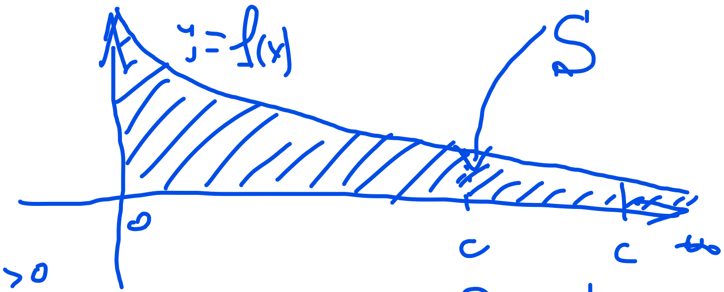
$[0,+\infty)$



10. Improper integrals: definition and properties

$\forall c>0$

Tasks leading to the notion of an improper integral

$\Phi(c)=\int_0^c f(x)\,dx$ $\lim\limits_{c\to+\infty}\Phi(c)=S\in\mathbb{R}$ $S=\int_0^{+\infty} f(x)\,dx$

3.8B/00:00 (04:13)

Starting to study a definite integral, we considered the problem of finding the area of a curvilinear trapezoid defined using some continuous function f on the segment $[a,b]$. We further proved that to solve this problem, it is necessary to calculate the integral $\int_a^b f(x)\,dx$.

Now suppose that the function f is defined and continuous on the entire positive semiaxis OX and it is positive and decreasing on this semiaxis. Is it possible to determine the area of the *infinite* region D bounded by the positive semiaxis OX, the line $x=0$, and the graph $y=f(x)$?

Let us choose some point $c>0$ and consider the part of the region D located to the left of the line $x=c$. This part is a curvilinear trapezoid defined on the segment $[0,c]$ and its area is $\Phi(c)=\int_0^c f(x)\,dx$.

As the value of c increases, the area of $\Phi(c)$ will increase too. If there exists a limit $\Phi(c)$ as $c\to+\infty$, then it is natural to consider this limit as the area of an infinite region D.

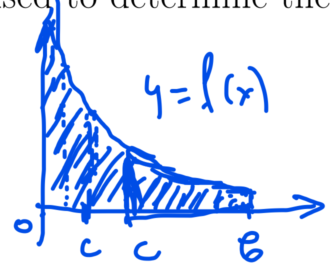
Consider another example. Suppose now that the function f is defined and continuous on the half-interval $(0,b]$, takes positive values on it and increases unlimitedly as $x\to+0$.

In this case, we get an infinite region D bounded by the segment $[0,b]$ of the axis OX, the lines $x=0$ and $x=b$, and the graph $y=f(x)$. To determine the area of the region D, we can choose the point $c\in(0,b)$ and consider the part of the region D bounded by the vertical lines $x=c$ and $x=b$. This part is a curvilinear trapezoid and its area is $\Phi(c)=\int_c^b f(x)\,dx$.

If there exists a limit $\Phi(c)$ as $c\to+0$, then this limit can be considered as the area of the infinite region D.

These examples show that improper integrals can be of two types: integrals over an infinite integration interval of a bounded function and integrals over a finite interval, but of a function that is unbounded on a given interval. In any of these cases, the passing to limit is used to determine the improper integral.

$\forall c\in(0,b)$ $\Phi(c)=\int_c^b f(x)\,dx$ $\lim\limits_{c\to+0}\Phi(c)=S\in\mathbb{R}$, $S=\int_0^b f(x)\,dx$



$\lim\limits_{x\to+0} f(x)=+\infty$

Definitions of an improper integral

Improper integral over a semi-infinite interval

3.8B/04:13 (10:02)

Definition 1 (definition of an improper integral over a semi-infinite interval).

Let a function f be defined on the set $[a, +\infty)$ and integrable on any segment $[a, c]$, $c > a$. If there exists a finite limit of the integral $\int_a^c f(x)\,dx$ as $c \to +\infty$, then they say that there exists an *improper integral* $\int_a^{+\infty} f(x)\,dx$ and its value is assumed to be equal to this limit:

$$\int_a^{+\infty} f(x)\,dx \stackrel{\text{def}}{=} \lim_{c\to+\infty} \int_a^c f(x)\,dx.$$

In this case, they say that the improper integral $\int_a^{+\infty} f(x)\,dx$ *converges.*

$\int \frac{dx}{x} = \ln x$

If the limit $\lim_{c\to+\infty} \int_a^c f(x)\,dx$ does not exist or is equal to infinity, then they say that the improper integral $\int_a^{+\infty} f(x)\,dx$ *diverges.*

An improper integral over a semi-infinite interval of the form $(-\infty, b]$ is defined in a similar way.

