13. Convergence tests for numerical series with non-negative terms

Comparison test

Criterion for convergence of numerical series with non-negative terms

3.10B/11:28 (07:21)

Theorem (criterion for convergence of numerical series with non-negative terms).

Let all terms of the series $\sum_{k=1}^{\infty} a_k$ be non-negative:

$$\forall\, k \in \mathbb{N} \quad a_k \geq 0.$$

Then this series converges if and only if the set of values of its partial sums $\sum_{k=1}^{n} a_k$ is bounded from above:

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N} \quad \sum_{k=1}^{n} a_k \leq M. \tag{1}$$

Proof.

Consider the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$. Since all the terms a_k are non-negative, we obtain that this sequence is non-decreasing:

$$\forall\, n \in \mathbb{N} \quad S_{n+1} = \sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1} \geq \sum_{k=1}^{n} a_k = S_n.$$

When studying the limit of a sequence, we proved that a non-decreasing sequence converges if and only if it is bounded from above.

Thus, the condition (1), which means that the partial sums S_n (with non-negative terms) are bounded from above, is equivalent to the convergence of the sequence $\{S_n\}$, and the convergence of this sequence is equivalent to the convergence of the numerical series. ☐

Comparison test for numerical series

3.10B/18:49 (06:49)

Theorem (comparison test for numerical series).
Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series for which the following condition holds:

$$\exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad 0 \leq a_k \leq b_k.$$

Then two statements are valid.

1. If the series $\sum_{k=1}^{\infty} b_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ also converges.
2. If the series $\sum_{k=1}^{\infty} a_k$ diverges, then the series $\sum_{k=1}^{\infty} b_k$ also diverges.

Proof.

Since the convergence of the series $\sum_{k=1}^{\infty} a_k$ is equivalent to the convergence of the series $\sum_{k=m}^{\infty} a_k$ and the same fact is true for series with terms b_k, it is enough to prove the theorem for the series $\sum_{k=m}^{\infty} a_k$ and $\sum_{k=m}^{\infty} b_k$, all terms of which are non-negative and satisfy the inequality $a_k leb_k$.

1. If the series $\sum_{k=m}^{\infty} b_k$ converges, then, by the criterion for the convergence of numerical series with non-negative terms, we have

necessary

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N}, n \geq m, \quad \sum_{k=m}^{n} b_k \leq M.$$

Then, due to the inequality $a_k \leq b_k$, we obtain that a similar estimate is also valid for partial sums of the series $\sum_{k=m}^{\infty} a_k$:

$$\sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} b_k \leq M.$$

sufficient part

Therefore, by the same criterion, the series $\sum_{k=m}^{\infty} a_k$ converges.

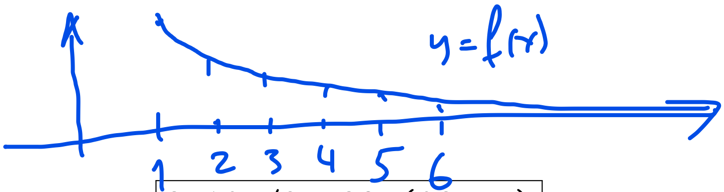
2. Let the series $\sum_{k=m}^{\infty} a_k$ diverge.

If we assume that the series $\sum_{k=m}^{\infty} b_k$ converges, then, by already proved statement 1, the series $\sum_{k=m}^{\infty} a_k$ should also converge. But this fact contradicts the condition. Therefore, the assumption made is false and the series $\sum_{k=m}^{\infty} b_k$ diverges. $\square$

Remark.

From the comparison test for numerical series, one can obtain a corollary similar to the corollary from the comparison test for improper integrals: if, for all $k \in \mathbb{N}$ starting from some m, the estimates $a_k > 0$, $b_k > 0$ are fulfilled and the limit relation $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$ holds, then the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or both diverge.

Integral test of convergence



Formulation of the integral test of convergence

3.10B/25:38 (03:55)

Theorem (integral test of convergence).

Let the function f be defined on the set $[1, +\infty)$, be non-negative and non-increasing, and $\lim_{x\to+\infty} f(x) = 0$.

Then the improper integral $\int_1^{+\infty} f(x)\,dx$ and the series $\sum_{k=1}^{\infty} f(k)$ either both converge or both diverge.

