$P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}\cdot x + a_n$

$\int P_n(x)\,dx = a_0\frac{x^{n+1}}{n+1} + a_1\frac{x^n}{n} + \dots + a_{n-1}\frac{x^2}{2} + a_nx + C$

$D=0 \quad x_1=x_2=\frac{-p}{2} \quad (x-x_1)^2$ $\quad x^2+px+q=0$

$D=p^2-4q \geq 0$

2. Integration of rational functions

$x^2+px+q=(x-x_1)(x-x_2)$ $\quad x_{1,2}=\frac{-p\pm\sqrt{D}}{2}$

$D<0$

Partial fraction decomposition of a rational function

`2.1B/36:15 (11:49)`

$\alpha\pm i\beta$

$P_3 \quad P_4 \quad \cancel{P_5}$

The *rational function* $R(x)$ is the ratio of two polynomials:

$$R(x) = \frac{P_m(x)}{Q_n(x)}.$$

In studying the question of integrating rational functions, the following facts from the course of algebra are used.

THEOREM 1 (ON THE FACTORIZATION OF A REAL POLYNOMIAL).

A polynomial $Q_n(x)$ of degree n with real coefficients can be decomposed into the following irreducible factors:

$Q_n(c_1)=0$ ....

$$Q_n(x) = a_0\underbrace{(x-c_1)^{\alpha_1}\dots(x-c_k)^{\alpha_k}}_{k}(x^2+p_1x+q_1)^{\beta_1}\dots(x^2+p_lx+q_l)^{\beta_l}. \tag{1}$$

Here a_0 is the coefficient of the highest degree of the polynomial $Q_n(x)$, $c_1, \dots, c_k$ are the real roots of the polynomial $Q_n(x)$ of multiplicity $\alpha_1, \dots, \alpha_k$, quadratic factors of the form $x^2+p_ix+q_i$ with real coefficients p_i, q_i have a negative discriminant: $p_i^2-4q_i < 0$; each factor $(x^2+p_ix+q_i)^{\beta_i}$ corresponds to a pair of complex conjugate roots of the polynomial $Q_n(x)$ of multiplicity β_i, $i=1,\dots,l$. In addition, the following relation holds:

$$\alpha_1+\dots+\alpha_k+2(\beta_1+\dots+\beta_l)=n.$$

THEOREM 2 (ON THE PARTIAL FRACTION DECOMPOSITION OF A REAL RATIONAL FUNCTION).

Let $R(x)$ be a rational function of the form $\frac{P_m(x)}{Q_n(x)}$ and decomposition (1) takes place for the polynomial $Q_n(x)$. Then $R(x)$ can be represented as follows:

$$R(x) = \tilde{P}(x) + \sum_{i=1}^{k}\sum_{j=1}^{\alpha_i}\frac{A_{ij}}{(x-c_i)^j} + \sum_{i=1}^{l}\sum_{j=1}^{\beta_i}\frac{B_{ij}x+D_{ij}}{(x^2+p_ix+q_i)^j}. \tag{2}$$

The term $\tilde{P}(x)$ appears if the degree m of the polynomial $P_m(x)$ is greater than or equal to the degree n of the polynomial $Q_n(x)$. This term $\tilde{P}(x)$ is

$\frac{P_2(x)}{Q_1(x)}$

$$\begin{array}{r|l} x^2+x+1 & x-2 \\ -(x^2-2x) & x+3 \\ \hline 3x+1 & \end{array}$$

$\frac{x^2+x+1}{x-2} = x+3+\frac{7}{x-2}$

$\frac{3x-6}{7}$

a polynomial of degree $m-n$ obtained by dividing the polynomial $P_m(x)$ by the polynomial $Q_n(x)$.

For all remaining terms in formula (2) (called *partial fractions*), the degree of the numerator is less than the degree of the denominator.

$d_1=2$

$c_1=1$

$D=2^2-4\cdot 2=4-8=-4<0$

Methods for finding the decomposition of a rational function

2.2A/00:00 (09:34)

$3<4$

Consider the following rational function as an example:

$$R(x)=\frac{5x^3+3x+2}{(x-1)^2(x^2+2x+2)}.$$

The degree of its numerator is less than the degree of the denominator, therefore, the term $\tilde{P}(x)$ will not be present in the decomposition. The decomposition will consist of three partial fractions:

