M. E. Abramyan

Lectures on differential calculus of functions of one variable

Textbook

MINISTRY OF SCIENCE AND HIGHER EDUCATION
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LECTURES ON DIFFERENTIAL CALCULUS OF FUNCTIONS OF ONE VARIABLE

Textbook for students
of science and engineering

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The textbook contains lecture material for the first semester of the course on mathematical analysis and includes the following topics: the limit of a sequence, the limit of a function, continuous functions, differentiable functions (up to Taylor's formula, L'Hospital's rule, and the study of functions by differential calculus methods). A useful feature of the book is the possibility of studying the course material at the same time as viewing a set of 22 video lectures recorded by the author and available on youtube.com. Sections and subsections of the textbook are provided with information about the lecture number, the start time of the corresponding fragment and the duration of this fragment. In the electronic version of the textbook, this information is presented as hyperlinks, allowing reader to immediately view the required fragment of the lecture.

The textbook is intended for students specializing in science and engineering.

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Contents

Preface

The book contains material of lectures of the first semester of a course in mathematical analysis, which was read by the author for several years at the I. I. Vorovich Institute of Mechanics, Mathematics, and Computer Science of the Southern Federal University (specialization 01.03.02 – "Applied Mathematics and Computer Science"). It includes the following topics: the limit of a sequence, the limit of a function, continuous functions, differentiable functions (up to Taylor's formula, L'Hospital's rule, and the study of functions by differential calculus methods).

The presented material is included, with some modifications, in most textbooks on mathematical analysis, so it makes sense to describe those features of the book that explain the purpose of its creation.

The book can be attributed to the category of "short textbooks", covering only the material that may usually be given in lectures. In this respect, it is similar to books [8–10, 13] and differs from the "detailed textbooks" that cover the subject with much greater completeness (along with the classic example [5], it can be also noted [6, 7, 11, 12, 15, 16]). All topics in the book are set out at a fairly high level of rigor, and all statements are provided with detailed proof. A significant part of the book consists of examples illustrating the concepts introduced and the results obtained.

The most important feature of the book, from the author's point of view, is its close relationship with the set of video lectures recorded in the 2015/16 academic year directly in the classroom. We can say that this book is a compendium of these lectures. Therefore, the reader may benefit most from the "parallel" study of the book and viewing the corresponding fragments of video lectures. Such an opportunity is particularly useful for undergraduate students who are poorly prepared for the perception of formal mathematical texts. In his lectures, the author pays a lot of attention to an informal description of the introduced concepts, as well as the ideas of proofs, actively uses drawings, conducts a dialogue with the student audience, asking suggestive questions and answering the students' questions, that is, he applies those methods that would be difficult to represent in the form of a mathematical text. At the same time, the presence of such a text and the ability to read it in parallel with viewing a video lecture allows, in the author's opinion, to provide a better understanding of the presented material. Also it is desirable that the content of the textbook and the content of the video lecture were well coordinated. The author sees the main feature of this book in such coordination with the set of video lectures.

In order to maximally facilitate the coordination of the material contained in the textbook with the corresponding fragment of the required video lecture, the author provided the sections and subsections of the textbook with links that include the number of the lecture, the time the fragment began and the duration of this fragment. It is especially convenient to use such links in the electronic version of the book, since in this case, it is suffice for the reader simply click on the specified hyperlink to immediately play the lecture starting from the required fragment. Such hyperlinks are highlighted in blue in the text of the electronic version of the book. Of course, a prerequisite is the availability of the Internet and the youtube.com site, which hosts video lectures.

One more source should be noted, which improves the perception of the material under study: subtitles for video lectures. In educational video materials of this kind, the role of subtitles is not reduced to providing viewing opportunities for those students who have a hearing impairment or do not speak Russian. When preparing subtitles, the author corrected certain phrases, supplementing them with explanatory words, and also reproduced board-written formulas. In some cases, large additional fragments of the explanatory text were added to the subtitles, especially in those (few) situations where the author made a mistake during the lecture. The subtitles also mark the beginning of each section of the lecture.

Another feature of the book is that it was simultaneously prepared in two versions: in Russian and English. An additional goal of the English-language version is to enable English-speaking students to familiarize themselves with the features of the presentation of educational material that are typical for the Russian educational literature. The order of topics and the selection of material used in English textbooks varies greatly; as examples closest to Russian textbooks, one can point out translated books [8, 9, 14], as well as [3]. English-speaking students also have the opportunity to “parallel” study the book and view video lectures, since all video lectures are subtitled in English.

Two versions of the book (Russian and English) can also be used by students to study the features of vocabulary and phraseology related with the subject being studied. Such an opportunity may be useful both for Russian-speaking students studying English, and for English-speaking students studying Russian.

In the English version of the book, the notations of the functions tan, arctan, sinh, arsinh, etc., accepted in publications in the English language, are used (instead of the notations tg, arctg, sh, arsh accepted in Russian texts). However, the notation C_n^k for the number of combinations has been preserved (despite the fact that the notation $\binom{n}{k}$ is traditionally used in English texts) to simplify the perception of formulas with this notation in video lectures. Completion of proofs is marked with $\square$.

The book contains no exercises for self-training. The classic source of such exercises is the book of problems [4]. In addition, one can note the textbook [1, 2], which is available in Russian and English version.

The book is equipped with an index containing all definitions and theorems. For references to the theorem, their detailed description is used, which is given before their formulation and is included in the index in the "Theorem" section. Theorems containing surnames in their titles are mentioned in the index also in positions corresponding to surnames. In the electronic version of the book, page numbers in the index, as well as in the table of contents, are hyperlinks allowing to go directly to this page.

If the material of the section or its part is absent in the video lectures or is presented in them somewhat differently, then this is indicated in the footnotes. There are very few such cases; we can note only two large fragments included in the book, but not in the video lectures, namely, the proof of L'Hospital's rule in Chapter 23 and the last two sections of Chapter 24.

The initial section named "Video lectures" provides complete information about the set of video lectures related to the book including their Internet links. That allows the reader to quickly access the required lecture even in the absence of an electronic version of the book. This section also briefly describes the notation used in subtitles for video lectures.

Video lectures

Using video lectures

If the framed text follows the title of the section or subsection, this means that a fragment of the video lecture is associated with this section or subsection. The framed text consists of three parts: the number of the video lecture, the time from which this fragment begins, and the duration of this fragment.

For example, the following text `1A/00:00 (09:55)` is located after the title of the first section of Chapter 1 (the section is devoted to the continuity axiom of real numbers). It means that this topic is discussed at the very beginning of lecture 1A, and the corresponding fragment of the lecture lasts 9 minutes 55 seconds. The last subsection of Chapter 24 connected with video lectures is the subsection devoted to the location of the tangent line at the inflection point. The correspondent text is `22B/10:37 (09:46)`, which means that this topic is discussed in the lecture 22B, starting at 10:37, and the discussion lasts 9 minutes 46 seconds.

In the electronic version of the book, all framed texts are hyperlinks. Clicking on such text allows you to immediately play the corresponding lecture, starting from the specified time.

When using the paper version of the book, hyperlinks, of course, are not available, therefore, an additional information is provided here, which will allow you to quickly start playing the required video lecture. A set of 22 video lectures is available on `youtube.com`. Each video lecture consists of two parts: A and B. The following list of lectures contains their titles and short links to each part.

1. Boundaries of sets
   - 1A: https://youtu.be/BDK1ONCWwCc
   - 1B: https://youtu.be/4Lg3JJ0PoYQ
2. Limit of a sequence
   - 2A: https://youtu.be/LK5ib5HberY
   - 2B: https://youtu.be/ZTQahfP3Iyk
3. Properties of the limit of a sequence
   - 3A: https://youtu.be/DfuFyplcXgg

   3B: https://youtu.be/sDqiPvQTXJE

4. Infinite limits
   4A: https://youtu.be/CXoPKQKi_VM
   4B: https://youtu.be/iWiRSYKhuVQ
5. Monotone sequences
   5A: https://youtu.be/Lv3SSXrYL1k
   5B: https://youtu.be/i-U5uIZtrmo
6. Subsequences
   6A: https://youtu.be/RcwY89d5dtM
   6B: https://youtu.be/ttrdB3DvYtg
7. Fundamental sequences
   7A: https://youtu.be/KWp5LcPfWXQ
   7B: https://youtu.be/_YIw4f9bBqE
8. The limit of a function
   8A: https://youtu.be/rsMlkECIHQ4
   8B: https://youtu.be/dgSu5rC_fEU
9. Properties of the limit of a function
   9A: https://youtu.be/h9TxJKC2-yM
   9B: https://youtu.be/0_p6hDI1yD4
10. One-sided limits
    10A: https://youtu.be/9wS-L_B91J4
    10B: https://youtu.be/sEtRpPMMFUw
11. Continuity of function at a point
    11A: https://youtu.be/_XDGLsjrCus
    11B: https://youtu.be/7cXTvHaG4i4
12. Continuity of function on a segment
    12A: https://youtu.be/JoiUa98DJzg
    12B: https://youtu.be/lFL2brIPQh8
13. Uniform continuity
    13A: https://youtu.be/VwL8dHA-aFM
    13B: https://youtu.be/3rR0H127Rcs
14. Points of discontinuity
    14A: https://youtu.be/U_BgRI07K-Y
    14B: https://youtu.be/CvbLVMD2oxM
15. O-notation
    15A: https://youtu.be/T2a2I55grwM

15B: https://youtu.be/Qdn7OleJZ4A

16. Differentiable functions
    16A: https://youtu.be/gaUSmE7fE40
    16B: https://youtu.be/ZHmxE-4QLjU
17. Differentiable functions–2
    17A: https://youtu.be/umY4z-k4hus
    17B: https://youtu.be/K6kewoPcZug
18. Leibniz's rule. Fermat's theorem, Rolle's theorem, Lagrange's theorem
    18A: https://youtu.be/iFFa4qHZIeg
    18B: https://youtu.be/pw09UNC0zUE
19. Taylor's formula
    19A: https://youtu.be/mh6Kd-01c-Q
    19B: https://youtu.be/qQsKibWy_Ng
20. Function expansions by Taylor's formula
    20A: https://youtu.be/XM0oD_tjCxs
    20B: https://youtu.be/9J0iH_oNZeo
21. Extrema of functions. Convex functions
    21A: https://youtu.be/BBhIKfiuBpk
    21B: https://youtu.be/U-u9AqcvMs0
22. Inflection points
    22A: https://youtu.be/NkbARe7z818
    22B: https://youtu.be/P8VLjFPLcsI

You can create a link that immediately plays the video lecture, starting from the specified time. Let's describe this additional feature using the previously mentioned fragment `22B/10:37 (09:46)` as an example. This is a fragment of part B of video lecture 22, its short link has the form `P8VLjFPLcsI`. We need to play a lecture starting at 10:37.

To do this, use the Internet link `https://www.youtube.com/watch?` specifying two options after it: a short link to the video lecture (option `v=`) and the start time of playback (option `t=`). The options themselves must be separated by the `&` character.

In our case, the full text of the Internet link will be as follows:

`https://www.youtube.com/watch?v=P8VLjFPLcsI&t=10m37s`

Pay attention to the time format: after the number of minutes, the letter `m` is indicated, and after the number of seconds, the letter `s` is indicated. If the number of seconds is 0, then only the number of minutes can be specified.

A set of hyperlinks to video lectures, which also contains the names of the corresponding chapters, sections, and subsections of this book, is presented on the website `mmcs.sfedu.ru` of the Institute of Mathematics, Mechanics, and Computer Science of the Southern Federal University (Moodle environment, link http://edu.mmcs.sfedu.ru/course/view.php?id=254). At the top of the page is a set of hyperlinks with names in Russian (Fig. 1), and then in English (Fig. 2).

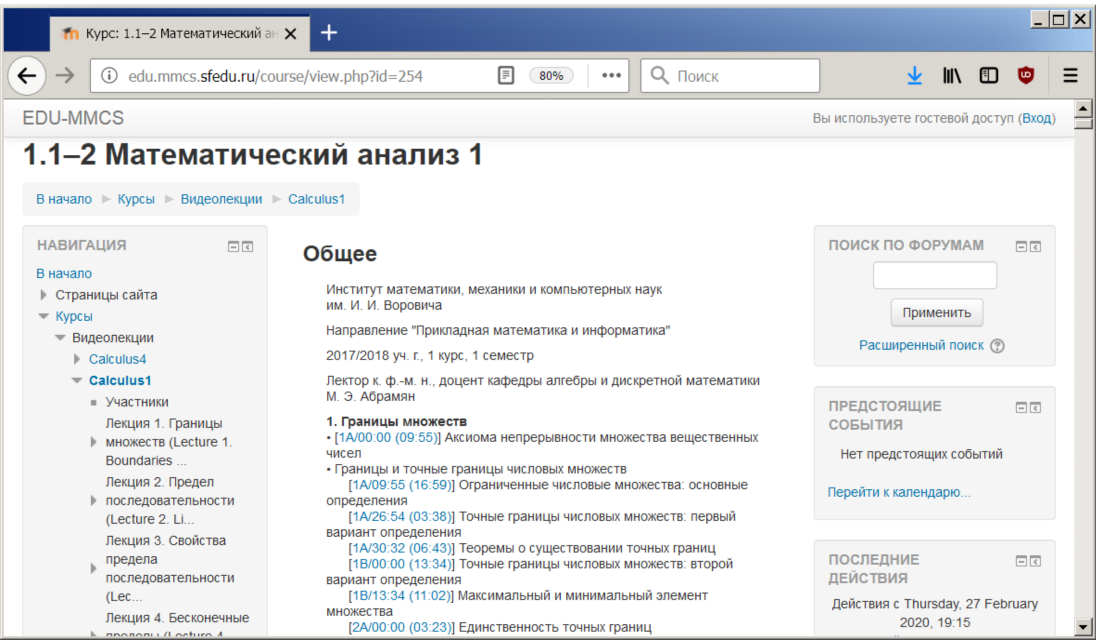

Fig. 1. Hyperlinks to video lectures on `edu.mmcs.sfedu.ru` (Russian version)

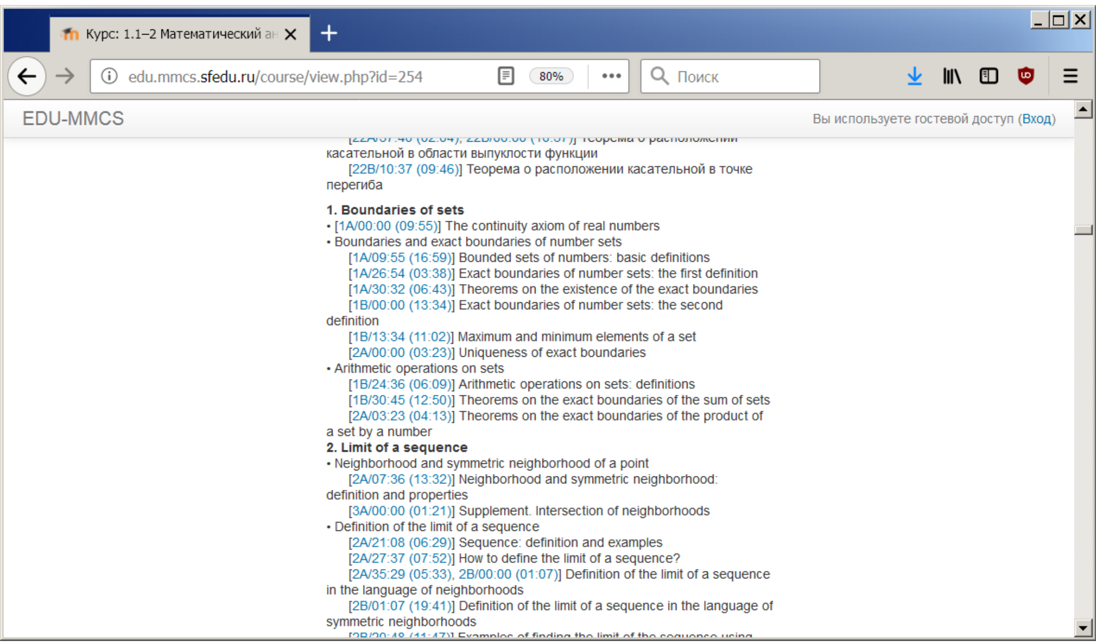

Fig. 2. Hyperlinks to video lectures on `edu.mmcs.sfedu.ru` (English version)

Using subtitles

When viewing video lectures, you may use subtitles in two languages: Russian and English. To enable or disable subtitles, you should use the first button in the group of buttons located in the lower right corner of the window of video lecture. The subtitle language is selected automatically, taking into account the settings of the computer.

To change the subtitle language, click the second button in this group (the gear is shown on this button). In the menu that appears, execute the command “Subtitles”. As a result, a list of available subtitle languages appears, in which you may select the option “Russian” or “English”.

Let’s describe the notation that is used in subtitles to highlight special types of text.

The title of the next part of the lecture is marked by double characters `=`, for example, `==THE CONTINUITY AXIOM FOR REAL NUMBERS==`.

