m-dimensional problems. Hometask

Let Ω be bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Functions f and g are continiously differentiable in Ω and continious up to the boundary:

$$f,\, g \in C^1(\Omega) \cap C(\overline{\Omega})$$

$\overline{n}$ denotes the outward normal unit vector to $\partial\Omega$.

Then the integration by parts formula holds:

$$\int\limits_{\Omega} \frac{\partial f}{\partial x_k} g(x)\, dx \; = -\int\limits_{\Omega} f(x)\frac{\partial g}{\partial x_k}\, dx \, + \int\limits_{\partial\Omega} f(x)g(x)n_k\, dS\,. \tag{1}$$

$1 \leqslant k \leqslant m$, n_k — k-th coordinate of normal vector.

Using the integration by parts we prove Green's first identity

$$\int\limits_{\Omega} \Delta f(x)\cdot g(x)\, dx \; = -\int\limits_{\Omega} (\nabla f(x), \nabla g(x))\, dx \, + \int\limits_{\partial\Omega} \frac{\partial f(x)}{\partial n} g(x)\, dS\,. \tag{2}$$

Using the integration by parts twice we prove Green's second identity:

$$\int\limits_{\Omega} \Delta f(x)\cdot g(x)\, dx \; = \int\limits_{\Omega} f(x)\Delta g(x)\, dx \, + \int\limits_{\partial\Omega} \left(\frac{\partial f(x)}{\partial n} g(x) - \frac{\partial g(x)}{\partial n} f(x)\right)\, dS\,. \tag{3}$$

Problem 3

Prove the following property of eigenvalues of Laplace operator. If

$$\Delta u + \lambda u = 0, x \in D; \quad u\big|_{\partial D} = 0 \tag{4}$$

and u is non-trivial solution to (4), then $\lambda > 0$.

Hint. First step. Rewrite the equation (4):

$$\lambda u = -\Delta u. \tag{5}$$

Multiply both sides of (5) by $u^*(x)$ and integrate over D.

$$\lambda \int\limits_{D} |u(x)|^2 dx = -\int\limits_{D} \Delta u \cdot u^*(x) dx. \tag{6}$$

Second step. Simplify right-hand side of the equation (6) using Green's identity and boundary conditions. Express λ from the obtained equation. You will show that λ is non-negative.

Third step. Suppose that λ is equal to zero and prove that it is not possible.