$(x-1)(x^2+2x+2)$ $\quad (x^2+2x+2$ $\quad (x-1)^2$

$$\int R(x)=\int\frac{A}{x-1}+\int\frac{B}{(x-1)^2}+\int\frac{Cx+D}{x^2+2x+2}.$$

It remains for us to find the coefficients of these fractions. To do this, we can use the so-called *method of equating coefficients*. Let us reduce the expression on the right-hand side to a common denominator and equate the resulting numerators:

$$5x^3+3x+2=A(x-1)(x^2+2x+2)+B(x^2+2x+2)+(Cx+D)(x-1)^2. \tag{3}$$

5 0 3 2

Now we remove parentheses on the right-hand side and group the terms with the same powers of x:

$$5x^3+3x+2=(A+C)x^3+(A+B-2C+D)x^2+$$
$$+(2B+C-2D)x+(-2A+2B+D).$$

Let us equate the coefficients at the same powers of x:

$$\begin{cases} 5=A+C,\\ 0=A+B-2C+D,\\ 3=2B+C-2D,\\ 2=-2A+2B+D. \end{cases}$$

As a result, we obtained a system of four linear equations in four unknowns. According to the theorem on the partial fraction decomposition of a rational function, this system has a solution. Having solved this system, we will find the required coefficients: $A=2$, $B=2$, $C=3$, $D=2$.

There is another way: we can consider the specific values of x. If we put $x = 1$ in equality (3), then two terms will disappear in its right-hand side and the only term will remain. Using this term, we can immediately find the coefficient B:

$$5 \cdot 1^3 + 3 + 2 = A(1-1)(1^2+2+2) + B(1^2+2+2) + (C+D)(1-1)^2,$$

$$10 = 5B,$$

$$B = 2.$$

This method is convenient if there are no quadratic factors in the factorization of a denominator. However, even in our case, this method allows us to simplify the resulting system by reducing the number of unknowns:

$$5 = A + C,$$

$$-2 = A - 2C + D,$$

$$-1 = C - 2D,$$

$$-2 = -2A + D.$$

Integration of terms in the partial fraction decomposition of a rational function

Simple cases based on the direct use of the table of integrals

`2.2A/09:34 (06:46)`

After finding the partial fraction decomposition of the rational function, it remains to integrate separately all the obtained terms.

1. The integral of the polynomial $\tilde{P}(x)$.

This integral is a polynomial whose free term is an arbitrary constant C.

2. The integral of a partial fraction of the form $\frac{A}{(x-c)^k}$ corresponding to the real root c of multiplicity k.

For $k \neq 1$, we have

$$\int \frac{A}{(x-c)^k}\,dx = \frac{A}{(1-k)(x-c)^{k-1}} + C.$$

$$A\int t^{-k}\,dt = A\frac{t^{-k+1}}{-k+1} + C = \frac{A}{(1-k)(x-c)^{k-1}} + C$$

For $k = 1$, we have

$$\int \frac{A}{x-c}\,dx = A\ln|x-c| + C.$$

$$x - c = t, \quad dx = dt$$

$$\int \frac{A\,dt}{t} = A\int\frac{dt}{t} = A\ln|t| + C =$$

$(a+b)^2 = a^2 + 2ab + b^2$ $\qquad$ $4/$ $\qquad$ $4q - p^2 > 0$

Using change of variable

2.2A/16:20 (09:56)

$D = p^2 - 4q < 0$

3. The integral of a partial fraction of the form $\frac{Bx+D}{(x^2+px+q)^k}$, provided that the discriminant of the quadratic polynomial is less than zero: $p^2 - 4q < 0$ (this fraction corresponds to complex conjugate roots of multiplicity k).

Let us transform the polynomial in the denominator by complete the square:

$$x^2 + px + q = \underbrace{x^2 + 2x \cdot \frac{p}{2} + \left(\frac{p}{2}\right)^2} - \left(\frac{p}{2}\right)^2 + q = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}.$$

> 0

Note that the expression $q - \frac{p^2}{4}$ is greater than zero, since, by assumption, the discriminant $p^2 - 4q$ is less than zero. Denote $q - \frac{p^2}{4} = \Delta^2$.

As a result of the transformation, we decrease the number of variables x in the integral:

$$\int \frac{(Bx+D)\,dx}{(x^2+px+q)^k} = \int \frac{(Bx+D)\,dx}{\left(\left(x+\frac{p}{2}\right)^2 + \Delta^2\right)^k}.$$

Let us change of variable: $t = x + \frac{p}{2}$. Differentials will not change: $dt = dx$. This variable changing will further simplify the denominator:

$$\underbrace{Bt} - \underbrace{\frac{Bp}{2} + D}_{D'}$$

$$\int \frac{(Bx+D)\,dx}{\left(\left(x+\frac{p}{2}\right)^2 + \Delta^2\right)^k} = \int \frac{\left(B\left(t-\frac{p}{2}\right)+D\right)dt}{(t^2+\Delta^2)^k}.$$