Initial stage of the proof

3.10B/29:33 (07:07)

We choose the points $k, k+1 \in \mathbb{N}$ and assume that the point $x \in \mathbb{R}$ is between k and $k+1$: $k \le x \le k+1$. Since the function f is non-increasing, the following double inequality holds:

$$f(k+1) \le f(x) \le f(k).$$

Let us integrate all the terms of the resulting double inequality from k to $k+1$; this operation will not change the sign of inequality:

$$f(k+1)\int_k^{k+1} dx \le \int_k^{k+1} f(x)\,dx \le f(k+1)\int_k^{k+1} dx.$$

The integrals on the left-hand and right-hand sides of this double inequality are equal to 1:

$$\int_k^{k+1} dx = x\Big|_k^{k+1} = k+1-k = 1.$$

Thus, the double inequality takes the form

$$f(k+1) \le \int_k^{k+1} f(x)\,dx \le f(k).$$

Now we summarize the inequalities obtained for $k = 1, \dots, n$:

$$\sum_{k=1}^{n} f(k+1) \le \sum_{k=1}^{n}\int_k^{k+1} f(x)\,dx \le \sum_{k=1}^{n} f(k).$$

Given the property of additivity of the integral with respect to the integration interval, the resulting double inequality can be rewritten as follows:

$$\sum_{k=1}^{n} f(k+1) \le \int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k).$$

Let us introduce the notation for the partial sum of the series: $S_n = \sum_{k=1}^{n} f(k)$. Using this notation, we finally obtain

$$S_{n+1} - f(1) \le \int_1^{n+1} f(x)\,dx \le S_n. \tag{2}$$

The final stage of the proof $\boxed{\texttt{3.10B/36:40 (07:13)}}$

Now we consider various situations related to the convergence or divergence of the initial integral and series.

1. Let the improper integral $\int_1^{+\infty} f(x)\,dx$ converge. Then, by virtue of the criterion for the convergence of improper integrals of non-negative functions, we have

$$\exists\, M > 0 \quad \forall\, c > 1 \qquad \int_1^{c} f(x)\,dx \le M.$$

Using this estimate for the integral and the left-hand side of estimate (2), we obtain

$$S_{n+1} - f(1) \le \int_1^{n+1} f(x)\,dx \le M.$$

Thus, we have proved that the partial sums of S_n are uniformly bounded:

$$\forall\, n \in \mathbb{N} \quad S_{n+1} \le M + f(1).$$

Therefore, by the criterion for the convergence of a numerical series with non-negative terms, the series $\sum_{k=1}^{\infty} f(k)$ converges.

2. Let the series $\sum_{k=1}^{\infty} f(k)$ converge. Then, by virtue of the criterion for the convergence of a numerical series with non-negative terms, we have

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N} \quad S_n \le M.$$

Choose an arbitrary real number $c > 1$ and consider the integral $\int_1^{c} f(x)\,dx$. For any number $c > 1$, there exists an integer n such that $c < n + 1$. Since the function $f(x)$ is non-negative, the estimate holds:

$$\int_1^{c} f(x)\,dx \le \int_1^{n+1} f(x)\,dx.$$

Using this estimate and the right-hand side of estimate (2), we obtain

$$\int_1^{c} f(x)\,dx \le \int_1^{n+1} f(x)\,dx \le S_n \le M.$$

We have proved that the integrals $\int_1^{c} f(x)\,dx$ are uniformly bounded:

$$\forall\, c > 1 \qquad \int_1^{c} f(x)\,dx \le M.$$

Therefore, by the criterion for the convergence of improper integrals of non-negative functions, the integral $\int_1^{+\infty} f(x)\,dx$ converges.

3. Let the series $\sum_{k=1}^{\infty} f(k)$ diverge. Assuming that the integral $\int_1^{+\infty} f(x)\,dx$ converges, we obtain that, by the result already proved in section 1, the series $\sum_{k=1}^{\infty} f(k)$ should also converge, but this contradicts the condition. Therefore, the integral diverges.