Students’ replicas are enclosed in square brackets `[ ]`. Words added for text coherence or additional explanations are enclosed in angle brackets `< >`.

In formula texts, the subscript is denoted by the `_` character, and the superscript is denoted by the `^` character. For example, a_0 is denoted as `a_0`, x^2 as `x^2`.

Instead of a circle on top, a circle on the right is used to indicate punctured neighborhoods. For example, the neighborhood $\overset{\circ}{U}_a$ is denoted as follows: `U°_a`.

Here are a few more examples.

The limit relation $\lim_{x\to a} f(x) = A$ is written as follows:

`lim_(x→a) f(x) = A`.

The formula $C_n^k + C_n^{k-1} = C_{n+1}^k$ from the lemma on the number of combinations property (see Chapter 20, formula (4)) is written as follows:

`C_n^k + C_n^(k-1) = C_(n+1)^k`.

Taylor’s formula for polynomials $P_n(x) = \sum_{k=0}^{n} \frac{P_n^{(k)}(x_0)}{k!}(X - x_0)^k$ (see Chapter 22, formula (2)) is written as follows:

`P_n(x) = ∑_(k=0)^n P^(k)_n(x_0)/k! · (x - x_0)^k`.

Preliminary information

Mathematical logic

Operations of mathematical logic.

The operations described below apply to *propositions*, i. e., statements regarding which it can be said whether they are true or false.

$\overline{A}$ – not A (*negation*).
$A \vee B$ – A or B (*disjunction*).
$A \wedge B$ – A and B (*conjunction*).
$A \Rightarrow B$ – A implies B (or B follows from A) (*implication*).
$A \Leftrightarrow B$ – A is equivalent to B (*equivalence*).

Some laws of mathematical logic.

$\overline{\overline{A}} \Leftrightarrow A$
$\overline{A \vee B} \Leftrightarrow \big(\overline{A} \wedge \overline{B}\big)$
$\overline{A \wedge B} \Leftrightarrow \big(\overline{A} \vee \overline{B}\big)$
$(A \Rightarrow B) \Leftrightarrow \big(\overline{A} \vee B\big)$
$(A \Leftrightarrow B) \Leftrightarrow \big((A \Rightarrow B) \wedge (B \Rightarrow A)\big)$

Sets

A *set* is a collection of objects. These objects are called *elements* or *members* of the set. We define a set if we are able to determine whether some object belongs to the set or not. A set cannot contain identical elements.

If x *is an element of the set* A (x *belongs to the set* A), then this is denoted as $x \in A$. If x is not an element of the set A, then this is denoted as $x \notin A$.

If each element of the set A is an element of the set B, then A is called the *subset* of the set B. This is denoted as $A \subset B$ or $B \supset A$.

The symbol $\varnothing$ denotes an *empty set* that does not contain elements. It is assumed that an empty set is a subset of *any* set.

The sets A and B are called *equal* ($A = B$) if they consist of the same elements:

$$(A = B) \Leftrightarrow \big((A \subset B) \wedge (B \subset A)\big).$$

Defining sets using properties.

Let x be some object and P be some property. If the object x has the property P, then we write $P(x)$.

For example, if P_{even} is the "to be an even number" property, then one can write $P_{even}(2)$ (since 2 is an even number).

The set of all objects x with the property P is denoted as $\{x : P(x)\}$.

For example, $\{x : P_{even}(x)\}$ is the set of all even numbers.

The set of all elements x belonging to the set M and having the property P is denoted as $\{x \in M : P(x)\}$.

Operations on sets.

We consider sets that are subsets of some *universal* set M.

Union of the sets A and B is the set of those elements from M that belong to either the set A or the set B:

$$A \cup B \stackrel{\text{def}}{=} \{x \in M : (x \in A) \vee (x \in B)\}.$$

Intersection of the sets A and B is the set of those elements from M that simultaneously belong to the set A and to the set B:

$$A \cap B \stackrel{\text{def}}{=} \{x \in M : (x \in A) \wedge (x \in B)\}.$$

Difference of the sets A and B is the set of those elements from M that belong to the set A and do not belong to the set B:

$$A \setminus B \stackrel{\text{def}}{=} \{x \in M : (x \in A) \wedge (x \notin B)\}.$$

Examples of sets.

$\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, \dots, n, \dots\}$ is the set of *natural* numbers.

$\mathbb{Z} \stackrel{\text{def}}{=} \{0, \pm1, \pm2, \dots, \pm n, \dots\}$ is the set of *integers*.

$\mathbb{Q} \stackrel{\text{def}}{=} \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\}$ is the set of *rational* numbers.

$\mathbb{R}$ is the set of *real* numbers (the union of rational and irrational numbers).

$[a, b] \stackrel{\text{def}}{=} \{x \in \mathbb{R} : a \le x \le b\}$ is the *segment* with endpoints $a, b \in \mathbb{R}$.

$(a, b) \stackrel{\text{def}}{=} \{x \in \mathbb{R} : a < x < b\}$ is the *interval* with endpoints $a, b \in \mathbb{R}$.

$[a, b) \stackrel{\text{def}}{=} \{x \in \mathbb{R} : a \le x < b\}$ is the *half-interval* with endpoints $a, b \in \mathbb{R}$.

$(a, b] \stackrel{\text{def}}{=} \{x \in \mathbb{R} : a < x \le b\}$ is the *half-interval* with endpoints $a, b \in \mathbb{R}$.

Quantifiers

$\exists$ – *"exists"* (*existential quantifier*).

$\forall$ – *"any"* or *"for any"* (*universal quantifier*).

The notation "$\exists\, x \in M \ \ P(x)$" means that there exists an element x that belongs to the set M and has the property P.

The notation "$\forall x \in M \ P(x)$" means that any element x belonging to the set M has the property P.

QUANTIFIERS AND THE LOGICAL NEGATION.

$$\overline{\exists x \in M \quad P(x)} \Leftrightarrow \big(\forall x \in M \quad \overline{P(x)}\big)$$

$$\overline{\forall x \in M \quad P(x)} \Leftrightarrow \big(\exists x \in M \quad \overline{P(x)}\big)$$

Absolute value and the integer part of a real number

Absolute value $|x|$ of the real number x is defined as follows:

$$|x| \stackrel{\text{def}}{=} \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

ABSOLUTE VALUE PROPERTIES.
1) $\forall x \in \mathbb{R} \quad \{x : |x| \leq a\} = \{x : -a \leq x \leq a\}$
2) $\forall x \in \mathbb{R} \quad -|x| \leq x \leq |x|$
3) $\forall x, y \in \mathbb{R} \quad |x + y| \leq |x| + |y|$ (the *triangle inequality*)
4) $\forall x, y \in \mathbb{R} \quad |x - y| \geq \big||x| - |y|\big|$

The *integer part* of a real number x is the integer k, for which the double inequality holds: $k \leq x < k + 1$. One can say that the integer part of x is the nearest integer to x that is less than or equal to x. The integer part of a number x is denoted as $[x]$.

The definition of the integer part implies a double inequality:

$$\forall x \in \mathbb{R} \quad [x] \leq x < [x] + 1.$$

Principle of mathematical induction

In order to prove that some statement $A(n)$ is true for all natural n, it is sufficient, by the *principle of mathematical induction*, to prove two auxiliary statements:

1) $A(1)$ is true (the *base case*);

2) if $A(k)$ is true, then $A(k+1)$ is also true (the *inductive step*).

One example of applying the principle of mathematical induction is the proof of Bernoulli's inequality given in Chapter 5.

Mappings and functions

The mapping f of the set X into the set Y (notation: $f : X \to Y$) is the correspondence, associating *to each* element of the set X *the unique* element of the set Y. The set X is called the *domain of definition*, or simply *domain* of the mapping, and the set Y is called the *target set*, or *codomain* of the mapping.

The common element $x \in X$ is called the *independent variable*, or *argument*, and $y \in Y$ is called the *dependent variable*. If f maps the element $x \in X$ into the element y (notation: $y = f(x)$), then the element y is called *the image* of the element x, and the element x is called *the inverse image*, or *preimage* of the element y.

A *range*, or *image* of a map f is the subset $f(X)$ of its codomain Y consisting of those and only those elements which are images of some elements from its domain X:

$$f(X) \stackrel{\text{def}}{=} \bigl\{y \in Y : \bigl(\exists\, x \in X \quad y = f(x)\bigr)\bigr\}.$$

A mapping $f : X \to Y$ is called *one-to-one* if for any element $y \in Y$ there exists a unique element $x \in X$ such that $f(x) = y$. If the mapping $f : X \to Y$ is one-to-one, then the *inverse mapping* $f^{-1} : Y \to X$ is defined for it, which acts according to the following rule: for any element $y \in Y$, $f^{-1}(y)$ is equal to the value of $x \in X$ for which $f(x) = y$. The inverse mapping is also one-to-one.

A *superposition* of mappings $f : X \to Y$ and $g : Y \to Z$ is a mapping denoted by $g \circ f$ and acting from X to Z according to the following rule:

$$\forall\, x \in X \quad (g \circ f)(x) = g\bigl(f(x)\bigr).$$

The mapping f is called the *internal mapping* of superposition $g \circ f$ and g is called its *external mapping*. The superposition operation is *associative*: $(h \circ g) \circ f = h \circ (g \circ f)$, but, generally speaking, it is *not commutative*: $f \circ g \neq g \circ f$ even if there exist superpositions $g \circ f$ and $f \circ g$.

A *function* f is a mapping acting from X to $\mathbb{R}$. In the case when $X \subset \mathbb{R}$, the function is called the *numerical* one. In what follows, by function we will always mean numerical functions.

The *basic elementary functions* are the following functions.

1. The *polynomial* $P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n : \mathbb{R} \to \mathbb{R}$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

2. The *fractional rational function* $R(x) = P(x)/Q(x)$, where P and Q are polynomials; $R : \mathbb{R} \setminus \{x \in \mathbb{R} : Q(x) = 0\} \to \mathbb{R}$.

3. The *power function* $x^\alpha : (0, +\infty) \to (0, +\infty)$ provided $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The domain and codomain of the power function can be expanded by decreasing the set of possible values of α (for example, the function x^α acts from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R} \setminus \{0\}$ for $\alpha \in \mathbb{Z}$, $\alpha < 0$).

4. The *exponential function* $a^x : \mathbb{R} \to (0, +\infty)$ provided $a > 0$, $a \neq 1$.

5. The *logarithmic function* $\log_a x : (0, +\infty) \to \mathbb{R}$ provided $a > 0$, $a \neq 1$.

6. *Trigonometric* and *inverse trigonometric functions*.

Graphs of some basic elementary functions are shown in Figs. 3 and 4. The graphs of the inverse trigonometric functions are shown in Fig. 7 in Chapter 14.

An *elementary function* is a numerical function obtained from the basic elementary functions using a finite number of arithmetic operations and superpositions.

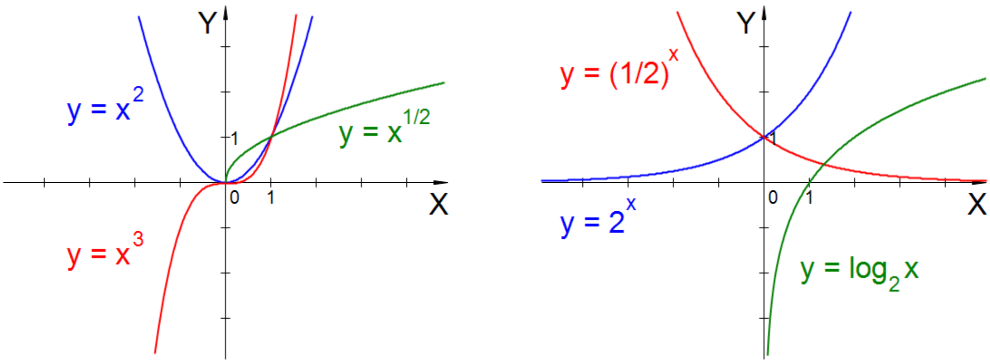

Fig. 3. Graphs of power, exponential, and logarithmic functions

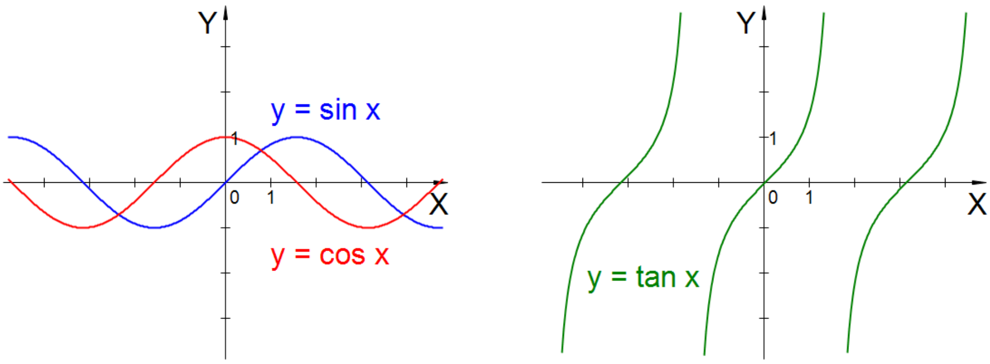

Fig. 4. Graphs of trigonometric functions

1. Boundaries of sets

The continuity axiom of real numbers   1A/00:00 (09:55)

Real numbers have a large amount of properties associated with arithmetic operations (addition, multiplication), as well as with comparison operations. These properties are studied in detail in the course of algebra. For our purposes, the property of real numbers, called the *continuity axiom*, will play a special role.

THE CONTINUITY AXIOM OF THE SET OF REAL NUMBERS.

Let X, Y be nonempty subsets of the set $\mathbb{R}$ with the following property: for any two elements $x \in X$, $y \in Y$ the inequality $x \le y$ holds. Then there exists a number $c \in \mathbb{R}$ such that for any elements $x \in X$, $y \in Y$ the inequality $x \le c \le y$ holds.

REMARK.

The set of rational numbers does not have this property. Indeed, consider two nonempty subsets of the set of *rational* numbers:

$$X = \left\{x \in \mathbb{Q} : 1 < x < \sqrt{2}\right\}, \quad Y = \left\{y \in \mathbb{Q} : \sqrt{2} < y < 2\right\}.$$

Obviously, the inequality $x \le y$ holds for any elements $x \in X$, $y \in Y$, but there is no *rational* number c satisfying the condition $x \le c \le y$ for all $x \in X$, $y \in Y$, since the number $\sqrt{2}$ is irrational.

Boundaries and exact boundaries of number sets

Bounded sets of numbers: basic definitions   1A/09:55 (16:59)

DEFINITION.

A number set X is called *upper-bounded* (or *bounded from above*) if there exists a real number M such that for any element x from the set X the estimate $x \le M$ is true:

$$\exists\, M \in \mathbb{R} \quad \forall\, x \in X \quad x \le M.$$

If a set X is *not* bounded from above, then this means that

$$\forall\, M \in \mathbb{R} \quad \exists\, x \in X \quad x > M.$$

A number set X is called *lower-bounded* (or *bounded from below*) if there exists a real number m such that for any element x from the set X the estimate $x \geq m$ holds:

$$\exists\, m \in \mathbb{R} \quad \forall x \in X \quad x \geq m.$$

If the set X is *not* bounded from below, then this means that

$$\forall\, m \in \mathbb{R} \quad \exists\, x \in X \quad x < m.$$

A number set X is called *bounded* if it is upper-bounded and lower-bounded:

$$\exists\, m, M \in \mathbb{R} \quad \forall x \in X \quad m \leq x \leq M.$$

The number M that appears in the definition of a upper-bounded set is called the *upper bound* of this set, and the number m that appears in the definition of a lower-bounded set is called the *lower bound* of this set.

If the set X is bounded, then

$$\exists\, M_0 > 0 \quad \forall x \in X \quad |x| \leq M_0.$$

As M_0, one can take the maximum of the numbers $|m|$ and $|M|$, where m and M are numbers from the definition of a bounded set.

Exact boundaries of number sets: the first definition

1A/26:54 (03:38)

Definition 1 of supremum and infimum.

If X is an upper-bounded set, then the smallest upper bound of the set X is called the *supremum* of the set X, or its *least upper bound* (or its *exact upper bound*) and is denoted as follows: $\sup X$.

If X is a lower-bounded set, then the largest lower bound of the set X is called the *infimum* of the set X, or its *greatest lower bound* (or its *exact lower bound*) and is denoted as follows: $\inf X$.

Theorems on the existence of the exact boundaries

1A/30:32 (06:43)

Theorem 1 (on the existence of the least upper bound).

A nonempty set bounded from above has the least upper bound.

Proof.

Let X be a given set. Denote by B the set of all its upper bounds. The set B is not empty, since by condition X is bounded from above. Then the estimate $x \leq y$ is true for any $x \in X$, $y \in B$.

Thus, the conditions of the continuity axiom of real numbers are satisfied, if we take the set X as the set A.

