)}\,x^\alpha}{\cancel{\alpha+1}} = x^\alpha$

$\alpha = -1$

$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad x \neq 0.$$

$\frac{-1}{-x} = \frac{1}{x}$ $\quad |x| = -x$ $\quad -1$



$|x| = x$ $\quad 1$

To prove the last formula, it suffices to differentiate the superposition $\ln|x| = \ln y \circ |x|$ for $x \neq 0$:

$$(\ln|x|)' = (\ln y)'\big|_{y=|x|} \cdot (|x|)' = \frac{1}{y}\bigg|_{y=|x|} \cdot \operatorname{sign} x = \frac{\operatorname{sign} x}{|x|} = \frac{1}{x}.$$

$$\int e^x\,dx = e^x + C.$$

$(e^x)' = e^x$

$(a^x)' = a^x \cdot \ln a$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C, \quad a > 0, \quad a \neq 1.$$

$\left(\frac{a^x}{\ln a}\right)' = \frac{\cancel{\ln a}\, a^x}{\cancel{\ln a}} = a^x$

$$\int \sin x\,dx = -\cos x + C.$$

$$\int \cos x\,dx = \sin x + C.$$

$(\sin x)' = \cos x$

$(-\cos x)' = -(-\sin x) = \sin x$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + C.$$

$(\tan x)' = \frac{1}{\cos^2 x}$

$\int \tan x\,dx = ?$

$f'(x) = \tan x$??

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$\int \frac{1}{1+x^2}\,dx = \arctan x + C$

$$\cancel{\int \sinh x\,dx = \cosh x + C.}$$

$$\cancel{\int \cosh x\,dx = \sinh x + C.}$$

The simplest properties of an indefinite integral

2.1A/29:31 (09:54), 2.1B/00:00 (02:28)

1. If the function f is integrable, then

$$\left(\int f(x)\,dx\right)' = f(x).$$

Proof.
Let $F(x)$ be the antiderivative of the function $f(x)$, then

$$\left(\int f(x)\,dx\right)' = (F(x)+C)' = F'(x) = f(x). \ \square$$

2. If the function f is differentiable, then

$(f+g)' = f' + g'$

$$\int f'(x)\,dx = f(x) + C.$$

$f'(x)$

Proof.
In this case, $f(x)$ is one of the antiderivatives of the function $f'(x)$, whence the formula to be proved follows. $\square$

3. Additivity of the indefinite integral.
Let f and g be integrable, then the function $f+g$ is also integrable and the formula holds:

$$\int (f+g)\,dx = \int f\,dx + \int g\,dx. \tag{1}$$

$\{\ldots\} = \{\ldots\} + \{\ldots\}$

$\supset a + b$

PROOF.

Equality (1) must be interpreted as the equality of two sets. Therefore, we should prove that the set from the left-hand side of equality (1) is equal to the set from the right-hand side of (1). Let F be some antiderivative of the function f, G be some antiderivative of the function g. Then $F+G$ is the antiderivative of the function $f+g$, since $(F+G)' = F' + G' = f + g$. Therefore, equality (1) can be rewritten in the form:

$$F + G + C = (F + C_1) + (G + C_2), \quad C, C_1, C_2 \in \mathbb{R}.$$

Obviously, if we choose the constants C_1 and C_2, that is, if we select some element of the right-hand set, then this element will also belong to the left-hand set (we can just put $C = C_1 + C_2$).

If we select some element $F + G + C$ of the left-hand set, then, by representing the constant C as the sum of two constants C_1 and C_2, we obtain that this element also belongs to the right-hand set.

Thus, we have proved the equality of these sets. □

$(\alpha f)' = \alpha \cdot f'$

4. HOMOGENEITY OF THE INDEFINITE INTEGRAL.

Let f be integrable, $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then the function αf is integrable and the formula holds:

$$\int \alpha f \, dx = \alpha \int f \, dx. \tag{2}$$

$\cancel{\int 0\,dx = 0}$

Formula (2) means that the constant factor can be taken out of the integral sign.

The proof is similar to the proof of property 3. □

REMARK.

In the case of $\alpha = 0$, formula (2) turns out to be incorrect, as we noted earlier that $\int 0 \, dx = C$.

If we combine the properties of additivity and homogeneity, then we get the property of linearity.

5. LINEARITY OF THE INDEFINITE INTEGRAL.

Let f and g be integrable, $\alpha, \beta \in \mathbb{R}$, with α and β not turning into 0 at the same time: $|a| + |b| \neq 0$. Then the function $\alpha f + \beta g$ is also integrable and the formula holds:

$$\int (\alpha f + \beta g) \, dx = \alpha \int f \, dx + \beta \int g \, dx.$$

$\int \left(\sum_{i=1}^{n} \alpha_i f_i \right) dx = \sum_{i=1}^{n} \alpha_i \int f_i \, dx$

EXAMPLE.