$\int \frac{dx}{x^\alpha}$

$\int_1^{+\infty} \frac{dx}{x^\alpha}$ $\quad \alpha \in \mathbb{R}$

Examples.

1. Consider the integral $\int_1^{+\infty} \frac{dx}{x^\alpha}$, $\alpha \in \mathbb{R}$.

We choose the value $c > 1$ and find the integral over a finite segment:

$c > 1$

$$\int_1^c \frac{dx}{x^\alpha} = \begin{cases} \left.\dfrac{x^{-\alpha+1}}{-\alpha+1}\right|_1^c = \dfrac{c^{-\alpha+1}}{-\alpha+1} - \dfrac{1}{-\alpha+1}, & \alpha \neq 1, \\ \ln x|_1^c = \ln c, \to +\infty & \alpha = 1. \end{cases}$$

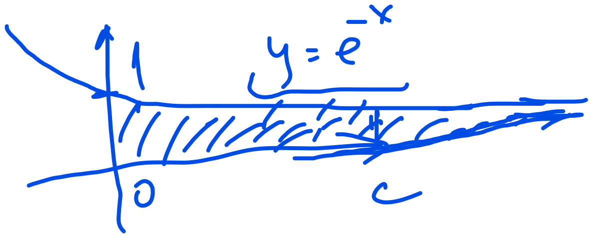
$\boxed{\alpha < 1}$ $\quad -\alpha + 1 > 0$ $\quad c \to +\infty$ $\quad -\alpha + 1 < 0$ $\quad \boxed{\alpha > 1}$

The function $\ln c$ approaches infinity as $c \to +\infty$. The function $c^{-\alpha+1}$ approaches infinity as $c \to +\infty$ if $\alpha < 1$ and approaches 0 if $\alpha > 1$. Consequently, the initial improper integral diverges for $\alpha \leq 1$ and converges for $\alpha > 1$ and, for the converging integral, the formula holds:

$$\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{c\to+\infty} \left(\frac{c^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1} \right) = \frac{1}{\alpha - 1}, \quad \alpha > 1.$$

2. Consider the integral $\int_0^{+\infty} e^{-x}\,dx$. In this case, for the segment $[0, c]$, we have

$$\int_0^c e^{-x}\,dx = -e^{-x}\Big|_0^c = -e^{-c} + 1.$$



Hence,

$$\int_0^{+\infty} e^{-x}\,dx = \lim_{c\to+\infty} (-e^{-c} + 1) = 1.$$

$\searrow 0$

Improper integral for an unbounded function and the definition of an improper integral in the general case 3.8B/14:15 (09:12)

Definition 2 (definition of an improper integral for an unbounded function).

Let the function f be defined on the half-interval $[a, b)$ and integrable on any segment $[a, c]$, $a < c < b$. If there exists a finite limit of the integral $\int_a^c f(x)\,dx$ as $c \to b - 0$, then they say that there exists an improper integral $\int_a^b f(x)\,dx$ and its value is assumed to be equal to this limit:

$\alpha \leq 0$

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{c\to b-0} \int_a^c f(x)\,dx.$$

$\alpha = -2$

$a \quad c \quad b$

In this case, they also say that the improper integral $\int_a^b f(x)\,dx$ converges.

If the limit $\lim_{c\to b-0} \int_a^c f(x)\,dx$ does not exist or is equal to infinity, then they say that the improper integral $\int_a^b f(x)\,dx$ diverges.

$\int_0^1 \frac{dx}{x^{-2}} = \int_0^1 x^2\,dx$

$\alpha = 2$

$\int_0^1 \frac{dx}{x^2}$

The improper integral for the function defined on the half-interval $(a, b]$ is defined in a similar way.

Example.

Consider the integral $\int_0^1 \frac{dx}{x^\alpha}$, $\alpha \in \mathbb{R}$. Obviously, for $\alpha \leq 0$, this integral is an usual (proper) integral, since the function $\frac{1}{x^\alpha}$ in this case is defined and continuous on the entire segment $[0, 1]$. The value of the integral for $\alpha < 0$ is equal to

$\frac{1}{x^\alpha}$

$x \to 0$

$[c, 1]$

$c > 0$

$$\int_0^1 \frac{dx}{x^\alpha} = \frac{x^{-\alpha+1}}{-\alpha+1}\bigg|_0^1 = \frac{1}{1-\alpha}.$$

$-\alpha + 1 > 0$

$\boxed{\alpha < 1}$

The formula $\int_0^1 \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}$ is also valid for the case $\alpha = 0$.