Now transform the numerator by grouping the free terms and denoting the difference $D - \frac{Bp}{2}$ by D':

$$\int \frac{\left(B\left(t-\frac{p}{2}\right)+D\right)dt}{(t^2+\Delta^2)^k} = \int \frac{(Bt+D')\,dt}{(t^2+\Delta^2)^k}.$$

Let us split the resulting integral into two:

$$\int \frac{(Bt+D')\,dt}{(t^2+\Delta^2)^k} = \int \frac{Bt\,dt}{(t^2+\Delta^2)^k} + \int \frac{D'\,dt}{(t^2+\Delta^2)^k}.$$

Thus, it remains for us to analyze the integrals of two types: $\int \frac{t\,dt}{(t^2+\Delta^2)^k}$ and $\int \frac{dt}{(t^2+\Delta^2)^k}$.

3a. Find the integral $\int \frac{t\,dt}{(t^2+\Delta^2)^k}$. We make the following variable change in it: $y = t^2 + \Delta^2$. Then $dy = 2t\,dt$ and as a result we get

$\frac{1}{2} dy = t\,dt$

$k = 1, 2, 3, \ldots$

$$\int \frac{t\,dt}{(t^2+\Delta^2)^k} = \frac{1}{2}\int \frac{dy}{y^k}. = \frac{1}{2}\ln|t^2+\Delta^2| + C$$

$k = 1$

The integral on the right-hand side can be found using the same formulas as the integrals considered in subsection 2.

$$\int \frac{dy}{y} = \ln|y| + C$$

Using recurrence relation

2.2A/26:16 (12:00)

3b. Now let us turn to the last integral: $\int \frac{dt}{(t^2+\Delta^2)^k}$.

In this case, we perform integration by parts, setting $u = \frac{1}{(t^2+\Delta^2)^k}$, $dv = dt$, $v = t$:

$$\int \frac{dt}{(t^2+\Delta^2)^k} = \frac{t}{(t^2+\Delta^2)^k} - \int \frac{2(-k)t^2\,dt}{(t^2+\Delta^2)^{k+1}} =$$
$$= \frac{t}{(t^2+\Delta^2)^k} + 2k\int \frac{t^2\,dt}{(t^2+\Delta^2)^{k+1}}.$$

In the numerator of the last integral, we add and subtract Δ^2:

$$\frac{t}{(t^2+\Delta^2)^k} + 2k\int \frac{(t^2+\Delta^2-\Delta^2)\,dt}{(t^2+\Delta^2)^{k+1}} =$$
$$= \frac{t}{(t^2+\Delta^2)^k} + 2k\int \frac{dt}{(t^2+\Delta^2)^k} - 2k\Delta^2\int \frac{dt}{(t^2+\Delta^2)^{k+1}}.$$

If we denote $I_k = \int \frac{dt}{(t^2+\Delta^2)^k}$, then we can write the resulting relation as follows:

$$I_k = \frac{t}{(t^2+\Delta^2)^k} + 2k(I_k - \Delta^2 I_{k+1}).$$

$I_4 \leftarrow I_3 \leftarrow I_2 \leftarrow I_1$

Express I_{k+1} in terms of I_k:

$$I_{k+1} = \frac{1}{2k\Delta^2}\left(\frac{t}{(t^2+\Delta^2)^k} + (2k-1)I_k\right).$$

We have obtained a recurrence relation that allows us to reduce the finding of the integral I_{k+1} to I_k. Applying it the required number of times, we can reduce the integral I_k to the integral I_1, which can be found explicitly:

$$I_1 = \int \frac{dt}{t^2+\Delta^2} = \frac{1}{\Delta}\arctan\frac{t}{\Delta} + C.$$

Theorem on the integration of a rational function

2.2B/00:00 (06:08)

Thus, we have shown that all the integrals arising during the integration of a rational function are expressed in terms of elementary functions and the following theorem holds.

Theorem (on the integration of a rational function).

Any rational function can be integrated in elementary functions.

$\frac{e^x}{x}$ $\int \frac{e^x}{x}\,dx =$ $\int \frac{\sin x}{x}\,dx =$

$\int \sin x\,dx = -\cos x + C$

This is an important fact, since there are elementary functions whose integrals are not expressed in terms of elementary functions. Examples of such functions are $\frac{e^x}{x}$, $\frac{\sin x}{x}$, $\frac{\cos x}{x}$.

Having proved the theorem on the integration of a rational function, we can use it to study the integrability of other types of functions. If we can reduce (for example, by changing a variable) a certain integrand to a rational function, then we can state that the original function is also integrated in elementary functions.

Remark.

When integrating rational functions, we "go beyond" the set of rational functions, because as a result of integrating rational functions, logarithms and arctangents can arise.