By the continuity axiom, we obtain:

$$\exists\, c \in \mathbb{R} \quad \forall\, x \in X, y \in B \quad x \le c \le y.$$

So, the number c is the least upper bound, because:
1) c is the upper bound of the set X, since $\forall\, x \in X\ x \le c$,
2) c is the smallest upper bound, since $\forall\, y \in B\ c \le y$. □
The following theorem can be proved in a similar way.
Theorem 2 (on the existence of the greatest lower bound).
A nonempty set bounded from below has the greatest lower bound.
Corollary.
A nonempty bounded set X has the least upper bound and greatest lower bound.

Exact boundaries of number sets: the second definition 1B/00:00 (13:34)

Definition 2 of supremum and infimum.
The number s is called the *supremum* of a number set X if
1) this number is the upper bound:

$$\forall\, x \in X \quad x \le s;$$

2) this number is the smallest upper bound:

$$\forall\, \varepsilon > 0 \quad \exists\, x \in X \quad x > s - \varepsilon.$$

The number i is called the *infimum* of a number set X if
1) this number is the lower bound:

$$\forall\, x \in X \quad x \ge i;$$

2) this number is the largest lower bound:

$$\forall\, \varepsilon > 0 \quad \exists\, x \in X \quad x < i + \varepsilon.$$

Obviously, definitions 1 and 2 are equivalent.
Example.
Let us prove that b is the least upper bound of the interval (a, b). By the definition of an interval, the point b is the upper bound (since if $x \in (a, b)$, then $a < x < b$). It remains to show that b is the smallest upper bound, i. e. that the following condition holds:

$$\forall\, \varepsilon > 0 \quad \exists\, x \in (a, b) \quad x > b - \varepsilon.$$

Indeed, for such x it is possible, for example, to take the point $b - \frac{\varepsilon}{2}$ (or any point of the interval (a, b) if $b - \frac{\varepsilon}{2} \le a$).

Maximum and minimum elements of a set

1B/13:34 (11:02)

DEFINITION.

Let X be a nonempty upper-bounded set. If the condition $\sup X \in X$ is fulfilled, then the element $\sup X$ is called the *maximum element* of the set X and denoted by $\max X$.

Let X be a nonempty lower-bounded set. If the condition $\inf X \in X$ is fulfilled, then the element $\inf X$ is called the *minimum element* of the set X and denoted by $\min X$.

Not each bounded nonempty set has a maximum or minimum element. For example, the interval (a, b) has neither a minimum nor a maximum element.

A set consisting of a finite number of numbers always has a minimum and maximum element. These elements can be found using the search algorithm for the minimum or maximum element.

THEOREM 1 (ON THE EXISTENCE OF A MAXIMUM ELEMENT IN AN UPPER-BOUNDED INTEGER SET).

If a nonempty set X contains only integers and is bounded from above, then it has a maximum element.

PROOF.

If X is upper-bounded, then it has the least upper bound s: $s = \sup X$. This means that $\forall\, x \in X\ x \le s$; in addition, for $\varepsilon = 1$, there exists an element $x_0 \in X$ such that $x_0 > s - 1$.

Let us show that $x_0 = \max X$. Since $x_0 \in X$, we get: $x_0 \le s$. The inequality $x_0 > s - 1$ can be transformed to the form $x_0 + 1 > s$, therefore all integers starting from $x_0 + 1$ do not belong to X. Thus, the estimate $x \le x_0$ holds for all $x \in X$, which means that x_0 is the upper bound of the set X and $x_0 \ge s$. From the inequalities $x_0 \le s$ and $x_0 \ge s$ it follows that x_0 coincides with the least upper bound s, therefore $x_0 = \max X$. □

The following theorem can be proved in a similar way.

THEOREM 2 (ON THE EXISTENCE OF A MINIMUM ELEMENT IN A LOWER-BOUNDED INTEGER SET).

If a nonempty set X contains only integers and is bounded from below, then it has a minimum element.

Uniqueness of exact boundaries

2A/00:00 (03:23)

THEOREM (ON THE UNIQUENESS OF EXACT BOUNDARIES).

If the set X has the least upper bound or the greatest lower bound, then this bound is uniquely determined.

Proof.

Let us prove this statement by contradiction. Suppose that the set X has two distinct least upper bounds: $a = \sup X$, $b = \sup X$, and $a \neq b$. Since $a \neq b$, we obtain that one of two inequalities holds: $a < b$ or $b < a$. If $a < b$ and $a = \sup X$, then the number b cannot be the least upper bound, and if $b < a$ and $b = \sup X$, then the number a cannot be the least upper bound. The obtained contradiction means that our assumption is false, and there exists the unique least upper bound.

The uniqueness of the greatest lower bound is proved similarly. □

Arithmetic operations on sets

Arithmetic operations on sets: definitions

1B/24:36 (06:09)

Definition.

Let X and Y be sets of real numbers. Then their *sum* $X + Y$ is defined as follows:

$$X + Y \stackrel{\text{def}}{=} \{z \in \mathbb{R} : (\exists\, x \in X, y \in Y \quad z = x + y)\}.$$

Example.

Let us find the sum of the sets $[0, 1]$ and $[2, 3]$ ($[0, 1]$ and $[2, 3]$ are segments).

For $x \in [0, 1]$, we have: $0 \leq x \leq 1$. For $y \in [2, 3]$, we have: $2 \leq x \leq 3$. Then $2 \leq x + y \leq 4$. Therefore, $[0, 1] + [2, 3] = [2, 4]$.

Definition.

Let X be the set of real numbers, $\lambda \in \mathbb{R}$. Then the *product* of the set X by the number λ is defined as follows:

$$\lambda X \stackrel{\text{def}}{=} \{z \in \mathbb{R} : (\exists\, x \in X \quad z = \lambda x)\}.$$

Remark.

Generally speaking, $X + X \neq 2X$. We give an example. Let $X = \{0, 1\}$. Then $X + X = \{0, 1, 2\}$, $2X = \{0, 2\}$. Therefore, $X + X \neq 2X$.

Theorems on the exact boundaries of the sum of sets

1B/30:45 (12:50)

Theorem 1 (on the least upper bound of the sum of sets).

Let X and Y be nonempty upper-bounded sets. Then

$$\sup(X + Y) = \sup X + \sup Y.$$

Proof.

1. Denote $s = \sup X + \sup Y$ and prove that s is an upper bound of the set $X + Y$.

We consider an arbitrary element z of the set $X + Y$: $z = x + y$ for some $x \in X$ and $y \in Y$.

Since $x \leq \sup X$, $y \leq \sup Y$, we obtain: $z = x + y \leq \sup X + \sup Y = s$.

Thus, for an arbitrary element $z \in X + Y$, the estimate $z \leq s$ holds, therefore, s is an upper bound.

2. Let us prove that s is the least upper bound of the set $X + Y$.

Let $\varepsilon > 0$ be an arbitrary positive number.

We will show that $s - \varepsilon$ is not the upper bound of the set $X + Y$, that is, there exists a number $z_0 = x_0 + y_0 \in X + Y$ such that $z_0 > s - \varepsilon$.

By the definition of the least upper bound of the set X, we have:

$$\exists\, x_0 \in X \quad x_0 > \sup X - \frac{\varepsilon}{2}.$$

By the definition of the least upper bound of the set Y, we have:

$$\exists\, y_0 \in Y \quad y_0 > \sup Y - \frac{\varepsilon}{2}.$$

Summing up these inequalities term by term, we obtain the required result:

$$z_0 = x_0 + y_0 > \sup X + \sup Y - \varepsilon = s - \varepsilon. \ \square$$

The following theorem can be proved in a similar way.

Theorem 2 (on the greatest lower bound of the sum of sets).

Let X and Y be nonempty lower-bounded sets. Then

$$\inf(X + Y) = \inf X + \inf Y.$$

Theorems on the exact boundaries of the product of a set by a number

2A/03:23 (04:13)

Theorem 1 (first theorem on the exact boundaries of the product of a set by a number).

Let X be a nonempty upper-bounded set, $\lambda > 0$. Then

$$\sup(\lambda X) = \lambda \sup X.$$

Proof.

1. Let $\lambda x \in \lambda X$.

Since for $x \in X$ we have $x \leq \sup X$, we obtain: $\lambda x \leq \lambda \sup X$.

Therefore, $\lambda \sup X$ is an upper bound of the set λX.

2. Let us choose $\varepsilon > 0$.

By the definition of the least upper bound of the set X, we have

$$\exists\, x' \in X \quad x' > \sup X - \frac{\varepsilon}{\lambda}.$$

Consequently,

$$\lambda x' > \lambda \left(\sup X - \frac{\varepsilon}{\lambda}\right) = \lambda \sup X - \varepsilon.$$

Thus, we found the element $\lambda x' \in \lambda X$ such that the inequality $\lambda x' > \lambda \sup X - \varepsilon$ holds for the selected ε. Therefore, $\lambda \sup X$ is the least upper bound of the set λX. □

The following theorem can be proved in a similar way.

Theorem 2 (second theorem on the exact boundaries of the product of a set by a number).

1. Let X be a nonempty lower-bounded set, $\lambda > 0$. Then

$$\inf(\lambda X) = \lambda \inf X.$$

2. Let X be a nonempty upper-bounded set, $\lambda < 0$. Then

$$\inf(\lambda X) = \lambda \sup X.$$

3. Let X be a nonempty lower-bounded set, $\lambda < 0$. Then

$$\sup(\lambda X) = \lambda \inf X.$$

2. Limit of a sequence

Neighborhood and symmetric neighborhood of a point

Neighborhood and symmetric neighborhood: definition and properties

2A/07:36 (13:32)

Definition.

Let A be a point on a number line: $A \in \mathbb{R}$. The *neighborhood* U_A of the point A is any interval (a, b) containing this point. The *symmetric ε-neighborhood* U_A^ε of the point A is the interval $(A - \varepsilon, A + \varepsilon)$, where $\varepsilon > 0$ is a number called the *radius* of the symmetric neighborhood.

The intersection of any nonempty finite set of neighborhoods of the point A is a neighborhood of the point A. The intersection of any nonempty finite set of symmetric neighborhoods of the point A is a symmetric neighborhood of the point A.

The union of any nonempty (not necessarily finite) set of neighborhoods of A is a neighborhood of A. The union of any nonempty (not necessarily finite) set of symmetric neighborhoods of A is a symmetric neighborhood of A.

Remark.

Any neighborhood (a, b) of the point A contains a symmetric neighborhood:

$$(a, b) \supset (A - \varepsilon, A + \varepsilon), \text{ where } \varepsilon = \min\{|A - a|, |A - b|\}.$$

Supplement. Intersection of neighborhoods

3A/00:00 (01:21)

In describing the properties of neighborhoods of points, we noted that the intersection of any nonempty finite set of neighborhoods of a given point is a neighborhood of this point. Now we show that in the case of an infinite set of neighborhoods, this statement is not true. To do this, it's enough to give an example.

Consider the set of intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$. All such intervals are neighborhoods of the point 0. However, their intersection consists of a single point 0. Indeed, for any point $x \neq 0$, there exists a number $n_0 \in \mathbb{N}$ such that $|x| \geq \frac{1}{n_0}$. So, the point x does not belong to the interval $\left(-\frac{1}{n_0}, \frac{1}{n_0}\right)$, and therefore it does not belong to the intersection of all such intervals for n from 1 to ∞.

Thus, the intersection of all intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$, consists of a single point 0. But a single point is not a neighborhood. So, we have shown that the intersection of an infinite number of neighborhoods of a point will not necessarily be its neighborhood.

Definition of the limit of a sequence

Sequence: definition and examples

2A/21:08 (06:29)

Definition.

The map $f : \mathbb{N} \to X$, where $\mathbb{N}$ is the set of natural numbers, is called the *sequence of elements* (or *terms*) $x_1 = f(1)$, $x_2 = f(2)$, ..., $x_n = f(n)$, ... and denoted by $\{x_n\}$. An element x_n is called the *common term* of the sequence.

A sequence is called a *numerical* one if $X = \mathbb{R}$.

Examples of sequences.

$\left\{\frac{1}{n}\right\} : 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$

$\left\{n^2\right\} : 1, 4, 9, 16, \dots, n^2, \dots$

How to define the limit of a sequence?

2A/27:37 (07:52)

If we consider the sequence $\left\{\frac{1}{n}\right\}$ and go through its elements in ascending order of their indices, then they will come closer and closer to the point 0. It is natural to assume that the number 0 will be the *limit* of the sequence $\left\{\frac{1}{n}\right\}$.

Another example of a sequence whose limit is 0 is the sequence $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\right\}$. This sequence is interesting in that its elements approach the point 0 from different sides.

If we consider the sequence $\left\{n^2\right\}$, then its elements will not approach any finite number, so it is natural to assume that this sequence has no finite limit.

What property of point 0 allows us to consider it as the limit of the sequences $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{(-1)^n}{n}\right\}$? To describe such a property, it is easiest to use the notion of a neighborhood of a point. The point A will be the limit of the sequence $\{x_n\}$ if for any neighborhood U_A of this point all elements of the sequence, except, perhaps, a finite number of its initial elements, will lie in this neighborhood. In other words, it is required that any neighborhood U_A contains an infinite number of elements of the sequence $\{x_n\}$, and outside it there is a finite number of elements.

It is easy to see that only the point 0 satisfies the indicated condition for the sequences $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{(-1)^n}{n}\right\}$.

In this definition, it is important not only that in any neighborhood there is an infinite number of elements of the sequence, but also that only a finite number remains outside the neighborhood. Without the second condition, it would turn out that the sequence $\{(-1)^n\} = \{1, -1, 1, -1, \dots\}$ has two limits: -1 and 1, however, the presence of several limits of one sequence would lead to problems in constructing the theory of limits.

Symmetric neighborhoods can also be used in the definition of the limit; this version of definition is often more convenient to use.

Definition of the limit of a sequence in the language of neighborhoods

2A/35:29 (05:33), 2B/00:00 (01:07)

Definition 1 of the sequence limit (in the language of neighborhoods).

The number $A \in \mathbb{R}$ is called the *limit* of a sequence $\{x_n\}$ if for any neighborhood U_A of the point A there exists a natural number $N \in \mathbb{N}$ such that all elements x_n with numbers greater than N will be contained in the neighborhood U_A. Formally we may write the previous condition as follows:

$$\forall\, U_A \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad x_n \in U_A.$$

Definition of the limit of a sequence in the language of symmetric neighborhoods

2B/01:07 (19:41)

Definition 2 of the sequence limit (in the language of symmetric neighborhoods).

The number $A \in \mathbb{R}$ is called the *limit* of a sequence $\{x_n\}$ if for any ε-neighborhood V_A^ε of the point A with radius $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that all elements x_n with numbers greater than N will be contained in the neighborhood V_A^ε:

$$\forall\, V_A^\varepsilon \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad x_n \in V_A^\varepsilon.$$

Theorem (on the equivalence of two definitions of the limit of a sequence).

Definitions 1 and 2 of the limit of a sequence are equivalent.

Proof.

Obviously, if A is the limit of a sequence in the sense of definition 1, then A is also the limit in the sense of definition 2, since any symmetric neighborhood is a neighborhood.

Let us prove the opposite. Let A be the limit of $\{x_n\}$ in the sense of definition 2. We show that A is the limit of $\{x_n\}$ in the sense of definition 1.

Let U_A be an arbitrary neighborhood of A. We can choose the symmetric neighborhood V_A^ε containing in U_A: $V_A^\varepsilon \subset U_A$.

According to definition 2, for the neighborhood of V_A^ε there exists $N \in \mathbb{N}$ such that $x_n \in V_A^\varepsilon$ for all $n > N$. But $V_A^\varepsilon \subset U_A$, so $x_n \in U_A$ for all $n > N$. Thus, since the choice of the neighborhood U_A is arbitrary, the point A is also the limit in the sense of definition 1. □

Definition 2 can be reformulated as follows.

Definition 3 of the sequence limit (in the language ε–N).

The number $A \in \mathbb{R}$ is called the *limit* of a sequence $\{x_n\}$ if for any number $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that for any $n > N$ the following inequality holds: $A-\varepsilon < x_n < A+\varepsilon$, or, equivalently, $|x_n - A| < \varepsilon$:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n - A| < \varepsilon.$$

Such a definition is called a definition in the language ε–N.

Limit notations: $\lim_{n\to\infty} x_n = A$, $\lim\limits_{n\to\infty} x_n = A$ or $x_n \to A$ as $n \to \infty$ ("x_n approaches A as n approaches infinity").

A sequence with a limit $A \in \mathbb{R}$ is called a *convergent* one (to the limit A).

Examples of finding the limit of the sequence using the definition

2B/20:48 (11:47)

1. $x_n = \frac{1}{n}$.

We will show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Let us select an arbitrary $\varepsilon > 0$ and find N such that for all $n > N$ the estimate $\left|\frac{1}{n} - 0\right| < \varepsilon$ holds, that is, $\frac{1}{n} < \varepsilon$.

The inequality $\frac{1}{n} < \varepsilon$ is equivalent to the inequality $n > \frac{1}{\varepsilon}$.

Let $N = \left[\frac{1}{\varepsilon}\right]$, where $[x]$ is the integer part of the number x.