4. Let the integral $\int_1^{+\infty} f(x)\,dx$ diverge. If we assume that the series $\sum_{k=1}^{\infty} f(k)$ converges, then, by the result already proved in section 2, the integral $\int_1^{+\infty} f(x)\,dx$ must also converge, but this contradicts the condition. Therefore, the series diverges. □

$\alpha > 0$ $\quad f = \frac{1}{x^{\alpha}}$

REMARK.

The limit relation $\lim_{x\to+\infty} f(x) = 0$ was not used in the proof. It is required in order to ensure that the necessary condition for the convergence of the series $\sum_{k=1}^{\infty} f(k)$ is satisfied, since if this condition is violated, the series will necessarily diverge (and, as follows from the proof, the integral $\int_1^{+\infty} f(x)\,dx$ will also diverge).

$\int_1^{+\infty} \frac{dx}{x^{\alpha}}$ conv $\alpha > 1$ $\quad$ diverg. $\alpha \le 1$ $\quad \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$

An example of applying the integral test of convergence

3.11A/00:00 (04:49)

Earlier, we found that the improper integral $\int_1^{+\infty} \frac{1}{x^{\alpha}}\,dx$ converges for $\alpha > 1$ and diverges for $\alpha \le 1$. Now we can extend this result to the corresponding series. For $\alpha > 0$, the function $f(x) = \frac{1}{x^{\alpha}}$ satisfies all the conditions of the previous theorem (it is non-negative and monotonously approaches 0 as $x \to +\infty$), therefore, by virtue of of the previous theorem, the series $\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$ converges for $\alpha > 1$ and diverges for $\alpha \in (0, 1]$. For $\alpha \le 0$, the series $\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$ also diverges, since, in this case, its common term $\frac{1}{k^{\alpha}}$ does not approach 0 as $k \to \infty$ and therefore the necessary convergence condition is not satisfied for the series. Thus, we have proved the following statement.

$\alpha = 1$ $\quad \sum_{k=1}^{\infty} \frac{1}{k}$ diverges

THEOREM (ON THE CONVERGENCE OF NUMERICAL SERIES WITH COMMON TERMS THAT ARE POWER FUNCTIONS).

The numerical series $\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$ converges for $\alpha > 1$ and diverges for $\alpha \le 1$. In particular, the series $\sum_{k=1}^{\infty} \frac{1}{k}$, called the *harmonic series*, diverges.

D'Alembert's test and Cauchy's test for convergence of a numerical series

Formulation of D'Alembert's test

3.11A/04:49 (04:01)

The tests considered in this section have no analogues for improper integrals.

Theorem (D'Alembert's test for convergence of a numerical series).

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms: $\forall\, k \in \mathbb{N}\;\; a_k > 0$.

1. Let the following condition be satisfied:

$$\exists\, q \in (0,1) \quad \exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad \frac{a_{k+1}}{a_k} \leq q.$$

Then the series $\sum_{k=1}^{\infty} a_k$ converges.

2. Let the following condition be satisfied:

$$\exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad \frac{a_{k+1}}{a_k} \geq 1.$$

Then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Proof of D'Alembert's test `3.11A/08:50 (08:54)`

1. Consider the terms of the initial series, starting with $k = m$. By condition, $\frac{a_{m+1}}{a_m} \leq q$, whence

$$a_{m+1} \leq q a_m.$$

The same inequality holds for the term a_{m+2}: $a_{m+2} \leq q a_{m+1}$. Given the previous inequality, we obtain

$$a_{m+2} \leq q a_{m+1} \leq q^2 a_m.$$

Obviously, for the terms a_{m+k}, $k \in \mathbb{N}$, the following estimate holds (which can be rigorously proved by mathematical induction):

$$a_{m+k} \leq q^k a_m. \tag{3}$$

Consider the series $\sum_{k=1}^{\infty} a_{m+k}$ and $\sum_{k=1}^{\infty} q^k a_m$.

The first series can be rewritten in the form $\sum_{k=m+1}^{\infty} a_k$, therefore, it coincides with the initial series, from which m first terms are removed. So, if the series $\sum_{k=1}^{\infty} a_{m+k}$ converges, then the initial series also converges, since the presence or absence of a finite number of initial terms of the series does not affect its convergence.