Using the simplest properties of the indefinite integral and the table of indefinite integrals, one can find the integrals of linear combinations of functions, for example:

$\ldots + C_1$ $\ldots + C_2$ $\boxed{5e^x + 6\cos x}$

$$\int (5e^x + 6\cos x)\,dx = 5\int e^x\,dx + 6\int \cos x\,dx = 5e^x + 6\sin x + C.$$

To verify the resulting relation, it suffices to differentiate the expression on the right-hand side.

Change of variables in an indefinite integral

`2.1B/02:28 (13:37)`

THEOREM (ON THE CHANGE OF VARIABLES).

Let $f(x)$ be an integrable function on (a,b) and one of its antiderivatives is the function $F(x)$. Let $\varphi(t)$ be a differentiable function on the interval (α,β) and $\varphi(t) \in (a,b)$ as $t \in (\alpha,\beta)$. Then

$$\int f\big(\varphi(t)\big)\varphi'(t)\,dt = F\big(\varphi(t)\big) + C. \qquad (3)$$

$df = f'(x)\cdot dx$

PROOF.

It is enough for us to verify that the right-hand side of equality (3) is the antiderivative of the integrand of the left-hand side of (3). We use the superposition differentiation theorem and the condition that $F'(x) = f(x)$:

$$\big(F(\varphi(t))\big)' = F'(x)|_{x=\varphi(t)}\,\varphi'(t) = f(x)|_{x=\varphi(t)}\,\varphi'(t) = f\big(\varphi(t)\big)\varphi'(t). \square$$

REMARK.

Considering that the expression $\underbrace{\varphi'(t)\,dt}_{=d\varphi}$ is the differential of the function φ, the left-hand side of equality (3) can be written as $\int f(\varphi)d\varphi$. $\int f(x)dx$

If we assume that φ is an independent variable, then equality (3) turns into the definition of an indefinite integral:

$$\int f(\varphi)d\varphi = F(\varphi) + C. \qquad (4)$$

However, the proved theorem means that equality (4) also holds for the case when φ is a *dependent variable*, that is, it represents a differentiable function of some independent variable (for example, t). In this case, the expression $d\varphi$ must be understood as the *differential* of the function.

The noted circumstance is an additional justification for including the expression dx in the notation of the indefinite integral. It should be noted that this notation is also convenient for calculating integrals by changing variables.

AN EXAMPLE OF APPLYING THE VARIABLE CHANGING THEOREM.

Find the integral $\int \tan x\,dx$:

$\cos x = t$

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x}. =$$

$(-\cos x)' = \sin x$

$\sin x\,dx = (-\cos x)'\,dx = -d(\cos x)$

$$= \int \frac{-d(\cos x)}{\cos x} = -\int \frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C$$

$x \mid \boxed{\cos x = y} \mid y$ $\quad dx \quad dx \quad dy$

$-\sin x\, dx = 1 \cdot dy$

$\sin x\, dx = -dy$

We introduce a new variable: $y = \cos x$. The variable y is the function φ from the variable changing theorem, that is, we can assume that y depends on x. Then dy is the differential of the function $\cos x$, therefore $dy = -\sin x\, dx$. Thus, by virtue of the remark on the variable changing theorem, the expression in the numerator of the initial integral can be replaced with $-dy$, and the expression in the denominator can be replaced with y. As a result of changing the variable $y = \cos x$, the initial integral is significantly simplified and can now be found using the table of indefinite integrals:

$$\int \frac{\sin x\, dx}{\cos x} = \int \frac{-dy}{y} = -\int \frac{dy}{y} = -\ln|y| + C.$$

It remains for us to return to the initial variable x. Finally we obtain

$$\int \tan x\, dx = -\ln|\cos x| + C.$$

Remarks.

1. When finding the last integral, we actually applied the formula (3), representing the initial integral as follows:

$$\int \tan x\, dx = \int f(\cos x) \cdot (\cos x)'\, dx, \quad f(y) = -\frac{1}{y}.$$

However, when performing a variable change in an indefinite integral, formula (3) is not used. Instead, in the integral, both the initial variable x and its differential dx are replaced, as was done in the above example.

2. Using a similar change of variable, one can also find the integral $\int \frac{dx}{\sin x}$. To do this, take into account that $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and additionally transform the resulting expression to obtain the result of differentiation of the function $\tan\frac{x}{2}$ in it.

$dy = \frac{dx}{2\cos^2\frac{x}{2}}$

$y = \tan\frac{x}{2}$

3. The resulting formula for the integral $\int \tan x\, dx$ makes sense on any interval that does not contain points $\frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$, that is, points at which the tangent function is not defined.