Taking into account that n is natural, we get that for all $n > \left[\frac{1}{\varepsilon}\right]$ the following estimate holds: $n \geq \left[\frac{1}{\varepsilon}\right] + 1$.

This estimate can be continued if we use the property of the integer part of a real number ($[x] \leq x < [x] + 1$):

$$n \geq \left[\frac{1}{\varepsilon}\right] + 1 > \frac{1}{\varepsilon}.$$

We have obtained that for all natural numbers $n > N$, where $N = \left[\frac{1}{\varepsilon}\right]$, the estimate $n > \frac{1}{\varepsilon}$ holds.

Therefore,

$$\forall\, \varepsilon > 0 \quad \exists\, N = \left[\frac{1}{\varepsilon}\right] \quad \forall\, n > N \quad \frac{1}{n} < \varepsilon.$$

This means that $\lim_{n\to\infty} \frac{1}{n} = 0$.

2. $x_n = \frac{(-1)^n}{n}$.

In this case, the limit will also be 0.

The proof is completely similar to the proof the sequence from the example 1, since the inequality $\left|\frac{(-1)^n}{n} - 0\right| < \varepsilon$ may be written in the same form as in the example 1: $\frac{1}{n} < \varepsilon$.

Example of a sequence without limit

2B/32:35 (08:29)

We can say that the number A is the limit of a sequence $\{x_n\}$ if *any* neighborhood of the number A contains *all* elements of the sequence except, perhaps, some *finite* amount of its starting elements.

In order to show that the number A is *not* the limit of a sequence $\{x_n\}$, it suffices to select *some* neighborhood of the number A, outside which there is an *infinite* number of elements of the sequence $\{x_n\}$.

Formally, the statement that the number A is *not* the limit of a sequence $\{x_n\}$ can be written by applying the negation operation to one of definitions of the limit, for example (for definition 3):

$$\overline{\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n - A| < \varepsilon,}$$

$$\exists\, \varepsilon > 0 \quad \forall\, N \in \mathbb{N} \quad \exists\, n > N \quad |x_n - A| \geq \varepsilon.$$

Let $\varphi_n = (-1)^n : -1, 1, -1, 1, \ldots$

Let us prove that this sequence has no limit. To do this, we use the above negation of the statement that the number A is the limit of the sequence $\{\varphi_n\}$.

Let $A = 1$. Choose $\varepsilon = \frac{1}{2}$. Then for *any* natural number N there exists an *odd* number $n > N$, for which $\varphi_n = -1$ and, therefore, this element of the sequence is *not* contained in the ε-neighborhood of the point 1. Therefore, the number $A = 1$ is not the limit of the sequence $\{\varphi_n\}$.

Let $A = -1$. Then, choosing $\varepsilon = \frac{1}{2}$, we obtain that for *any* natural number N there exists an *even* number $n > N$, for which $\varphi_n = 1$ and, therefore, this element of the sequence is *not* contained in the ε-neighborhood of the point -1. Therefore, the number $A = -1$ is also not the limit of the sequence $\{\varphi_n\}$.

Let A be a number other than 1 and -1. Let $\varepsilon = \min\{|A-1|, |A+1|\}$. Then for the ε-neighborhood of the point A, *all* elements of the sequence $\{\varphi_n\}$ will be out of this neighborhood. Therefore, all such numbers also cannot be the limit of the sequence $\{\varphi_n\}$.

The simplest properties of the limit of a sequence

The uniqueness theorem for the limit of a convergent sequence

3A/01:21 (13:39)

Theorem (on the uniqueness of the limit of a convergent sequence).

A convergent sequence cannot have two different limits.

Proof.

We prove the theorem by contradiction. Suppose that A and B are different limits of the given sequence $\{x_n\}$:

$$\lim_{n\to\infty} x_n = A, \quad \lim_{n\to\infty} x_n = B, \quad A \neq B.$$

Then the points A and B have disjoint neighborhoods U_A and U_B: $U_A \cap U_B = \varnothing$.

By the definition of the limit of a sequence, we have for the neighborhood U_A:

$$\exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \quad x_n \in U_A. \tag{1}$$

Similarly, for the neighborhood U_B, we have:

$$\exists\, N_2 \in \mathbb{N} \quad \forall\, n > N_2 \quad x_n \in U_B. \tag{2}$$

Let $N = \max\{N_1, N_2\}$. Then, by virtue of relations (1) and (2), $x_n \in U_A \cap U_B$ for $n > N$.

But the neighborhoods of U_A and U_B do not intersect. That means that for $n > N$ $x_n \in \varnothing$, which is impossible. The obtained contradiction means that our assumption was incorrect, and the sequence $\{x_n\}$ cannot have two different limits. □

A theorem on the boundedness of a convergent sequence

3A/15:00 (12:09)

Definition.

A sequence $\{x_n\}$ is called *bounded* if there exists $M > 0$ such that for all $n \in \mathbb{N}$ the estimate $|x_n| \le M$ holds:

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N} \quad |x_n| \le M.$$

Theorem (on the boundedness of a convergent sequence).
A convergent sequence is bounded.

Proof.
Let $A = \lim_{n\to\infty} x_n$. Then for $\varepsilon = 1$ we have:

$$\exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n - A| < 1.$$

Applying the triangle inequality for the absolute value of sum, we get:

$$|x_n| = |(x_n - A) + A| \le |x_n - A| + |A| < 1 + |A|.$$

Thus, for any $n > N$ we have $|x_n| < M_1$, where $M_1 = 1 + |A|$.

In addition, the set $\{|x_1|, |x_2|, \ldots, |x_N|\}$ is finite and therefore has the maximum element with the value M_2. So, the estimate $|x_n| \le M_2$ holds for all $n \le N$.

Taking $M = \max\{M_1, M_2\}$, we get:

$$\forall\, n \in \mathbb{N} \quad |x_n| \le M. \ \square$$

Remark.
The converse assertion is not true: the bounded sequence is not necessarily convergent. As an example, we can use the previously considered sequence $\{\varphi_n\} = \{(-1)^n\}$. Obviously, it is bounded, since $\forall\, n \in \mathbb{N}\ |\varphi_n| \le 1$, but we have proved that it has no limit.

3. Properties of the limit of a sequence

Infinitesimal sequences: definition and properties

3A/27:09 (18:15)

Definition.

The sequence $\{x_n\}$ is called *infinitely small* sequence, or *infinitesimal*, if $\lim_{n\to\infty} x_n = 0$.

Theorem (on properties of infinitesimals).

1. If $\{x_n\}$ and $\{y_n\}$ are infinitesimals, then the sequence $\{x_n + y_n\}$ is infinitesimal.

2. If $\{x_n\}$ is an infinitesimal and the sequence $\{y_n\}$ is bounded, then the sequence $\{x_n y_n\}$ is infinitesimal.

Proof.

1. Let $\varepsilon > 0$ be an arbitrary number. Then

$$\exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \quad |x_n| < \frac{\varepsilon}{2},$$
$$\exists\, N_2 \in \mathbb{N} \quad \forall\, n > N_2 \quad |y_n| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then for any $n > N$

$$|x_n + y_n| \le |x_n| + |y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So we obtain that

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n + y_n| < \varepsilon.$$

This means that $\lim_{n\to\infty}(x_n + y_n) = 0$.

2. Since $\{y_n\}$ is bounded, we have:

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N} \quad |y_n| \le M.$$

Since $\lim_{n\to\infty} x_n = 0$, then

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n| < \frac{\varepsilon}{M}.$$

Therefore,

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n y_n| = |x_n| \cdot |y_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon. \ \square$$

Example.

$\lim_{n\to\infty} \frac{\sin n}{n} = 0$, since $\{\sin n\}$ is the bounded sequence, and $\left\{\frac{1}{n}\right\}$ is infinitesimal.

A criterion for convergence in terms of an infinitesimal

3B/00:00 (13:57)

THEOREM (CRITERION FOR THE EXISTENCE OF A FINITE LIMIT IN TERMS OF INFINITESIMALS).

The sequence $\{x_n\}$ has the limit $A \in \mathbb{R}$ if and only if the sequence $\{x_n - A\}$ is infinitesimal:

$$\bigl(\lim_{n\to\infty} x_n = A\bigr) \Leftrightarrow \bigl(\{x_n - A\} \text{ is infinitesimal}\bigr).$$

REMARK.

In this theorem, the assertion in the direction from left to right ($\Rightarrow$) corresponds to the *necessary condition* for the existence of a limit ("if the limit of the sequence $\{x_n\}$ exists, then it is *necessary* that the sequence $\{x_n - A\}$ is infinitesimal"), and the assertion in the direction from right to left ($\Leftarrow$) corresponds to the *sufficient condition* for the existence of a limit ("if the sequence $\{x_n - A\}$ is infinitesimal, then this is suffice that the limit of the sequence $\{x_n\}$ exists").

PROOF.

Both the first and second conditions of the theorem mean the same thing, namely:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n - A| < \varepsilon. \ \square$$

Arithmetic properties of the limit of a sequence

Formulation of the theorem on arithmetic properties of the limit and proof for the limit of the sum

3B/13:57 (11:16)

THEOREM (ON ARITHMETIC PROPERTIES OF THE LIMIT OF A SEQUENCE).

Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences, assume that $\{x_n\}$ converges to A, $\{y_n\}$ converges to B. Then the following three properties hold.

1. The sum $\{x_n + y_n\}$ of these sequences is a convergent sequence, and the limit of the sum is $A + B$:

$$\lim_{n\to\infty} (x_n + y_n) = A + B.$$

2. The product $\{x_n y_n\}$ of these sequences is a convergent sequence, and the limit of the product is AB:

$$\lim_{n\to\infty} x_n y_n = AB.$$

3. Under the additional conditions $y_n \neq 0$ for any $n \in \mathbb{N}$ and $B \neq 0$, the quotient $\left\{\frac{x_n}{y_n}\right\}$ of these sequences is a converging sequence, and the limit of the quotient is $\frac{A}{B}$:

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{A}{B}.$$

Brief verbal formulation of the theorem.

The limit of the sum is equal to the sum of the limits, the limit of the product is equal to the product of the limits, the limit of the quotient (under some additional assumptions) equals the quotient of the limits.

Proof.

1. Consider the sequence $\{(x_n + y_n) - (A + B)\}$. We have for it:

$$(x_n + y_n) - (A + B) = (x_n - A) + (y_n - B). \tag{1}$$

Since $\lim_{n\to\infty} x_n = A$, by the necessary condition of the previous criterion, we obtain that the sequence $\{x_n - A\}$ is infinitesimal.

Similarly, since $\lim_{n\to\infty} y_n = B$, by the necessary condition of the previous criterion, we obtain that the sequence $\{y_n - B\}$ is also infinitesimal.

Then, by the property 1 of infinitesimals, we get that the sequence $\{(x_n - A) + (y_n - B)\}$ is infinitesimal as the sum of infinitesimals.

It follows from (1) that the sequence $\{(x_n + y_n) - (A + B)\}$ is also infinitesimal, and, by a sufficient condition of the previous criterion, we get that $\lim_{n\to\infty}(x_n + y_n) = A + B$.

Proof for the limit of the product

3B/25:13 (06:00)

2. Consider the sequence $\{x_n y_n - AB\}$. We have for it:

$$x_n y_n - AB = x_n y_n - Ay_n + Ay_n - AB = (x_n - A)y_n + A(y_n - B). \tag{2}$$

The sequences $\{x_n - A\}$ and $\{y_n - B\}$ are infinitesimal, and the sequence $\{y_n\}$ is bounded since it is convergent.

Then, by the property 2 of infinitesimals, we obtain that the sequence $\{(x_n - A)y_n\}$ is infinitesimal as the product of an infinitesimal $\{x_n - A\}$ and a bounded sequence $\{y_n\}$. The sequence $\{A(y_n - B)\}$ is also infinitesimal as the product of the infinitesimal $\{y_n - B\}$ by the bounded sequence $\{A\}$ with constant elements A. The sequence $\{(x_n - A)y_n + A(y_n - B)\}$ is infinitesimal by the property 1 of infinitesimals.

It follows from (2) that the sequence $\{x_n y_n - AB\}$ is also infinitesimal, and, by a sufficient condition of the previous criterion, we obtain $\lim_{n\to\infty}(x_n y_n) = AB$.

Proof for the limit of the quotient

3B/31:13 (11:13)

3. Consider the sequence $\left\{\frac{x_n}{y_n} - \frac{A}{B}\right\}$. We have for it:

$$\frac{x_n}{y_n} - \frac{A}{B} = \frac{x_n B - A y_n}{y_n B} = \frac{1}{B y_n}(x_n B - AB + AB - A y_n) =$$
$$= \frac{1}{B} \cdot \frac{1}{y_n}\big((x_n - A)B - A(y_n - B)\big). \tag{3}$$

Reasoning in the same way as in the proof of the property 2, we see that the sequence $\{(x_n - A)B - A(y_n - B)\}$ is infinitesimal.

Since by condition $B \neq 0$, a constant sequence with elements $\frac{1}{B}$ is bounded.

So, it suffices to prove, taking into account (3), that the sequence $\left\{\frac{1}{y_n}\right\}$ is also bounded.

Choose $\varepsilon = \frac{|B|}{2}$. This is a positive number, since $B \neq 0$. Taking into account that $\lim_{n\to\infty} y_n = B$, we get for the choosen ε:

$$\exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |y_n - B| < \frac{|B|}{2}.$$

Let us represent y_n in the form $B - (B - y_n)$ and use the absolute value property 4 (the lower bound for the absolute value of difference) taking into account that $|y_n - B| < \frac{|B|}{2}$:

$$|y_n| = |B - (B - y_n)| \geq \big||B| - |B - y_n|\big| > \left||B| - \frac{|B|}{2}\right| = \frac{|B|}{2}.$$

Thus, for all $n > N$ we obtain the estimate $|y_n| > \frac{|B|}{2}$, which can be rewritten as $\left|\frac{1}{y_n}\right| < \frac{2}{|B|}$. Therefore, the elements of the sequence $\left\{\frac{1}{y_n}\right\}$ for $n > N$ are bounded by the same value $\frac{2}{|B|}$. The starting part of this sequence $\left\{\frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_N}\right\}$ is also bounded because it contains a finite number of elements. Therefore, the sequence $\left\{\frac{1}{y_n}\right\}$ is bounded. □

Passing to the limit in inequalities

The first theorem

4A/00:00 (13:32)

THEOREM 1 (FIRST THEOREM ON PASSING TO THE LIMIT IN INEQUALITIES FOR SEQUENCES).

Let $\{x_n\}$ and $\{y_n\}$ be the sequences such that the following property holds:

$$\exists\, N_0 \in \mathbb{N} \quad \forall\, n > N_0 \quad x_n \leq y_n. \tag{4}$$

Let $\lim_{n\to\infty} x_n = A$, $\lim_{n\to\infty} y_n = B$. Then $A \leq B$.

Proof.

We prove this theorem by contradiction: assume $A > B$. Then the points A and B have disjoint neighborhoods U_A and U_B respectively: $U_A \cap U_B = \varnothing$. Moreover, the neighborhood U_A is located on the numerical line to the *right* of the neighborhood U_B.

By definition 1 of the limit of a sequence, we have for selected neighborhoods:

$$\exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \quad x_n \in U_A,$$

$$\exists\, N_2 \in \mathbb{N} \quad \forall\, n > N_2 \quad y_n \in U_B.$$

Let $N = \max\{N_0, N_1, N_2\}$. Then for all $n > N$, taking into account (4), the following three conditions must be satisfied:

$$x_n \in U_A, \quad y_n \in U_B, \quad x_n \le y_n.$$

But from the first two conditions, in view of choice of neighborhoods U_A and U_B, the inequality $x_n > y_n$ follows, and this contradicts the third condition. The obtained contradiction means that our assumption is false and $A \le B$. □

Remark.

If the strict inequality $x_n < y_n$ holds for the elements of the given sequences, then this does not imply that $A < B$. In this case, as before, only the non-strict inequality $A \le B$ is guaranteed for the limit values.

Examples.

1. For elements of the sequences $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$, the strict inequality holds:

$$\forall\, n \in \mathbb{N} \quad -\frac{1}{n} < \frac{1}{n}.$$

However, their limits are equal: $\lim_{n\to\infty}(-\frac{1}{n}) = \lim_{n\to\infty}\frac{1}{n} = 0$.

2. For elements of the sequences $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{1}{n^2}\right\}$, the strict inequality holds starting with $n = 2$:

$$\forall\, n \ge 2 \quad \frac{1}{n^2} < \frac{1}{n}.$$

However, their limits are also equal: $\lim_{n\to\infty}\frac{1}{n^2} = \lim_{n\to\infty}\frac{1}{n} = 0$.

The second theorem

4A/13:32 (12:50)

Theorem 2 (second theorem on passing to the limit in inequalities for sequences).

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences, for elements of which the following condition is satisfied:

$$\exists\, N_0 \in \mathbb{N} \quad \forall\, n > N_0 \quad x_n \le y_n \le z_n. \tag{5}$$

Let $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = A$. Then the sequence $\{y_n\}$ is also convergent and $\lim_{n\to\infty} y_n = A$.

