m-dimensional problems

Let Ω be bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Functions f and g are continiously differentiable in Ω and continious up to the boundary:

$$f,\, g \in C^1(\Omega) \cap C(\overline{\Omega})$$

$\overline{n}$ denotes the outward normal unit vector to $\partial\Omega$.

Then the integration by parts formula holds:

$$\int\limits_{\Omega} \frac{\partial f}{\partial x_k} g(x)\, dx \;=\; -\int\limits_{\Omega} f(x) \frac{\partial g}{\partial x_k}\, dx \;+ \int\limits_{\partial\Omega} f(x) g(x) n_k \, dS\,. \tag{1}$$

$1 \leqslant k \leqslant m$, n_k — k-th coordinate of normal vector.

Problem 1

Using the integration by parts prove First Green's identity:

$$\int\limits_{\Omega} \Delta f(x) \cdot g(x)\, dx \;=\; -\int\limits_{\Omega} (\nabla f(x), \nabla g(x))\, dx \;+ \int\limits_{\partial\Omega} \frac{\partial f(x)}{\partial n} g(x)\, dS\,. \tag{2}$$

Prove Second Green's identity:

$$\int\limits_{\Omega} \Delta f(x) \cdot g(x)\, dx \;=\int\limits_{\Omega} f(x) \Delta g(x)\, dx \;+ \int\limits_{\partial\Omega} \left(\frac{\partial f(x)}{\partial n} g(x) - \frac{\partial g(x)}{\partial n} f(x) \right) dS\,. \tag{3}$$

Prove an important formula:

$$\int\limits_{\Omega} \Delta u(x)\, dx \;= \int\limits_{\partial\Omega} \frac{\partial u(x)}{\partial n}\, dS\,. \tag{4}$$

Problem 2

Let $u \in \mathbb{C}^2(D)$. D is bounded domain in $\mathbb{R}^m$, $\partial D \in \mathbb{C}^1$. Let $u\big|_{\partial D} = 0$. Prove:

$$\int\limits_{D} \Delta u \cdot u \, dx \leqslant 0. \tag{5}$$

When does the equality take place?

Problem 3

Prove that, if

$$\Delta u + \lambda u = 0, x \in D; \quad u\big|_{\partial D} = 0 \tag{6}$$

and u is non-trivial solution, then $\lambda > 0$.

Problem 4

Prove that, if

$$\Delta u + \lambda u = 0, x \in D; \quad \frac{\partial u}{\partial n}\Big|_{\partial D} = 0 \tag{7}$$

and u is non-trivial solution, then $\lambda \geqslant 0$.

Problem 5

Deduce first Green's identity and second Green's identity for biharmonic operator $\Delta^2 u$ in $D \subset \mathbb{R}^m$. Using Green's second identity find the adjoint differential expression for biharmonic operator.

Remark.

$$\Delta^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots \frac{\partial^2}{\partial x_m^2}\right)^2 \tag{8}$$

Problem 6

Let $u \in \mathbb{C}^4(\Omega)$, Ω is bounded domain in $\mathbb{R}^m$, $\partial\Omega \in \mathbb{C}^1$. Suppose that $u\Big|_{\partial\Omega} = \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0$. Prove:

$$\int\limits_{\Omega} \Delta^2 u \cdot u \, dx \geqslant 0. \tag{9}$$

When does the equality take place?

Definition L^* is called the adjoint differential expression with respect to L, if following identity takes place:

$$\int\limits_{\Omega} Lf(x) \cdot g(x) \, dx = \int\limits_{\Omega} f(x) L^* g(x) \, dx + \int\limits_{\partial\Omega} M(f(x), g(x)) \, dS \, .$$

Problem 7

Find the adjoint differential expression for the heat operator

$$Lu = u_t - \Delta u \tag{10}$$