$\cos\frac{x}{2}$ $\quad \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{\cos\frac{x}{2}\, dx}{2\sin\frac{x}{2}\cos^2\frac{x}{2}}$

Formula of integration by parts

Derivation of the formula of integration by parts

$= \int \frac{dx}{2\left(\tan\frac{x}{2}\right)\cos^2\frac{x}{2}} = \int \frac{dy}{y} = \ln|y| + C = \ln\left|\tan\frac{x}{2}\right| + C$

2.1B/16:05 (07:28)

The integral of the product of functions is not equal to the product of the integrals. This is due to the more complicated form of the formula for differentiating the product, compared with the formula for differentiating the sum:

$\int (f+g)\, dx = \int f\, dx + \int g\, dx \quad \Big| \quad \int f \cdot g\, dx \overset{?}{=} \cancel{\int f\, dx \int g\, dx}$

$$(uv)' = u'v + uv'. \tag{5}$$

Nevertheless, using formula (5) for differentiating the product, we can obtain the *formula of integration by parts*, which in some cases allows us to simplify the calculation of the integral of the product.

Let us express the product uv' from equality (5):

$$uv' = (uv)' - u'v.$$

Integrating the last equality and using the linearity of the indefinite integral (the simplest property 5), we obtain

$$\int uv'\,dx = \int \big((uv)' - u'v\big)\,dx = \int (uv)'\,dx - \int u'v\,dx. \tag{6}$$

Given the simplest property 2 of the indefinite integral, we have

$$\int (uv)'\,dx = uv + C.$$

Since the remaining term $\int u'v\,dx$ on the right-hand side of equality (6) also contains an arbitrary constant, we can add the constant C to this arbitrary constant and not specify it explicitly. Finally, we obtain the following formula:

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx.}$$

This formula is called the *formula of integration by parts*. It holds if the functions u and v are differentiable and there exists at least one of the integrals included in it (in this case, there necessarily exists another integral).

So, the formula of integration by parts allows us to express the integral of the product of the functions u and v' in terms of the integral of the product of u' and v. It is used in situations where the integral on its right-hand side is easier to find than the integral on the left-hand side.

The formula of integration by parts can also be written in the following form:

$$\int u\,dv = uv - \int v\,du.$$

$v'dx \downarrow v$

$(\ln x)' = \frac{1}{x}$

Examples of applying the formula of integration by parts

2.1B/23:33 (12:42)

1. Let us find the integral $\int \ln x\,dx$. We put $u(x) = \ln x$, $dv = dx$, whence $v(x) = x$. Then

$\int \ln x\,dx = x\ln x - x + C$

$$\int \ln x\,dx = x\ln x - \int x(\ln x)'\,dx =$$
$$= x\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx = x\ln x - \int dx = x\ln x - x + C.$$

2. Let us find the integral $\int e^x \sin x\,dx$.

We put $u(x) = \sin x$, $dv = e^x\,dx$, whence $v(x) = e^x$. Then

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx. \tag{7}$$

Transform the integral on the right-hand side of (7) by the formula of integration by parts with $u(x) = \cos x$, $v(x) = e^x$:

$$\int e^x \cos x\,dx = e^x\cos x - \int e^x(-\sin x)\,dx = e^x\cos x + \int e^x \sin x\,dx.$$

Substituting the found integral in (7), we obtain

$$\int e^x \sin x\,dx = e^x \sin x - e^x\cos x - \int e^x\sin x\,dx.$$

So, after completing two integrations by parts, we get the initial integral $\int e^x \sin x\,dx$. If we denote one of the antiderivatives of the initial integrand by the symbol I, then the last equality can be written as follows:

$$I = e^x(\sin x - \cos x) - I. \tag{8}$$

Solving equation (8) with respect to I, we obtain

$$I = \frac{e^x}{2}(\sin x - \cos x).$$

The final formula takes the form

$$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

degree $P = 2$ | 1
degree $P' = 1$ | 0

A similar technique can be applied when integrating functions of a more general form $e^{bx}\sin ax$ and $e^{bx}\cos ax$.

Remark.

v

It is also advisable to apply the formula of integration by parts in the case of an integrand of the form $\overbrace{P(x)}^{u}\overbrace{f(x)}^{v'}$, where $P(x)$ is a polynomial and $f(x)$ is a function for which there exists a simple antiderivative (such as $\sin x$, $\cos x$, a^x). In this case, we put $u(x) = P(x)$; as a result of differentiation of the function $u(x)$, a polynomial of a lesser degree will be obtained. The integration by parts is again applied to the obtained integral, and the process is repeated until the polynomial at the next differentiation turns into a constant.

If there is a function $\ln x$ in the integrand, we can put $u(x) = \ln x$, since we obtain a simpler function $\frac{1}{x}$ after differentiation.

Sometimes it is convenient to apply the formula of integration by parts, not dividing the integrand into two factors, but setting $dv = dx$, $v(x) = x$ (as in example 1).