For $\alpha > 0$, we have an improper integral, since the function $\frac{1}{x^\alpha}$ is unbounded in a neighborhood of the point 0. So, we choose the value $c \in (0, 1)$ and find the integral over the finite segment:

$c \to +0$

$$\int_c^1 \frac{dx}{x^\alpha} = \begin{cases} \dfrac{x^{-\alpha+1}}{-\alpha+1}\bigg|_c^1 = \dfrac{1}{-\alpha+1} - \dfrac{c^{-\alpha+1}}{-\alpha+1}, & \alpha \neq 1, \\ \ln x\big|_c^1 = -\ln c, & \alpha = 1. \end{cases}$$

1

$\ln x$

$x \to +0$

The function $-\ln$ c approaches infinity as $c \to +0$. The function $c^{-\alpha+1}$ approaches infinity as $c \to +0$ if $\alpha > 1$ and approaches 0 if $\alpha < 1$. Consequently, the initial improper integral diverges for $\alpha \geq 1$ and converges for $0 < \alpha < 1$ and the formula holds for the converging integral:

$$\int_0^1 \frac{dx}{x^\alpha} = \lim_{c\to+0}\left(\frac{1}{-\alpha+1} - \frac{c^{-\alpha+1}}{-\alpha+1}\right) = \frac{1}{1-\alpha}, \qquad 0 < \alpha < 1.$$

$\int_0^{+\infty} \frac{dx}{x^\alpha}$

converges

$\boxed{\alpha < 1}$

$\int_0 \quad \int_1^{+\infty}$

So, the integral $\int_0^1 \frac{dx}{x^\alpha}$ exists for $\alpha < 1$, it is equal to $\frac{1}{1-\alpha}$, and, for $0 < \alpha < 1$, it must be understood in an improper sense.

In the future, it will be convenient for us to simultaneously consider improper integrals over semi-infinite intervals and improper integrals of unbounded functions. So, let us give a general definition of an improper integral.

Definition 3 (the definition of an improper integral in the general case).

$c \to +\infty$

Let the function f be defined on the half-interval $[a, b)$ and integrable on any segment $[a, c]$, $a < c < b$. The point b is either finite or equal to $+\infty$. If there exists a finite limit of the integral $\int_a^c f(x)\,dx$ as $c \to b - 0$, then they say that there exists an improper integral $\int_a^b f(x)\,dx$ and its value is assumed to be equal to this limit:

$c \to b$ b

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{c \to b-0} \int_a^c f(x)\,dx.$$

In this case, they also say that the improper integral $\int_a^b f(x)\,dx$ converges.

If the limit $\lim_{c \to b-0} \int_a^c f(x)\,dx$ does not exist or is equal to infinity, then they say that the improper integral $\int_a^b f(x)\,dx$ diverges.

An improper integral with a singularity at the left endpoint a of the integration interval is defined in a similar way; the left endpoint may be equal to $-\infty$.

If an improper integral has a singularity at both endpoints of the integration interval (a, b), then it is considered as the sum of the integrals over the intervals $(a, d]$ and $[d, b)$ for some point $d \in (a, b)$ and is convergent if and only if improper integrals converge over each of the intervals $(a, d]$ and $[d, b)$. We return to the discussion of integrals with several singularities at the end of the next chapter.

Properties of improper integrals

Linearity of the improper integral with respect to the integrand

3.8B/23:27 (05:12)

Theorem 1 (on the linearity of an improper integral with respect to integrand).

Let the functions f and g be defined on $[a, b)$, $\alpha, \beta \in \mathbb{R}$. Let there exist improper integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$. Then there exists an improper integral $\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx$ and the following formula holds:

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx. \tag{1}$$

PROOF.

Let $c \in (a,b)$. From the definition of the improper integral, we obtain that there exist integrals $\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$. Then, due to the linearity of the usual (proper) definite integral with respect to integrands, the integral $\int_a^c \big(\alpha f(x) + \beta g(x)\big)\,dx$ also exists and the equality holds:

$\int_a^b f(x)dx$ $\quad$ $\int_a^b g(x)dx$

$\exists \lim\limits_{c\to b-0}$

$$\int_a^c \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int_a^c f(x)\,dx + \beta \int_a^c g(x)\,dx.$$

$c \to b-0$

In the resulting equality, we pass to the limit as $c \to b-0$. By the definition of an improper integral, the limits of $\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$ exist and are equal to $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$, respectively. Using the arithmetic properties of the limit, we obtain that the limit on the left-hand side also exists and is equal to $\alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$.

Thus, we proved that the integral $\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx$ converges and we also proved formula (1). $\square$

Additivity of an improper integral with respect to the integration interval and change of variables in an improper integral `3.8B/28:39 (05:47)`

THEOREM 2 (ON THE ADDITIVITY OF AN IMPROPER INTEGRAL WITH RESPECT TO THE INTEGRATION INTERVAL).