The second series can be transformed as follows: $\sum_{k=1}^{\infty} q^k a_m = a_m \sum_{k=1}^{\infty} q^k$. Since, by condition, $q \in (0,1)$, we obtain, by virtue of the formula for the sum of infinite geometric progression, that the series $\sum_{k=1}^{\infty} q^k$ converges.

Considering estimate (3) and applying the comparison test for numerical series, we obtain that the series $\sum_{k=1}^{\infty} a_{m+k}$ also converges and therefore the initial series $\sum_{k=1}^{\infty} a_k$ converges too.

2. As in the proof of section 1, we consider the terms of the initial series, starting with $k = m$. By condition, $\frac{a_{m+1}}{a_m} \geq 1$, whence

$$a_{m+1} \geq a_m.$$

Similarly, we obtain the estimate $a_{m+2} \geq a_{m+1} \geq a_m$. The same estimate will be valid for all terms a_{m+k} for $k \in \mathbb{N}$:

$$a_{m+k} \geq a_m.$$

We have obtained that the terms of the initial series, starting with a_m, are bounded from below by the positive value a_m. This means that the sequence $\{a_k\}$ cannot approach 0 as $k \to \infty$. Indeed, choosing the number $\varepsilon > 0$ equal to the minimum of a finite set of positive numbers $a_1, a_2, \ldots, a_m$, we get that the ε-neighborhood of zero does not contain any element of the sequence $\{a_k\}$. But, by the definition of the limit equal to A, any neighborhood of the point A should contain all elements of the sequence except, perhaps, a finite number of its initial elements.

Since the necessary convergence condition is not satisfied for the series $\sum_{k=1}^{\infty} a_k$, this series diverges. □

The limit D'Alembert test `3.11A/17:44 (07:23)`

Corollary (the limit D'Alembert test).

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms: $\forall\, k \in \mathbb{N}\ \ a_k > 0$. Suppose that there exists a limit $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = q$. If $q < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges; if $q > 1$, then the series diverges.

$q = 1$?

Proof.

Using the limit definition in the language ε–N, we can write

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, k > N \quad \left|\frac{a_{k+1}}{a_k} - q\right| < \varepsilon.$$

1. If $q < 1$, then choosing $\varepsilon = \frac{1-q}{2} > 0$, we get that, for all $k > N$, the inequality $\frac{a_{k+1}}{a_k} - q < \frac{1-q}{2}$ holds, from which the estimate follows:

$$\frac{a_{k+1}}{a_k} < q + \frac{1-q}{2} = \frac{1+q}{2} = q'.$$

Since $q < 1$, we obtain that $q' < 1$, therefore the condition of statement 1 of D'Alembert's test is satisfied for the initial series. Consequently, the series converges.

2. If $q > 1$, then choosing $\varepsilon = \frac{q-1}{2} > 0$, we get that, for all $k > N$, the inequality $\frac{a_{k+1}}{a_k} - q > -\frac{q-1}{2}$ holds, from which the estimate follows:

$$\cancel{\frac{a_{k+1}}{a_k} > q - \frac{q-1}{2} = \frac{q+1}{2} > 1.}$$

Thus, for the initial series, the condition of statement 2 of D'Alembert's test is satisfied, therefore the series diverges. □

REMARKS.

1. If the limit $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ is 1, then nothing can be said about the convergence or divergence of the series and further investigation is required.

2. If the limit $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ is equal to $+\infty$, then, by similar reasoning, we can prove that the series diverges.

An example of applying D'Alembert's test `3.11A/25:07 (02:48)`

$e^x = \sum_{k=0}^{N} \frac{x^k}{k!} + \bar{o}(x^N)$

Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Recall that, by definition, it is supposed that $0! = 1$. Here x is an arbitrary real number. Denote $a_k = \frac{x^k}{k!}$ and consider the following limit:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} = \lim_{k\to\infty} \frac{x^{k+1}k!}{x^k(k+1)!} = \lim_{k\to\infty} \frac{x}{k+1} = 0. = q < 1$$

The limit exists and its value is less than 1, therefore, due to the limit D'Alembert test, this series converges for any value of the parameter $x \in \mathbb{R}$.

REMARK.

In what follows, we prove that the sum of the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is equal to e^x.

Cauchy's test `3.11A/27:55 (07:22)`

THEOREM (CAUCHY'S TEST FOR CONVERGENCE OF A NUMERICAL SERIES).