Proof.

Let us choose an arbitrary value $\varepsilon > 0$ and write the limit definition in the language ε–N for the sequences $\{x_n\}$ and $\{z_n\}$:

$$\exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \quad |x_n - A| < \varepsilon,$$

$$\exists\, N_2 \in \mathbb{N} \quad \forall\, n > N_2 \quad |z_n - A| < \varepsilon.$$

Note that, by the absolute value property 1, the inequality $|x_n - A| < \varepsilon$ is equivalent to the double inequality $A - \varepsilon < x_n < A + \varepsilon$, and the inequality $|z_n - A| < \varepsilon$ is equivalent to the double inequality $A - \varepsilon < z_n < A + \varepsilon$.

Let $N = \max\{N_0, N_1, N_2\}$. Then, for all $n > N$, taking into account (5) and the double inequalities, the following chain of inequalities is fulfilled:

$$A - \varepsilon < x_n \le y_n \le z_n < A + \varepsilon.$$

Thus, for $n > N$, we get $A - \varepsilon < y_n < A + \varepsilon$, or, equivalently, $|y_n - A| < \varepsilon$.

This means that the sequence $\{y_n\}$ converges to A. $\square$

4. Infinite limits

Neighborhoods of the points at infinity 4A/26:22 (04:59)

We expand the number line by adding to it the *points at infinity* $+\infty$, $-\infty$, and ∞.

Definition.

The *neighborhood of the point* $+\infty$ is any open ray of the form $(E, +\infty)$, where $E \in \mathbb{R}$: $(E, +\infty) \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x > E\}$.

The *neighborhood of the point* $-\infty$ is any open ray of the form $(-\infty, E)$, where $E \in \mathbb{R}$: $(-\infty, E) \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x < E\}$.

The neighborhoods of the points at infinity are denoted as $U_{+\infty}$, $U_{-\infty}$.

The intersection of any two neighborhoods of the point $+\infty$ is a neighborhood of this point. The intersection of any two neighborhoods of the point $-\infty$ is a neighborhood of this point.

The *neighborhood of the point* ∞ is any set of the form $(-\infty, E_1) \cup (E_2, +\infty)$, where $E_1, E_2 \in \mathbb{R}$, $E_1 < E_2$. Notation: U_∞.

A *symmetric neighborhood of the point* ∞ is any set of the form $(-\infty, -R) \cup (R, +\infty)$, where $R \in \mathbb{R}$, $R > 0$. This set can also be written as $\{x \in \mathbb{R} : |x| > R\}$. Notation: U_∞^R.

Infinitely large sequences

Definitions and examples 4A/31:21 (04:57), 4B/00:00 (11:24)

Definition 1 of the infinite limit of a sequence (in the language of neighborhoods).

It is said that *the sequence* $\{x_n\}$ *approaches* $+\infty$ ($\lim_{n\to\infty} x_n = +\infty$) if

$$\forall\, U_{+\infty} \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad x_n \in U_{+\infty}.$$

It is said that *the sequence* $\{x_n\}$ *approaches* $-\infty$ ($\lim_{n\to\infty} x_n = -\infty$) if

$$\forall\, U_{-\infty} \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad x_n \in U_{-\infty}.$$

It is said that *the sequence* $\{x_n\}$ *approaches* ∞ ($\lim_{n\to\infty} x_n = \infty$) if

$$\forall\, U_{\infty} \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad x_n \in U_{\infty}.$$

DEFINITION 2 OF THE INFINITE LIMIT OF A SEQUENCE (IN THE LANGUAGE E–N).

It is said that *the sequence* $\{x_n\}$ *approaches* $+\infty$ if

$$\forall E \quad \exists N \in \mathbb{N} \quad \forall n > N \quad x_n > E.$$

It is said that *the sequence* $\{x_n\}$ *approaches* $-\infty$ if

$$\forall E \quad \exists N \in \mathbb{N} \quad \forall n > N \quad x_n < E.$$

It is said that *the sequence* $\{x_n\}$ *approaches* ∞ if

$$\forall E > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |x_n| > E.$$

It is easy to prove that definitions 1 and 2 are equivalent.

Sequences that have infinite limits are called *infinitely large*.

EXAMPLES.

$$\lim_{n\to\infty} n = +\infty, \qquad \lim_{n\to\infty} (-n) = -\infty, \qquad \lim_{n\to\infty} (-1)^n n = \infty.$$

REMARKS.

1. If a sequence approaches $+\infty$ or $-\infty$, then it approaches ∞ as well; the converse assertion is, generally speaking, not true (see the examples above).

2. The expression "a convergent sequence" is used only if the sequence has a finite limit.

The uniqueness theorem for the limit of an infinitely large sequence

4B/11:24 (08:34)

THEOREM (ON THE UNIQUENESS OF THE LIMIT OF AN INFINITELY LARGE SEQUENCE).

1. A sequence cannot simultaneously have an infinite and a finite limit.
2. A sequence cannot simultaneously have limits equal to $+\infty$ and $-\infty$.

PROOF.

The theorem can be proved in the same way as the uniqueness theorem for the limit of a convergent sequence, if we take into account that the finite and infinite points (and the points $+\infty$ and $-\infty$) have disjoint neighborhoods. □

Arithmetic properties of infinitely large sequences

4B/19:58 (15:10)

We previously proved that if sequences $\{x_n\}$ and $\{y_n\}$ converge to the numbers A and B, respectively, then the sequence $\{x_n + y_n\}$ converges to $A+B$. Briefly, this statement is formulated as follows: the limit of the sum is equal to the sum of the limits. However, if the terms have infinite limits, then in some situations it is impossible to find the limit of the sum (and even prove its existence) if we know only the limits of the terms. Such situations are called *indeterminate forms*. For example, indeterminate form occurs when $\{x_n\}$ approaches $+\infty$, and $\{y_n\}$ approaches $-\infty$. In this case, the sum $\{x_n + y_n\}$ can have both a finite and an infinite limit (and it can also have no limit at all): the result depends on the initial sequences. Such indeterminate form can be denoted as $\bigl(+\infty + (-\infty)\bigr)$.

Table 1 shows the limit values of the sum of the sequences $\{x_n + y_n\}$ for various limits of the sequences $\{x_n\}$ (which are shown in the first column of the table) and $\{y_n\}$ (which are shown in the first row of the table). Indeterminate forms are marked with "?".

Table 1

The limits of the sum of sequences

$(+)$	B	$+\infty$	$-\infty$	∞
A	$A+B$	$+\infty$	$-\infty$	∞
$+\infty$	$+\infty$	$+\infty$	?	?
$-\infty$	$-\infty$	?	$-\infty$	?
∞	∞	?	?	?

Examples.

Let us give examples that correspond to the indeterminate form $\bigl(+\infty + (-\infty)\bigr)$. These examples show that in this situation, the limit of the sum of sequences cannot be determined by the limits of each term.

1. $x_n = n, \quad \lim\limits_{n\to\infty} x_n = +\infty; \qquad y_n = -n, \quad \lim\limits_{n\to\infty} y_n = -\infty;$

$$x_n + y_n = 0, \quad \lim_{n\to\infty}(x_n + y_n) = 0.$$

2. $x_n = n^2, \quad \lim\limits_{n\to\infty} x_n = +\infty; \qquad y_n = -n, \quad \lim\limits_{n\to\infty} y_n = -\infty;$

$$x_n + y_n = n^2 - n = n(n-1), \quad \lim_{n\to\infty}(x_n + y_n) = +\infty.$$

If the value of the cell of table is not equal to "?", then this means that the limit of the sum of sequences is determined by limit values of the terms. Let us formulate and prove a theorem related to one of such cases.

Theorem (on the arithmetic property of infinitely large sequences).

Let $\lim_{n\to\infty} x_n = A \in \mathbb{R}$, $\lim_{n\to\infty} y_n = +\infty$.

Then $\lim_{n\to\infty}(x_n + y_n) = +\infty$.

Proof.

We need to prove that

$$\forall E \quad \exists N \in \mathbb{N} \quad \forall n > N \quad x_n + y_n > E. \tag{1}$$

Let us select an arbitrary E and show how to choose the number N so that for all $n > N$ the required estimate is satisfied.

By condition, $\lim_{n\to\infty} x_n = A$. It means that

$$\forall\, \varepsilon > 0 \quad \exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \quad |x_n - A| < \varepsilon.$$

Let $\varepsilon = 1$. Then

$$\exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \quad |x_n - A| < 1, \text{ or } A - 1 < x_n < A + 1. \qquad (2)$$

By condition, $\lim_{n\to\infty} y_n = +\infty$. It means that

$$\forall\, E' \quad \exists\, N_2 \in \mathbb{N} \quad \forall\, n > N_2 \quad y_n > E'.$$

Let $E' = E - A + 1$ (recall that we previously selected some value of E). Then

$$\exists\, N_2 \in \mathbb{N} \quad \forall\, n > N_2 \quad y_n > E - A + 1. \qquad (3)$$

Let $N = \max\{N_1, N_2\}$. Then, by virtue of (2) and (3), two estimates are valid for all $n > N$:

$$x_n > A - 1,$$
$$y_n > E - A + 1.$$

Summing up these inequalities term by term, we obtain that for all $n > N$ the estimate (1) holds:

$$x_n + y_n > E. \ \square$$

We also give a table (see Table 2) with the values of the limit of the product $\{x_n y_n\}$ when the limits of the sequences $\{x_n\}$ (first column) and $\{y_n\}$ (first row) are known. The properties of the limits of products indicated in the table can be proved in the same way as the properties of the limits of sums considered above. As in Table 1, indeterminate forms are marked with the symbol "?". In this case, the indeterminate form arises when a sequence approaching zero is multiplied by a sequence having an infinite limit $\big((0\cdot(+\infty)\big)$, $\big(0\cdot(-\infty)\big)$, $(0\cdot\infty)\big)$.

Table 2

The limits of the product of sequences

$(\cdot)$	$B = 0$	$B > 0$	$B < 0$	$+\infty$	$-\infty$	∞
$A = 0$	0	0	0	?	?	?
$A > 0$	0	AB	AB	$+\infty$	$-\infty$	∞
$A < 0$	0	AB	AB	$-\infty$	$+\infty$	∞
$+\infty$	?	$+\infty$	$-\infty$	$+\infty$	$-\infty$	∞
$-\infty$	?	$-\infty$	$+\infty$	$-\infty$	$+\infty$	∞
∞	?	∞	∞	∞	∞	∞

Similar situations arise when finding the limit of quotient. In this case, examples of indeterminate forms are $\left(\frac{0}{0}\right)$ and $\left(\frac{\infty}{\infty}\right)$.

5. Monotone sequences

Bounded and monotone sequences: definitions

5A/00:00 (08:57)

Definition.

The sequence $\{x_n\}$ is called *bounded from above*, or *upper-bounded* if

$$\exists\, M \in \mathbb{R} \quad \forall\, n \in \mathbb{N} \quad x_n \le M.$$

The sequence $\{x_n\}$ is called *bounded from below*, or *lower-bounded* if

$$\exists\, M \in \mathbb{R} \quad \forall\, n \in \mathbb{N} \quad x_n \ge M.$$

The sequence $\{x_n\}$ is called *bounded* if

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N} \quad |x_n| \le M.$$

Definition.

The sequence $\{x_n\}$ is called *increasing* if

$$\forall\, n \in \mathbb{N} \quad x_{n+1} > x_n.$$

The sequence $\{x_n\}$ is called *non-decreasing* if

$$\forall\, n \in \mathbb{N} \quad x_{n+1} \ge x_n.$$

The sequence $\{x_n\}$ is called *decreasing* if

$$\forall\, n \in \mathbb{N} \quad x_{n+1} < x_n.$$

The sequence $\{x_n\}$ is called *non-increasing* if

$$\forall\, n \in \mathbb{N} \quad x_{n+1} \le x_n.$$

Non-increasing and non-decreasing sequences are called *monotone* ones, and increasing and decreasing sequences are called *strictly monotone* ones.

Convergence of monotone sequences

A test of convergence for monotone bounded sequences

5A/08:57 (15:50)

Theorem (a test of convergence for monotone bounded sequences).

1. If a sequence is non-decreasing and bounded from above, then it converges.

2. If the sequence is non-increasing and bounded from below, then it converges.

Proof.

Let us prove the statement 1. Let the sequence $\{x_n\}$ be non-decreasing and bounded from above. Boundedness from above means that

$$\exists\, M \quad \forall\, n \in \mathbb{N} \quad x_n \le M.$$

Since the set X of values of the numerical sequence $\{x_n\}$ is bounded from above, it has the least upper bound. Denote $A = \sup X = \sup\{x_1, x_2, \ldots, x_n, \ldots\}$ and show that $\lim_{n\to\infty} x_n = A \in \mathbb{R}$. So, we need to prove that

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n - A| < \varepsilon.$$

Let us select an arbitrary $\varepsilon > 0$ and show how to choose the required value N for it. From the definition of the least upper bound, we obtain two following conditions:

$$\forall\, n \in \mathbb{N} \quad x_n \le A, \tag{1}$$

$$\exists\, N \in \mathbb{N} \quad x_N > A - \varepsilon. \tag{2}$$

From the definition of non-decreasing sequence we get that

$$\forall\, n \in \mathbb{N} \quad x_{n+1} \ge x_n. \tag{3}$$

Then the following chain of inequalities holds for all $n > N$:

$$A - \varepsilon \overset{(2)}{<} x_N \overset{(3)}{\le} x_n \overset{(1)}{\le} A < A + \varepsilon.$$

If we discard the intermediate terms of the constructed chain of inequalities, then we get:

$$A - \varepsilon < x_n < A + \varepsilon, \text{ or, which is the same, } |x_n - A| < \varepsilon.$$

The statement 1 is proved.

The statement 2 may be proved similarly using the definition of the greatest lower bound. □

Convergence criterion for monotone sequences

5A/24:47 (04:52)

Theorem (convergence criterion for monotone sequences).

A monotone sequence is convergent if and only it is bounded:

if $\{x_n\}$ is monotone, then
$\big(\{x_n\}$ is convergent$\big) \Leftrightarrow \big(\{x_n\}$ is bounded$\big)$.

PROOF.

The necessity ($\Rightarrow$) is satisfied due to the corresponding property of convergent sequences (in this case, the monotonicity property is not required).

The sufficiency ($\Leftarrow$) is satisfied by the theorem just proved: if a sequence is monotonic and bounded, then it converges (recall that boundedness means simultaneous boundedness both from above and from below). □

REMARK.

The assertion of the theorem remains valid if the given sequence is monotonic starting from some number n_0.

Examples of application of the convergence theorem for monotone sequences

Example 1: $x_n = 1/q^n$ 5A/29:39 (08:43), 5B/00:00 (02:36)

THEOREM 1 (ON THE CONVERGENCE OF THE SEQUENCE $\{1/q^n\}$).

If $q > 1$, then $\lim\limits_{n\to\infty} \dfrac{1}{q^n} = 0$.

PROOF.

Denote $x_n = \frac{1}{q^n}$ and consider the ratio $\frac{x_{n+1}}{x_n}$:

$$\frac{x_{n+1}}{x_n} = \frac{q^n}{q^{n+1}} = \frac{1}{q} < 1, \text{ hence, } \forall\, n \in \mathbb{N} \quad x_{n+1} < x_n.$$

Our sequence $\{x_n\}$ is decreasing and bounded from below by 0. Therefore, by the theorem proved above, it has a finite limit:

$$\lim_{n\to\infty} \frac{1}{q^n} = A \in \mathbb{R}.$$

Let us show that $A = 0$.

It is clear that if the limit of the sequence $\{x_n\}$ is A, then the limit of the sequence $\{x_{n+1}\}$ is also equal to A (the sequence $\{x_{n+1}\}$ is obtained from the sequence $\{x_n\}$ by removing the element x_1). Hence,

$$A = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{q^{n+1}} = \frac{1}{q} \lim_{n\to\infty} \frac{1}{q^n} = \frac{1}{q} \cdot A.$$

Transforming the equality $A = \frac{A}{q}$ to the form $A\left(1 - \frac{1}{n}\right) = 0$ and taking into account that $q > 1$, we get that $A = 0$. □

Example 2: $x_n = n/q^n$ 5B/02:36 (11:54)

THEOREM 2 (ON THE CONVERGENCE OF THE SEQUENCE $\{n/q^n\}$).

If $q > 1$, then $\lim_{n\to\infty} \frac{n}{q^n} = 0$.

PROOF.

Denote $x_n = \frac{n}{q^n}$ and consider the ratio $\frac{x_{n+1}}{x_n}$:

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)q^n}{q^{n+1}n} = \frac{1}{q} \cdot \frac{n+1}{n}.$$

Notice, that

$$\lim_{n\to\infty} \frac{1}{q} \cdot \frac{n+1}{n} = \frac{1}{q} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{q} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{1}{q}.$$

Since $\frac{1}{q} < 1$, we can choose a neighborhood U of the point $\frac{1}{q}$ such that U is located to the left of point 1 (it is enough, for example, to take a symmetric neighborhood of a point $\frac{1}{q}$ of radius $1 - \frac{1}{q}$). Then, by definition of the limit,

$$\exists\, N \in \mathbb{N} \quad \forall\, n > N \quad \frac{1}{q} \cdot \frac{n+1}{n} \in U \text{ and therefore } \frac{1}{q} \cdot \frac{n+1}{n} < 1.$$

We have obtained that starting from some number n_0, the inequality $\frac{x_{n+1}}{x_n} < 1$ holds, that is, $x_{n+1} < x_n$.