Let the function f be defined on $[a,b)$ and there exists an improper integral $\int_a^b f(x)\,dx$. Then, for any point $d \in (a,b)$, the improper integral $\int_d^b f(x)\,dx$ converges and the equality holds:

$$\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx.$$

The proof of this theorem is carried out similarly to the proof of Theorem 1, using the additivity property of the usual definite integral with respect to the integration segment and arithmetic properties of the limit. $\square$

THEOREM 3 (ON THE CHANGE OF VARIABLES IN AN IMPROPER INTEGRAL).

Let the function f be defined on $[a,b)$ and there exists an improper integral $\int_a^b f(x)\,dx$. Let the function φ act from $[\alpha,\beta)$ on $[a,b)$, be continuously differentiable on $[\alpha,\beta)$, $\varphi'(t) > 0$ for $t \in [\alpha,\beta)$, $\varphi(\alpha) = a$ and $\lim_{t\to\beta-0} \varphi(t) = b$. Then there exists an improper integral $\int_\alpha^\beta f\big(\varphi(t)\big)\varphi'(t)\,dt$ and the equality holds:

$c \to b-0$

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt. \tag{2}$$

PROOF[1].

Let $\gamma \in (\alpha, \beta)$. Then, by the theorem on the change of variables in a usual (proper) definite integral, the equality holds:

$$\int_a^{\varphi(\gamma)} f(x)\,dx = \int_\alpha^\gamma f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt. \tag{3}$$

The left-hand side of equality (3) can be represented as a superposition, where the external function has the argument c and the internal function has the argument γ:

$$\int_a^{\varphi(\gamma)} f(x)\,dx = \left(\int_a^c f(x)\,dx\right) \circ \varphi(\gamma).$$

The conditions of the theorem imply the following limit equalities: $\lim_{\gamma\to\beta-0}\varphi(\gamma) = b$, $\lim_{c\to b-0}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx$, and $\varphi(t) \neq b$ when $t \in [\alpha, \beta)$. Thus, all the conditions of the superposition limit theorem are satisfied and, by virtue of this theorem, the limit of the superposition $\int_a^{\varphi(\gamma)} f(x)\,dx$ as $\gamma \to \beta - 0$ is equal to the limit of the external function:

$$\lim_{\gamma\to\beta-0}\int_a^{\varphi(\gamma)} f(x)\,dx = \lim_{c\to b-0}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx.$$

Therefore, the limit of the right-hand side of equality (3) as $\gamma \to \beta - 0$ also exists and is equal to $\int_a^b f(x)\,dx$.

Thus, we proved that the improper integral $\int_\alpha^\beta f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt$ converges and we also obtained formula (2). $\square$

Integration formula by parts for improper integrals. Theorem on the coincidence of the integral in the proper and improper sense

`3.8B/34:26 (07:19)`

THEOREM 4 (ON INTEGRATION BY PARTS OF AN IMPROPER INTEGRAL).

Let the functions u and v be defined and continuously differentiable on $[a, b)$ and there exists a limit $\lim_{x\to b} u(x)v(x)$. Then the improper integrals $\int_a^b uv'\,dx$ and $\int_a^b u'v\,dx$ either both converge or both diverge, and if they converge, then the following relation holds, which is called the *integration formula by parts* for improper integrals:

[1]There is no proof of this theorem in video lectures.

$$\int_a^b uv'\,dx = (uv)\big|_a^b - \int_a^b u'v\,dx.$$

In this formula, the notation $(uv)\big|_a^b$ means the following difference: $\lim_{x\to b} u(x)v(x) - u(a)v(a)$.

Proof.

The proof of this theorem is carried out similarly to the proof of Theorem 1, using the integration formula by parts for the usual definite integral and arithmetic properties of the limit. □

Theorem 5 (on the coincidence of the integral in the proper and improper sense).

If the function f is defined and integrable on the interval $[a, b]$, then the following equality holds, where the usual (proper) integral is indicated on the left-hand side:

c

$\lim_{c\to b-0} \int_a^c f(x)\,dx$

usual def int

$$\int_a^b f(x)\,dx = \lim_{c\to b-0} \int_a^c f(x)\,dx. \qquad (4)$$

improper int

Thus, if a proper integral exists, then it coincides with the corresponding integral understood in an improper sense (i. e., determined by passing to the limit).

Proof.

The integral $\int_a^c f(x)\,dx$, which is considered as a function $\Phi(c)$ of the argument c, is an integral with a variable upper limit:

$$\Phi(c) = \int_a^c f(x)\,dx.$$

Since, by condition, the function f is integrable over the segment $[a, b]$, we obtain, by virtue of the properties of the integral with a variable upper limit, that the function $\Phi(c)$ is continuous on this segment. Hence,

$$\lim_{c\to b-0} \Phi(c) = \Phi(b).$$

— from continuity of integral with var. upper limit

Taking into account the definition of the function $\Phi(c)$, we obtain the relation (4). □

f - integr $\to$ Φ - continuous

f - continuous $\to$ Φ - differentiable