Let $\sum_{k=1}^{\infty} a_k$ be a series with non-negative terms: $\forall\, k \in \mathbb{N} \;\; a_k \geq 0$.

1. Let the following condition be satisfied:

$$\exists\, q \in (0,1) \quad \exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad \sqrt[k]{a_k} \leq q.$$

$\sqrt[k]{a_k}$ $\qquad \frac{a_{k+1}}{a_k}$

Then the series $\sum_{k=1}^{\infty} a_k$ converges.

2. Let the following condition be satisfied:

$$\exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad \sqrt[k]{a_k} \geq 1.$$

Then the series $\sum_{k=1}^{\infty} a_k$ diverges.

PROOF.

1. Consider the terms of the initial series, starting with $k = m$. By condition, $\sqrt[k]{a_k} \leq q$; let us raise both sides of this inequality to the power of k:

$$a_k \leq q^k. \tag{4}$$

$\sum_{k=m}^{\infty} q^k$ – conv. $0 < q < 1$

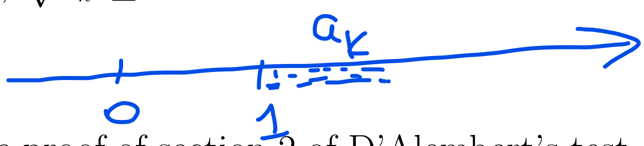
$\sum_{k=m}^{\infty} a_k$ conv

Estimate (4) is valid for terms of the series $\sum_{k=m}^{\infty} a_k$ and $\sum_{k=m}^{\infty} q^k$. Since, by condition, $q \in (0,1)$, we obtain, by virtue of the formula for the sum of infinite geometric progression, that the series $\sum_{k=m}^{\infty} q^k$ converges.

Taking into account estimate (4) and applying the comparison test for numerical series, we obtain that the series $\sum_{k=m}^{\infty} a_k$ also converges and therefore the original series $\sum_{k=1}^{\infty} a_k$ converges too.

2. As in the proof of section 1, we consider the terms of the initial series, starting with $k = m$. By condition, $\sqrt[k]{a_k} \geq 1$. We raise both sides of this inequality to the power of k:

$$a_k \geq 1.$$



$\cancel{a_k \to 0}$

Arguing in the same way as in the proof of section 2 of D'Alembert's test, we obtain that the sequence $\{a_k\}$ cannot approach 0 as $k \to \infty$, and therefore the necessary convergence condition is not satisfied for the series $\sum_{k=1}^{\infty} a_k$. So, this series diverges. □

Corollary (the limit Cauchy test).

Let $\sum_{k=1}^{\infty} a_k$ be a series with non-negative terms: $\forall\, k \in \mathbb{N}\ \ a_k \geq 0$. Suppose that there exists a limit $\lim_{k\to\infty} \sqrt[k]{a_k} = q$. If $q < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges; if $q > 1$, then the series diverges.

The proof is carried out in the same way as the proof of the limit D'Alembert test. □

Remarks.

1. If the limit $\lim_{k\to\infty} \sqrt[k]{a_k}$ is 1, then nothing can be said about the convergence or divergence of the series and further investigation is required.

2. If the limit $\lim_{k\to\infty} \sqrt[k]{a_k}$ is equal to $+\infty$, then we can prove that the series diverges.

An example of applying Cauchy's test

`3.11A/35:17 (06:06)`

Consider the series $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k^2}$. Denote $a_k = \left(1+\frac{1}{k}\right)^{-k^2}$ and consider the following limit:

$e \approx 2.7..$

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left(1+\frac{1}{k}\right)^{-k^2}} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{-k} = \frac{1}{e}. \quad < 1$$

In the last step, we used the second remarkable limit $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k} = e$. Thus, the limit $\lim_{k\to\infty} \sqrt[k]{a_k}$ exists and its value $\frac{1}{e}$ is less than 1. Therefore, by virtue of the limit Cauchy test, this series converges.

Note that the series $\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{-k}$ diverges, since its common term $\left(1+\frac{1}{k}\right)^{-k}$ does not approach 0 as $k \to \infty$ (as shown above, the limit of the common term is $\frac{1}{e}$).

$\frac{1}{e} \neq 0$