Thus, starting from some index, our sequence $\{x_n\}$ is decreasing and, in addition, is bounded from below by 0.

Therefore, by the proved above test of convergence for monotone bounded sequences, it has a finite limit:

$$\lim_{n\to\infty} \frac{n}{q^n} = A \in \mathbb{R}.$$

Let us show that $A = 0$. Acting in the same way as in theorem 1, we obtain:

$$A = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{n+1}{q^{n+1}} = \frac{1}{q} \lim_{n\to\infty} \frac{n+1}{n} \frac{n}{q^n} = \frac{1}{q} \cdot A.$$

From the equality $A = \frac{A}{q}$, taking into account that $q > 1$, it follows that $A = 0$. $\square$

REMARK.

A stronger statement can be proved: if $\alpha > 0$, $q > 1$, then

$$\lim_{n\to\infty} \frac{n^\alpha}{q^n} = 0.$$

Other examples

5B/14:30 (12:26)

THEOREM 3 (ON THE CONVERGENCE OF THE SEQUENCE $\{\sqrt[n]{n}\}$).

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

PROOF.

Let us choose an arbitrary $\varepsilon > 0$ and denote $q = 1 + \varepsilon > 1$.

We have already proved that $\lim_{n\to\infty} \frac{n}{q^n} = 0$, therefore,

$$\exists\, N \in \mathbb{N} \quad \forall\, n > N \quad \frac{n}{q^n} < 1.$$

Let us transform the last estimate taking into account that $q = 1 + \varepsilon$:

$$n < (1 + \varepsilon)^n.$$

Since the estimate $n \ge 1$ holds for all positive integers n, we obtain a double estimate:

$$1 \le n < (1 + \varepsilon)^n.$$

Let us extract the root of degree n from all parts of this estimate:

$$1 \le \sqrt[n]{n} < 1 + \varepsilon.$$

Thus, for all $n > N$, the following inequalities are valid:

$$1 - \varepsilon < 1 \le \sqrt[n]{n} < 1 + \varepsilon, \text{ or } |\sqrt[n]{n} - 1| < \varepsilon.$$

We obtain that for an arbitrary $\varepsilon > 0$ there exists an integer N such that for all $n > N$ the following estimate holds:

$$|\sqrt[n]{n} - 1| < \varepsilon.$$

By the definition of the limit in the language ε–N, we get $\lim_{n\to\infty} \sqrt[n]{n} = 1$. □

THEOREM 4 (ON THE CONVERGENCE OF THE SEQUENCE $\{\sqrt[n]{a}\}$).

If $a > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$.

PROOF.

The case $a \ge 1$ is considered similarly to theorem 3: for an arbitrary $\varepsilon > 0$ we set $q = 1 + \varepsilon > 1$ and use the auxiliary sequence $\left\{\frac{a}{q^n}\right\}$ whose limit is 0 (see theorem 1).

The case $0 < a < 1$ easily reduces to the already considered case: if $0 < a < 1$, then $\frac{1}{a} > 1$ and, in addition, $\sqrt[n]{a} = \frac{1}{\sqrt[n]{1/a}}$. Therefore

$$\lim_{n\to\infty}\sqrt[n]{a}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{1/a}}=\frac{1}{\lim_{n\to\infty}\sqrt[n]{1/a}}=\frac{1}{1}=1.\ \square$$

THEOREM 5 (ON THE CONVERGENCE OF THE SEQUENCE $\{q^n/n!\}$).

If $q>0$, then $\lim\limits_{n\to\infty}\frac{q^n}{n!}=0$, where $n!=1\cdot 2\cdot 3\ldots n$ is the *factorial* of n.

The proof is similar to the proof of theorem 2. $\square$

REMARK.

Thus, of the three types of sequences considered above ($\{n^\alpha\}$ when $\alpha>0$, $\{q^n\}$ when $q>1$, $\{n!\}$), the sequence $\{n^\alpha\}$ is growing most slowly and the sequence $\{n!\}$ has the highest rate of growth.

The limit of the sequence $(1+1/n)^n$ and Bernoulli's inequality

5B/26:56 (13:26), 6A/00:00 (11:32)

THEOREM 6 (ON THE CONVERGENCE OF THE SEQUENCE $\{(1+1/n)^n\}$).

The sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent.

Before proving the theorem, we prove an auxiliary statement.

LEMMA (BERNOULLI'S INEQUALITY).

For any $\alpha>-1$ and $n\in\mathbb{N}$, the following estimate holds:

$$(1+\alpha)^n\ge 1+n\alpha. \tag{4}$$

The estimate (4) is called the *Bernoulli's inequality.*

PROOF OF THE LEMMA.

We carry out the proof by the *principle of mathematical induction.*

1. The base case. For $n=1$, non-strict inequality (4) turns into equality $1+\alpha=1+\alpha$ and, therefore, is valid.

2. The inductive step. We suppose that the estimate (4) is valid for $n=k$ and we should prove that in this case it will also be valid for $n=k+1$. Let us transform the left-hand side of the estimate (4) for $n=k+1$:

$$(1+\alpha)^{k+1}=(1+\alpha)^k(1+\alpha).$$

By the inductive hypothesis, the estimate $(1+\alpha)^k\ge 1+k\alpha$ is true, therefore

$$(1+\alpha)^k(1+\alpha)\ge(1+k\alpha)(1+\alpha).$$

The sign of the inequality will not change, since we multiply both sides of the inequality by the positive value $(1+\alpha)$ (because, by the condition, $\alpha>-1$). Let us transform the resulting expression:

$$(1 + k\alpha)(1 + \alpha) = 1 + (k + 1)\alpha + k\alpha^2 \geq 1 + (k + 1)\alpha.$$

The last estimate is fulfilled due to the fact that $k\alpha^2 \geq 0$.

We have obtained a chain of inequalities that begins with the expression $(1 + \alpha)^{k+1}$ and ends with the expression $1 + (k + 1)\alpha$. Hence,

$$(1 + \alpha)^{k+1} \geq 1 + (k + 1)\alpha.$$

We proved that if the estimate (4) is assumed to be valid for $n = k$, then it will be valid for $n = k + 1$.

By the principle of mathematical induction, we conclude that the estimate (4) is valid for all positive integers n. □

Proof of theorem 6.

Consider the auxiliary sequence $x'_n = \left(1 + \frac{1}{n}\right)^{n+1}$ and show that it is decreasing. To do this, we estimate the relation $\frac{x'_{n-1}}{x'_n}$ for $n > 1$:

$$\frac{x'_{n-1}}{x'_n} = \frac{\left(1 + \frac{1}{n-1}\right)^n}{(1 + \frac{1}{n})^{n+1}} = \frac{\left(\frac{n}{n-1}\right)^n}{\left(\frac{n+1}{n}\right)^{n+1}} = \frac{\left(\frac{n}{n-1}\right)^n}{\left(\frac{n+1}{n}\right)^n \frac{n+1}{n}} =$$

$$= \left(\frac{n^2}{(n-1)(n+1)}\right)^n \frac{1}{\frac{n+1}{n}} = \left(\frac{n^2}{n^2 - 1}\right)^n \frac{n}{n+1}.$$

The base of the power can be transformed as follows:

$$\frac{n^2}{n^2 - 1} = \frac{n^2 - 1 + 1}{n^2 - 1} = 1 + \frac{1}{n^2 - 1}.$$

Since $n > 1$, the fraction $\frac{1}{n^2-1}$ is positive, and we can apply the Bernoulli's inequality for $\alpha = \frac{1}{n^2-1}$:

$$\left(\frac{n^2}{n^2 - 1}\right)^n \frac{n}{n+1} = \left(1 + \frac{1}{n^2 - 1}\right)^n \frac{n}{n+1} \geq \left(1 + \frac{n}{n^2 - 1}\right) \frac{n}{n+1}.$$

Let us perform the final transformations taking into account that $\frac{n}{n^2-1} > \frac{1}{n}$:

$$\left(1 + \frac{n}{n^2 - 1}\right) \frac{n}{n+1} > \left(1 + \frac{1}{n}\right) \frac{n}{n+1} = \frac{n+1}{n} \cdot \frac{n}{n+1} = 1.$$

So, we obtain that $\frac{x'_{n-1}}{x'_n} > 1$, i. e., $x'_{n-1} > x'_n$, and the sequence $\{x'_n\}$ is decreasing. In addition, all its members are positive, therefore, the sequence is bounded from below by 0. Therefore, it has a limit.

Taking into account the definition of x'_n we obtain:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) =$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n. \ \square$$

Remark.

The limit of the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is denoted by the letter e. Since $x'_1 = \left(1+\frac{1}{1}\right)^2 = 4$, the double estimate $0 < e < 4$ holds for the constant e.

Increasing of the sequence $(1+1/n)^n$ 6A/11:32 (08:40)

Another version of the proof of theorem 6 is possible, which is based on the fact that the sequence $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing and bounded from above.

We will not prove the boundedness of the sequence $\{x_n\}$, however, we show that it is increasing.

For this, we use another auxiliary estimate, which we accept without proof: for any positive numbers $a_1, a_2, \ldots, a_n$, the following estimate holds:

$$\sqrt[n]{a_1 a_2 \ldots a_n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}. \qquad (5)$$

Moreover, equality is achieved if and only if $a_1 = a_2 = \cdots = a_n$.

The left-hand side of the estimate (5) is the *geometric mean* of the numbers $a_1, a_2, \ldots, a_n$, and the right-hand side is their *arithmetic mean*. Thus, the estimate (5) establishes the relationship between the geometric mean and arithmetic mean for a set of positive numbers.

Consider the following expression:

$$\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n} = \sqrt[n+1]{1\cdot\left(1+\frac{1}{n}\right)^n}.$$

After adding a new factor equal to 1, we get $n+1$ factors under the root. Given the estimate (5), the original expression is estimated from above by the corresponding arithmetic mean, moreover, there is a strict inequality, since the factor 1 is not equal to the other factors:

$$\sqrt[n+1]{1\cdot\left(1+\frac{1}{n}\right)^n} < \frac{1+n\left(1+\frac{1}{n}\right)}{n+1}.$$

The last fraction may be transformed as follows:

$$\frac{1+n\left(1+\frac{1}{n}\right)}{n+1} = \frac{1+n+1}{n+1} = 1+\frac{1}{n+1}.$$

Thus, we get an inequality:

$$\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n} < 1+\frac{1}{n+1}.$$

Let us raise both sides of this inequality to the power of $n+1$:

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}.$$

We have received that the inequality $x_n < x_{n+1}$ is valid for any positive integer n, and thereby we have proved that the sequence $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing.

The first element x_1 of this sequence is $\left(1+\frac{1}{1}\right)^1 = 2$. Since we already proved that the increasing sequence $\{x_n\}$ has a limit (equal to e), the estimate $e > x_1 = 2$ should be fulfilled for this limit. Therefore, we can refine the previously obtained double estimate for e: $2 < e < 4$.

Calculations show that the value of the constant e is approximately 2.7183.

6. Nested segments theorem and Bolzano–Cauchy theorem on the limit point

Nested segments theorem

Sequence of nested segments and sequence of contracting segments: definitions and examples 6A/20:12 (07:11)

DEFINITION.

A sequence of segments $\{[a_n, b_n]\}$ is called the *sequence of nested segments* if each next segment is nested in the previous one, that is, $\forall\, n \in \mathbb{N}$ $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$. A sequence of nested segments is called the *sequence of contracting segments* if their lengths approach zero: $\lim_{n\to\infty}(b_n - a_n) = 0$.

AN EXAMPLE OF A SEQUENCE OF NESTED SEGMENTS.

$$\left\{\left[-1-\frac{1}{n}, 1+\frac{1}{n}\right]\right\} = \left\{[-2, 2], \left[-\frac{3}{2}, \frac{3}{2}\right], \left[-\frac{4}{3}, \frac{4}{3}\right], \dots\right\}.$$

Note that this sequence is not a sequence of contracting segments.

AN EXAMPLE OF A SEQUENCE OF CONTRACTING SEGMENTS.

$$\left\{\left[0, \frac{1}{n}\right]\right\} = \left\{[0, 1], \left[0, \frac{1}{2}\right], \left[0, \frac{1}{3}\right], \dots\right\}.$$

Nested segments theorem 6A/27:23 (13:30)

THEOREM (NESTED SEGMENTS THEOREM).

1. If $\{[a_n, b_n]\}$ is a sequence of nested segments, then there is at least one point c belonging to all segments of the sequence:

$$\exists\, c \in \mathbb{R} \quad \forall\, n \in \mathbb{N} \quad c \in [a_n, b_n].$$

2. If $\{[a_n, b_n]\}$ is a sequence of contracting segments, then the point c belonging to all segments of the sequence is unique.

PROOF.

1. First, we prove the existence of the point c. It follows from the axiom of continuity.

Consider two sets: $A = \{a_1, a_2, \dots\}$ is the set of left endpoints of segments, $B = \{b_1, b_2, \dots\}$ is the set of right endpoints of segments.

For any $n, m \in \mathbb{N}$, the estimate $a_n \le b_m$ is valid. Indeed, any left endpoint cannot be larger than any right endpoint, because otherwise the corresponding segments will not be nested.

Therefore, any element of the set A does not exceed any element of the set B. Then, by virtue of the axiom of continuity of the set of real numbers,

$$\exists\, c \in \mathbb{R} \quad \forall\, n, m \in \mathbb{N} \quad a_n \le c \le b_m.$$

In particular, choosing $m = n$, we obtain: $\forall\, n \in \mathbb{N}$ $a_n \le c \le b_n$, that is, $c \in [a_n, b_n]$.

Thus, the point c simultaneously belongs to all segments of the sequence $\{[a_n, b_n]\}$.

2. Let us prove the uniqueness of the point c if $\{[a_n, b_n]\}$ is a sequence of contracting segments.

Suppose that there are two points c_1 and c_2 that are common to all segments. Without loss of generality, we can assume that $c_1 \le c_2$.

By our assumption, the following triple inequality holds:

$$\forall\, n \in \mathbb{N} \quad a_n \le c_1 \le c_2 \le b_n.$$

This inequality means that the distance from c_1 to c_2 does not exceed the distance from a_n to b_n:

$$0 \le c_2 - c_1 \le b_n - a_n.$$

Passing to the limit in this inequality as $n \to \infty$, we obtain:

$$0 \le c_2 - c_1 \le 0.$$

Therefore, $c_1 - c_2 = 0$, $c_1 = c_2$. Thus, for a sequence of contracting segments, any two common points of all segments coincide, that is, such a common point is unique. □

Remark.

For a sequence of nested *intervals*, this theorem is false. For example, for a sequence of intervals $\left\{\left(0, \frac{1}{n}\right)\right\}$, there does not exist a point that would be common to all intervals of this sequence.

Limit points of a set. Bolzano–Cauchy theorem

Limit points of a set: two definitions, proof of their equivalence

6B/00:00 (09:53)

Definition 1 of the limit point.

Let X be some set of real numbers. A point c is called the *limit point* of the set X if for any neighborhood of the point c there is at least one point of the set X contained in this neighborhood and not equal to c:

$$\forall U_c \quad \exists x \in X \setminus \{c\} \quad x \in U_c.$$

Remark.

It is not necessary for the limit point of the set X to belong to this set.

Example.

All limit points of the interval (a, b) form a segment $[a, b]$.

Definition 2 of the limit point.

A point c is called the *limit point* of the set X if for any neighborhood of the point c there exists an infinite number of points of the set X contained in this neighborhood.

Theorem (on the equivalence of the definitions of the limit point).

Definitions 1 and 2 of the limit point are equivalent.

Proof.

It is clear that if c is the limit point in the sense of definition 2, then it is also the limit point in the sense of definition 1.

Let us prove the converse statement: if c is the limit point in the sense of definition 1, then it is also the limit point in the sense of definition 2.

Let c be the limit point of the set X in the sense of definition 1. To prove the required statement, it suffices to choose some neighborhood U_c of the point c and show that definition 2 holds for this neighborhood, that is, it contains an infinite number of elements of the set X.

By definition 1, in the chosen neighborhood U_c there exists a point $x_1 \neq c$ belonging to the set X.

We select a new neighborhood U_2 of the point c, which is embedded in U_c and does not contain the point x_1. Such a neighborhood can always be selected (for example, we can take the intersection of the neighborhood U_c and a symmetric neighborhood of a point c of radius $|c - x_1|$). By definition 1, in this new neighborhood U_2 there also exists a point $x_2 \neq c$ belonging to the set X.

Now we select the neighborhood U_3 of the point c, which is embedded in U_2 and does not contain the point x_2, and in it we select the point $x_3 \neq c$ belonging to the set X. And so on.

The described process of constructing neighborhoods and choosing points x_n allows us to obtain an infinite number of points that belong to the set X and are contained in a neighborhood of U_c. Therefore, this neighborhood satisfies definition 2. □

The limit point theorem (Bolzano–Cauchy) 6B/09:53 (16:30)

Theorem (Bolzano–Cauchy theorem on the limit point).

Any infinite bounded set of real numbers has at least one limit point.

Proof.

We prove this theorem on the basis of the nested segments theorem.

Let X be an infinite bounded set. Since the set X is bounded, therefore, there is a segment $[a_0, b_0]$ that contains all the elements of the set X.

Indeed, by the definition of a bounded set,

$$\exists\, M > 0 \quad \forall\, x \in X \quad |x| \le M.$$

So we can take $[-M, M]$ as a segment $[a_0, b_0]$.

Let us show that the segment $[a_0, b_0]$ contains at least one limit point of the set X.

We divide the segment $[a_0, b_0]$ into two half-length segments by the point $\frac{a_0+b_0}{2}$. At least one of these new segments contains an infinite number of elements of the set X (if this were not so, and both segments contain a finite number of elements of the set X, then the entire set X would be finite, which contradicts the condition).

Now we select one of the obtained segments that contains an infinite number of elements of the set X and denote it by $[a_1, b_1]$ (if both segments contain an infinite number of elements X, then we can select any of them).

Taking into account that the length of the segment $[a_1, b_1]$ is equal to half the length of the segment $[a_0, b_0]$, we get: $b_1 - a_1 = \frac{b_0-a_0}{2}$.

We repeat the described actions for the segment $[a_1, b_1]$: we select half-length segment that contains an infinite number of elements of the set X and denote it by $[a_2, b_2]$. Then $b_2 - a_2 = \frac{b_1-a_1}{2} = \frac{b_0-a_0}{2^2}$. And so on. As a result of repeating these steps, we get a sequence of nested segments $\{[a_n, b_n]\}$, each of which contains an infinite number of elements of the set X.

For the segment length $[a_n, b_n]$, the relation $b_n - a_n = \frac{b_0-a_0}{2^n}$ is true. Thus, the length of the segment $[a_n, b_n]$ approaches 0 when n approaches infinity, therefore, this sequence of nested segments is a sequence of contracting segments.

By the nested segments theorem, there exists a point c which belongs to all segments $[a_n, b_n]$.

Let us show that this point c is the limit point of the set X. For this, according to definition 2, it suffices to prove that any neighborhood U_c of this point contains an infinite number of elements of the set X.

We select some neighborhood U_c of the point c. Let d be the distance from the point c to the nearest endpoint of the neighborhood U_c, $d > 0$.

Since the segments $[a_n, b_n]$ are contracting ones, we can choose the segment $[a_k, b_k]$ whose length will be less than $\frac{d}{2}$. The selected segment necessarily contains the point c, therefore, this segment will be entirely contained in the neighborhood U_c.

The segment $[a_k, b_k]$, like any segment of the sequence $\{[a_n, b_n]\}$, contains an infinite number of elements of the set X, so the neighborhood U_c, which contains the segment $[a_k, b_k]$, also has an infinite number of elements of X. Therefore, by definition 2, the point c is the limit point of the set X. □

7. Subsequences. Cauchy criterion

Subsequences. Bolzano–Weierstrass theorem

Subsequences: definition and examples

6B/26:23 (05:32)

DEFINITION.

Let $\{x_n\}$ be a sequence and let $\{n_k\}$ be an increasing sequence of indices (natural numbers): $1 \le n_1 < n_2 < n_3 < \dots < n_k < \dots$ Then the sequence $\{y_k\}$ with the elements $y_k = x_{n_k}$, $k \in \mathbb{N}$, is called the *subsequence* of the sequence $\{x_n\}$.

Thus, in order to obtain a subsequence, we can remove any number of elements from the original sequence, but it is necessary that an infinite number of elements remain and that the order of the remaining elements does not change.

EXAMPLES.

For a sequence $\left\{\frac{1}{n}\right\}$, we can take as a subsequence, for example, $\left\{\frac{1}{2n}\right\}$ or $\left\{\frac{1}{2n-1}\right\}$.

For a sequence $\left\{\frac{(-1)^n}{n}\right\}$, we can take as a subsequence, for example, $\left\{\frac{1}{2n}\right\}$ or $\left\{-\frac{1}{2n-1}\right\}$.

For a sequence $\{(-1)^n\}$, we can take as a subsequence, for example, $\{1, 1, 1, \dots\}$ or $\{-1, -1, -1, \dots\}$.

The theorem on subsequences of a convergent sequence

6B/31:55 (11:27)

LEMMA (ON THE INDICES OF SUBSEQUENCE ELEMENTS).

For elements of the sequence of indices $\{n_k\}$ from the definition of a subsequence, the following estimate holds:

$$\forall\, k \in \mathbb{N} \quad n_k \ge k.$$

PROOF (BY MATHEMATICAL INDUCTION)[1].

1. In the case $k = 1$, the inequality holds, because, by the definition of the sequence of indices, $n_1 \ge 1$.

[1] In video lectures, there is no proof of this lemma.

2. Suppose that the inequality holds for some k, that is, $n_k \geq k$, and prove that then it also holds for $k+1$. Indeed, since $n_{k+1} > n_k$, we get: $n_{k+1} > n_k \geq k$, that is, $n_{k+1} > k$. But this means that $n_{k+1} \geq k+1$. □

Theorem (on subsequences of a convergent sequence).

If a sequence has a limit (finite or infinite), then any subsequence of it has the same limit.

Proof[2].

Let us prove this theorem for the case of a finite limit (the case of infinite limit can be proved in a similar way).

Let the sequence $\{x_n\}$ converge to A and $\{y_k\}$ be some subsequence of $\{x_n\}$: $y_k = x_{n_k}$, $k \in \mathbb{N}$.

We prove that the subsequence $\{y_k\}$ also converges to A.

For the sequence $\{x_n\}$, using the limit definition, we have:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |x_n - A| < \varepsilon.$$

For all $k > N$, by virtue of the lemma on the indices of subsequence elements, we obtain $n_k \geq k > N$, therefore, $n_k > N$ and $|x_{n_k} - A| < \varepsilon$.

By definition, $x_{n_k} = y_k$, and the last inequality can be written in the form $|y_k - A| < \varepsilon$.

Thus, for all $k > N$, the estimate $|y_k - A| < \varepsilon$ holds. We have shown that

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall k > N \quad |y_k - A| < \varepsilon.$$

This means that $\lim_{k\to\infty} y_k = A$. □

The Bolzano–Weierstrass theorem on a convergent subsequence

7A/00:00 (15:54)

Theorem (Bolzano–Weierstrass theorem on a convergent subsequence).

A convergent subsequence can be extracted from any bounded sequence.

Proof.

Let $\{x_n\}$ be a bounded sequence. Consider the set of values of its elements:

$$X = \{x_n : n \in \mathbb{N}\}.$$

If some elements of the sequence have the same value x, then this value will be included in the set X exactly once, because all elements of the set are different.

For example, for the sequence $\{(-1)^n\}$, we have $X = \{-1, 1\}$.

[2] In video lectures, a version of the proof is given in which the lemma on the indices of the elements of a subsequence is not used.

Two cases are possible:

1) the set X consists of a finite number of elements (as in the above example);

2) the set X consists of an infinite number of elements.

We first consider case 1. It is possible only if there are an infinite number of elements with the same value c.

We suppose that the elements x_n with the value c have the indices $n_1 < n_2 < \dots < n_k < \dots$, and there are an infinite number of these indices. Consider a subsequence of $\{y_k\}$, composed of elements of the sequence $\{x_n\}$ with these indices: $y_k = x_{n_k}$. This is a constant sequence, since all its elements are equal to c. Therefore, it has a limit equal to c.

In the above example, for the sequence $\{(-1)^n\}$, we can choose a constant sequence consisting of units: $\{1, 1, 1, \dots\}$.

Now we consider case 2 and suppose X is an infinite set. Since, by hypothesis, the initial sequence is bounded, the set X is also a bounded set. Therefore, X is an infinite bounded set, and, by the Bolzano–Cauchy theorem, it has at least one limit point c.

We construct a subsequence $\{y_k\}$ of the sequence $\{x_n\}$, which converges to the limit point c.

Let us choose a symmetric neighborhood of the point c of radius 1:

$$U_c^1 = (c - 1, c + 1).$$

By definition 2 of the limit point, U_c^1 contains an infinite number of elements of the set X. We select one of these elements and suppose that this element has the index n_1: x_{n_1}.

The element x_{n_1} will be the first element of our subsequence: $y_1 = x_{n_1}$. For y_1, the following estimate holds: $|y_1 - c| < 1$.

Now choose a neighborhood of the point c of radius $\frac{1}{2}$:

$$U_c^{1/2} = \left(c - \frac{1}{2}, c + \frac{1}{2}\right).$$

In the neighborhood of $U_c^{1/2}$, there is also an infinite number of elements of X. We choose an element with the index n_2 from them, and we require that the estimate $n_2 > n_1$ be fulfilled. The required element x_{n_2} necessarily exists, since there are a finite number of indices that do not suit us (from 1 to n_1) and an infinite number of elements X in a neighborhood of $U_c^{1/2}$.

Denote $y_2 = x_{n_2}$. For y_2, the following estimate holds: $|y_2 - c| < \frac{1}{2}$.

Continuing this process, we get a subsequence $\{y_k\}$ of the sequence $\{x_n\}$, for whose elements the estimate $|y_k - c| < \frac{1}{k}$ holds.

Let us show that $\{y_k\}$ converges to c. We must prove that

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall k > N \quad |y_k - c| < \varepsilon.$$

Choose an arbitrary $\varepsilon > 0$. There exists a natural N such that $\frac{1}{N} < \varepsilon$. By constructing the sequence $\{y_k\}$, the following estimate holds for all $k > N$:

$$|y_k - c| < \frac{1}{k} < \frac{1}{N} < \varepsilon.$$

Therefore, by the definition of the limit, the sequence $\{y_k\}$ converges to the point c. □

Corollary

7A/15:54 (08:57)

Theorem (corollary of the Bolzano–Weierstrass theorem). Any sequence contains a subsequence that has a finite or an infinite limit.

Proof.

If the given sequence $\{x_n\}$ is bounded, then, by the Bolzano–Weierstrass theorem, we can extract a convergent subsequence from it.

It remains to consider the case when the given sequence is not bounded. We show that in this case a subsequence having the limit ∞ can be obtained from the given sequence.

Let us write a condition meaning that the sequence $\{x_n\}$ is not bounded, applying the negation operation to the definition of a bounded sequence:

$$\overline{\exists M > 0 \quad \forall n \in \mathbb{N} \quad |x_n| \le M},$$

$$\forall M > 0 \quad \exists n \in \mathbb{N} \quad |x_n| > M.$$

We describe the process of constructing the required subsequence of $\{y_k\}$.

Choose $M = 1$. Then there is an index n_1 such that $|x_{n_1}| > 1$. Let $y_1 = x_{n_1}$.

Choose $M = 2$ and exclude from consideration all the elements of the original sequence $\{x_n\}$ with indices less than or equal to n_1. The rest of the sequence is still unbounded, so there is an index $n_2 > n_1$ such that $|x_{n_2}| > 2$. Let $y_2 = x_{n_2}$.

Continuing this process, we obtain a subsequence $\{y_k\}$ of the sequence $\{x_n\}$, for whose elements the estimate $|y_k| > k$ holds. It is easy to prove that $\lim_{k\to\infty} y_k = \infty$. □

An example of an unbounded sequence with no limit

7A/24:51 (04:55)

From the fact that the sequence is not bounded, it does not follow that it has an infinite limit. For example, the following sequence is unbounded:

$$\left\{n^{(-1)^n}\right\} = \left\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \dots\right\}.$$

However, this sequence has no limit, since it has an infinite number of elements contained both in an arbitrary neighborhood of the point 0 and in an arbitrary neighborhood of the point $+\infty$. At the same time, we can extract from it a subsequence converging to 0, as well as a subsequence approaching $+\infty$.

Fundamental sequences. Cauchy criterion for sequence convergence

Fundamental sequences: definition

7A/29:46 (04:25)

Definition.

The sequence $\{x_n\}$ is called the *fundamental sequence*, or the *Cauchy sequence*, if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N, n > N \quad |x_m - x_n| < \varepsilon. \tag{1}$$

The Cauchy criterion for sequence convergence

7A/34:11 (05:20), 7B/00:00 (26:41)

Theorem (Cauchy criterion for sequence convergence).

The sequence $\{x_n\}$ is convergent if and only if it is a fundamental sequence:

$$\left(\lim_{n\to\infty} x_n = A \in \mathbb{R}\right) \Leftrightarrow (\{x_n\} \text{ is the fundamental sequence}).$$

Proof.

1. Let us prove the necessity.

Given: $\lim_{n\to\infty} x_n = A \in \mathbb{R}$.

Prove: $\{x_n\}$ is a fundamental sequence, that is, it satisfies (1).

We use the definition of the limit of a sequence in the language ε–N:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |x_n - A| < \frac{\varepsilon}{2}.$$

The same is true for an arbitrary number $m > N$:

$$\forall m > N \quad |x_m - A| < \frac{\varepsilon}{2}.$$

Assuming $m > N$, $n > N$, we get:

$$|x_m - x_n| = |x_m - A + A - x_n| \le |x_m - A| + |A - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, for $\{x_n\}$, condition (1) is satisfied, therefore, $\{x_n\}$ is a fundamental sequence. The necessity is proven.

2. Now we prove sufficiency.

Given: $\{x_n\}$ is a fundamental sequence, that is, it satisfies (1).

Prove: the sequence $\{x_n\}$ is convergent.

First, we prove that the sequence $\{x_n\}$ is bounded.

If we put $\varepsilon = 1$ in condition (1), then we get:

$$\exists\, N \in \mathbb{N} \quad \forall\, m > N, n > N \quad |x_m - x_n| < 1. \tag{2}$$

Let us choose some $m_0 > N$. Then

$$|x_n| = |x_n - x_{m_0} + x_{m_0}| \le |x_n - x_{m_0}| + |x_{m_0}|.$$

If $n > N$, then the first term in the right-hand side of the resulting inequality is estimated from above by 1, by virtue of (2). The second term does not depend on n. Thus, for all $n > N$ we get the estimate:

$$|x_n| < 1 + |x_{m_0}| = M.$$

We have found that there exists a value M such that for all $n > N$ the estimate $|x_n| < M$ holds. This means that the set of elements of the sequence $\{x_n\}$ with numbers greater than N is bounded.

But the set of remaining elements $\{x_1, x_2, \dots, x_N\}$ is finite and, therefore, also bounded. Therefore, the entire sequence $\{x_n\}$ is bounded (you can compare this proof with the proof of the theorem on the boundedness of a converging sequence).

We have proved that the sequence $\{x_n\}$ is bounded. Therefore, the Bolzano–Weierstrass theorem holds for it. By virtue of this theorem, there exists a convergent subsequence $\{y_k\}$ of the sequence $\{x_n\}$:

$$\lim_{k\to\infty} y_k = A, \text{ where } y_k = x_{n_k}, \quad 1 \le n_1 < n_2 < \dots < n_k < \dots$$

Now we prove that the limit A of the subsequence $\{y_k\}$ is also the limit of the given sequence $\{x_n\}$. For this, we must show that

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n - A| < \varepsilon. \tag{3}$$

Let us select some $\varepsilon > 0$. For this ε, due to the fundamentality of the sequence $\{x_n\}$, condition (1) is satisfied:

$$\exists\, N_1 \in \mathbb{N} \quad \forall\, m > N_1, n > N_1 \quad |x_m - x_n| < \frac{\varepsilon}{2}. \tag{4}$$

In addition, we know that the subsequence $\{y_k\}$ has a limit A, so for the same ε, by definition of the limit, we get:

$$\exists\, N_2 \in \mathbb{N} \quad \forall\, k > N_2 \quad |y_k - A| < \frac{\varepsilon}{2}.$$

The last inequality can be rewritten as follows:

$$|x_{n_k} - A| < \frac{\varepsilon}{2}. \quad (5)$$

Put $N = \max\{N_1, N_2\}$ and show that condition (3) is fulfilled for all $n > N$, that is, for all $n > N$ the estimate $|x_n - A| < \varepsilon$ holds. We choose $k_0 = N + 1$ and transform the expression $|x_n - A|$ as follows:

$$|x_n - A| = |x_n - x_{n_{k_0}} + x_{n_{k_0}} - A| \leq |x_n - x_{n_{k_0}}| + |x_{n_{k_0}} - A|. \quad (6)$$

Since $k_0 = N + 1 > N \geq N_2$, the last term in the resulting sum, by virtue of (5), can be estimated as follows:

$$|x_{n_{k_0}} - A| < \frac{\varepsilon}{2}. \quad (7)$$

To estimate the first term $|x_n - x_{n_{k_0}}|$, we note that, by virtue of the lemma on the indices of subsequence elements, the index n_{k_0} can be estimated as follows: $n_{k_0} \geq k_0 > N \geq N_1$. Then for $n > N \geq N_1$ we have $n_{k_0} > N_1$, $n > N_1$, therefore, by virtue of (4),

$$|x_n - x_{n_{k_0}}| < \frac{\varepsilon}{2}. \quad (8)$$

Given estimates (7) and (8) in inequality (6), we finally obtain:

$$|x_n - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, condition (3) is satisfied and, therefore, $\lim_{n\to\infty} x_n = A$. The sufficiency is proved. □

Examples of applying the Cauchy criterion[3].

To prove that the sequence $\{x_n\}$ does not have a finite limit, it suffices to show that $\{x_n\}$ is not a fundamental sequence, that is, that the negation of the condition (1) is fulfilled:

$$\exists\, \varepsilon > 0 \quad \forall\, N \in \mathbb{N} \quad \exists\, m > N, n > N \quad |x_m - x_n| \geq \varepsilon. \quad (9)$$

1. Let us show that the sequence $\{x_n\} = \{(-1)^n\}$ is not a fundamental sequence.

Let $\varepsilon = 1$. For any positive integer N, we choose $m = 2N > N$ and $n = 2N + 1 > N$. Then

$$|x_m - x_n| = |(-1)^{2N} - (-1)^{2N+1}| = |1 - (-1)| = 2 > 1 = \varepsilon.$$

Thus, condition (9) is satisfied, the sequence is not fundamental and, therefore, it does not have a finite limit. Recall that we have already established this fact (see Chapter 2).

2. Let us show that the sequence $\{x_n\} = \left\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\right\}$ is not a fundamental sequence.

[3] In video lectures, there are no such examples.

Let $\varepsilon = \frac{1}{2}$. For any positive integer N, we choose $m = 4N > N$ and $n = 2N > N$. Then

$$|x_m - x_n| = \left|1 + \frac{1}{2} + \cdots + \frac{1}{4N} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{2N}\right)\right| =$$

$$= \frac{1}{2N+1} + \frac{1}{2N+2} + \cdots + \frac{1}{4N}.$$

The last sum contains $2N$ terms, and each term can be estimated from below by $\frac{1}{4N}$:

$$\frac{1}{2N+k} \geq \frac{1}{4N}, \quad k = 1, 2, \ldots, 2N.$$

Therefore, we finally get:

$$|x_m - x_n| = \frac{1}{2N+1} + \frac{1}{2N+2} + \cdots + \frac{1}{4N} \geq 2N \cdot \frac{1}{4N} = \frac{1}{2} = \varepsilon.$$

Thus, condition (9) is satisfied, the sequence is not fundamental and, therefore, it has no finite limit.

Since this sequence increases and has no finite limit, it is unbounded due to the convergence criterion for monotone sequences. So, the sequence is unbounded and increasing, therefore its limit is $+\infty$.

8. The limit of a function

Definition and uniqueness of the limit of a function

Definitions of the limit of a function and their equivalence

7B/26:41 (19:39)

By U_A, V_A, we will denote, as before, the neighborhood of the point A. Recall the definition of a symmetric neighborhood of the point A of radius ε:

$$U_A^\varepsilon \stackrel{\text{def}}{=} \{x \in \mathbb{R} : |x - A| < \varepsilon\}.$$

Definition.

A *punctured neighborhood* $\overset{\circ}{U}_a$ and a *punctured symmetric neighborhood* $\overset{\circ}{U}{}_a^\varepsilon$ of the point a are defined as follows:

$$\overset{\circ}{U}_a \stackrel{\text{def}}{=} U_a \setminus \{a\}, \qquad \overset{\circ}{U}{}_a^\varepsilon \stackrel{\text{def}}{=} \{x \in \mathbb{R} : 0 < |x - A| < \varepsilon\}.$$

Definition 1 of the function limit (in the language of neighborhoods).

Let $f : E \to \mathbb{R}$ be a function defined on $E \subset \mathbb{R}$, let a be the limit point of the set E. We say that the function f *has a limit*, as $x \to a$, equal to $A \in \mathbb{R}$ and write it in the form $\lim_{x\to a} f(x) = A$ if for any neighborhood U_A of the point A there exists a punctured neighborhood $\overset{\circ}{V}_a$ of the point a such that for any x belonging to the intersection $E \cap \overset{\circ}{V}_a$, $f(x)$ belongs to U_A:

$$\forall\, U_A \quad \exists\, \overset{\circ}{V}_a \quad \forall\, x \in E \cap \overset{\circ}{V}_a \quad f(x) \in U_A.$$

Definition 2 of the function limit (in the language of symmetric neighborhoods).

The function f *has a limit*, as $x \to a$, equal to $A \in \mathbb{R}$ if for any symmetric neighborhood U_A^ε of the point A there exists a punctured symmetric neighborhood $\overset{\circ}{V}{}_a^\delta$ of the point a such that for any x belonging to the intersection $E \cap \overset{\circ}{V}{}_a^\delta$, $f(x)$ belongs to U_A^ε:

$$\forall\, U_A^\varepsilon \quad \exists\, \overset{\circ}{V}{}_a^\delta \quad \forall\, x \in E \cap \overset{\circ}{V}{}_a^\delta \quad f(x) \in U_A^\varepsilon.$$

Definition 3 of the function limit (in the language ε–δ).

The function f *has a limit*, as $x \to a$, equal to $A \in \mathbb{R}$ if for any number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for any x from E satisfying the condition $0 < |x - a| < \delta$, the estimate $|(f(x) - A| < \varepsilon$ holds:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in E, 0 < |x - a| < \delta, \quad |f(x) - A| < \varepsilon.$$

Theorem (on the equivalence of three definitions of the limit of a function).

Definitions 1, 2, and 3 of the limit of a function are equivalent.

Proof[4].

The equivalence of definitions 2 and 3 follows from the definition of symmetric neighborhood.

Let us prove the equivalence of definitions 1 and 2. It is clear that if A is a limit in the sense of definition 1 then A is also a limit in the sense of definition 2 (since any symmetric neighborhood of a point is also its ordinary neighborhood). Let us prove that if A is a limit in the sense of definition 2 then A is also a limit in the sense of definition 1.

Let U_A be an arbitrary neighborhood. Then there exists a symmetric neighborhood $U_A^\varepsilon \subset U_A$. For a symmetric neighborhood U_A^ε, according to definition 2, there exists a neighborhood $\overset{\circ}{V}{}_a^\delta$ such that for all $x \in E \cap \overset{\circ}{V}{}_a^\delta$, $f(x)$ belongs to the neighborhood U_A^ε. By the embedding $U_A^\varepsilon \subset U_A$, this means that $f(x)$ also belongs to the neighborhood U_A. Thus, for any neighborhood of the point A, definition 1 holds. The equivalence of definitions 1 and 2 is proved.

Since each of definitions 1 and 3 is equivalent to definition 2, definitions 1 and 3 are also equivalent. □

Remark.

In definition of the limit of a function, a punctured neighborhood of the point a is used, since the value of the function at this point should not affect the value of the limit. It should also be noted that the function may not be defined at the point a: the definition does not require that a belongs to E, but it requires that a be the limit point of this set.

The uniqueness theorem for the limit of a function

8A/00:00 (12:21)

Theorem (on the uniqueness of the limit of a function).

If a function has a limit at a point, then this limit is unique.

[4] In video lectures, there is no proof of this theorem.

Proof.

Let $f : E \to \mathbb{R}$ be a function, let a be the limit point of the set E. Let $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} f(x) = B$. We should prove that $A = B$.

Let us prove this fact by contradiction: suppose that $A \neq B$.

We will use the definition of the limit in the language of neighborhoods. Let us choose neighborhoods U_A and U_B so that they do not intersect: $U_A \cap U_B = \varnothing$. This can always be done, since $A \neq B$.

By definition 1 of the limit of a function, we obtain the following relations for these neighborhoods:

$$\exists \overset{\circ}{V}{}'_a \quad \forall x \in E \cap \overset{\circ}{V}{}'_a \quad f(x) \in U_A, \tag{1}$$

$$\exists \overset{\circ}{V}{}''_a \quad \forall x \in E \cap \overset{\circ}{V}{}''_a \quad f(x) \in U_B. \tag{2}$$

Consider the point x_0 that belongs to the intersection of three sets: $x_0 \in E \cap \overset{\circ}{V}{}'_a \cap \overset{\circ}{V}{}''_a$. Such a point exists, since the intersection a $\overset{\circ}{V}{}'_a \cap \overset{\circ}{V}{}''_a$ of the punctured neighborhoods of the point is itself a punctured neighborhood of this point, and the intersection of any punctured neighborhood of the point a and the set E is not empty, since a is the limit point of E.

Then for this point x_0, due to (1) and (2), we have: $f(x_0) \in U_A$ and $f(x_0) \in U_B$. So $f(x_0) \in U_A \cap U_B$. But $U_A \cap U_B = \varnothing$, therefore $f(x_0) \in \varnothing$, which is impossible. The resulting contradiction means that $A = B$. □

Criterion for the existence of the limit of a function in terms of sequences

Formulation of the criterion

8A/12:21 (03:49)

Theorem (criterion for the existence of the limit of a function in terms of sequences).

Let $f : E \to \mathbb{R}$ be a function, let a be the limit point of the set E. For the function f to have the limit $\lim_{x\to a} f(x) = A$, it is necessary and sufficient that for any sequence $\{x_n\}$ satisfying the conditions $x_n \in E \setminus \{a\}$ and $\lim_{n\to\infty} x_n = a$, the limit of the sequence $\{f(x_n)\}$ exists and is equal to A:

$$\forall \{x_n\}, x_n \in E \setminus \{a\}, \lim_{n\to\infty} x_n = a, \quad \exists \lim_{n\to\infty} f(x_n) = A. \tag{3}$$

Thus, the statement of the theorem can be written as follows:

$$\Bigl(\lim_{x\to a} f(x) = A\Bigr) \Leftrightarrow \bigl(\text{condition (3) holds}\bigr).$$

Proof of necessity

8A/16:10 (11:56)

Given: $\lim_{x\to a} f(x) = A$. Prove: condition (3) holds.

Let us choose an arbitrary sequence $\{x_n\}$ satisfying all the required conditions:

$$x_n \in E \setminus \{a\}, \quad \lim_{n\to\infty} x_n = a.$$

We should show that the limit of the sequence $\{f(x_n)\}$ exists and is equal to A. Let us write the definition of the limit of a function in the language ε–δ:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in E, 0 < |x - a| < \delta, \quad |f(x) - A| < \varepsilon. \tag{4}$$

Let us choose an arbitrary value $\varepsilon > 0$, find the corresponding $\delta > 0$ for it, and write down the definition of the limit of the sequence $\{x_n\}$ for this δ:

$$\exists\, N \in \mathbb{N} \quad \forall\, n > N \quad 0 < |x_n - a| < \delta. \tag{5}$$

In the obtained double estimate (5), the right-hand side ($|x_n - a| < \delta$) is satisfied because the sequence $\{x_n\}$ has the limit a, and the left-hand side ($0 < |x_n - a|$) is satisfied due to the condition $x_n \in E \setminus \{a\}$.

Since for all $n > N$, by virtue of (5), the condition $0 < |x_n - a| < \delta$ holds, we obtain, by virtue of (4), that the estimate $|f(x_n) - A| < \varepsilon$ also holds for these n.

Thus, we have shown that for an arbitrarily chosen $\varepsilon > 0$ one can, using the auxiliary value δ, find a number N such that for all $n > N$ the estimate $|f(x_n) - A| < \varepsilon$ holds:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |f(x_n) - A| < \varepsilon.$$

The last relation means that $\lim_{n\to\infty} f(x_n) = A$.

Proof of sufficiency

8B/00:00 (13:41)

Given: condition (3) holds. Prove: $\lim_{x\to a} f(x) = A$.

We will prove this fact by contradiction. Suppose that the number A is not the limit of the function $f(x)$ as $x \to a$. Let us write down what this means by applying the logical negation operation to the definition of the limit in the language of symmetrical neighbors:

$$\overline{\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in \overset{\circ}{V}{}_a^\delta \cap E \quad f(x) \in U_A^\varepsilon,}$$

$$\exists\, \varepsilon > 0 \quad \forall\, \delta > 0 \quad \exists\, x \in \overset{\circ}{V}{}_a^\delta \cap E \quad f(x) \notin U_A^\varepsilon. \tag{6}$$

Let us construct the sequence $\{x_n\}$ choosing the values δ equal to $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ in the relation (6). For $\delta = 1$, there exists a number $x_1 \in \overset{\circ}{V}{}_a^1 \cap E$ for which the condition $f(x_1) \notin U_A^\varepsilon$ is fulfilled; for $\delta = \frac{1}{2}$, there exists a number $x_2 \in \overset{\circ}{V}{}_a^{1/2} \cap E$ for which the condition $f(x_2) \notin U_A^\varepsilon$ is fulfilled; $\ldots$; for $\delta = \frac{1}{n}$, there exists a number $x_n \in \overset{\circ}{V}{}_a^{1/n} \cap E$ for which the condition $f(x_n) \notin U_A^\varepsilon$ is fulfilled, and so on.

Since for any $n \in \mathbb{N}$ for the elements of the constructed sequence $\{x_n\}$, the condition $x_n \in \overset{\circ}{V}{}_a^{1/n} \cap E$ holds, we obtain that the elements x_n belong to the set $E \setminus \{a\}$ and $\lim_{n\to\infty} x_n = a$. At the same time, the sequence $\{f(x_n)\}$ does not have a limit equal to A, because there exists a symmetric neighborhood U_A^ε of the point A that does not contain any element of the sequence $\{f(x_n)\}$.

We have constructed the sequence $\{x_n\}$ for which condition (3) is violated, however, this condition must be satisfied for any sequence with the specified properties. This contradiction means that our assumption that the number A is not the limit of the function $f(x)$, as $x \to a$, is false, and $\lim_{x\to a} f(x) = A$. $\square$

Remarks.

1. The proved criterion means that condition (3) can be considered as another definition of the limit of a function. A feature of such a definition is that it defines the limit of a function through the limits of auxiliary sequences.

2. This criterion makes it easy to prove the *absence* of the limit of the function f at the point a. To prove this, it is enough to give two sequences $\{x'_n\}$ and $\{x''_n\}$ with elements from $E \setminus \{a\}$ that converge to a and such that the sequences $\{f(x'_n)\}$ and $\{f(x''_n)\}$ do not converge to the same limit.

Examples of functions with and without limits

8B/13:41 (20:11)

1. Consider the function $f(x) = x \sin \frac{1}{x}$, $E = \mathbb{R} \setminus \{0\}$.

We will show that $\lim_{x\to 0} x \sin \frac{1}{x} = 0$ using the definition of the limit of a function in the language ε–δ.

For any $x \neq 0$, the estimate $\left|x \sin \frac{1}{x}\right| \leq |x|$ holds. Let us choose an arbitrary value $\varepsilon > 0$ and set $\delta = \varepsilon$. Then for x such that $0 < |x| < \delta$, we obtain

$$\left|x \sin \frac{1}{x}\right| \leq |x| < \delta = \varepsilon.$$

Thus, for any $\varepsilon > 0$ there exists $\delta > 0$ (which can be taken as the value of ε itself) such that for all x satisfying the condition $0 < |x| < \delta$, the estimate $\left|x \sin \frac{1}{x}\right| < \varepsilon$ holds. Therefore, the definition 3 of the limit of a function is fulfilled for the function $x \sin \frac{1}{x}$ when $a = 0$, $A = 0$.

2. Let us define the function $\operatorname{sign} x$ ("signum of x") as follows:

$$\operatorname{sign} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

We will show that this function has no limit at the point 0. To do this, we consider two sequences that do not take the value 0 and converge to 0:

$$x'_n = \frac{1}{n}, \quad x''_n = -\frac{1}{n}.$$

By the definition of the function $\operatorname{sign} x$,

$$\lim_{n\to\infty} \operatorname{sign} x'_n = \lim_{n\to\infty} \operatorname{sign} \frac{1}{n} = \lim_{n\to\infty} 1 = 1,$$

$$\lim_{n\to\infty} \operatorname{sign} x''_n = \lim_{n\to\infty} \operatorname{sign}\left(-\frac{1}{n}\right) = \lim_{n\to\infty} (-1) = -1.$$

Thus, $\lim_{n\to\infty} \operatorname{sign} x'_n \neq \lim_{n\to\infty} \operatorname{sign} x''_n$. Since condition (3) is not satisfied, we conclude, by virtue of the necessary part of the previous criterion, that the function $\operatorname{sign} x$ has no limit at the point 0 (although it has a value equal to 0 at this point).

3. Consider the function $f(x) = |\operatorname{sign} x|$.

This function is equal to 1 everywhere, except for the point 0, at which it is equal to 0. Therefore, for any sequence $\{x_n\}$ that converges to 0 and does not take the value 0, we obtain: $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1$. Therefore, condition (3) is satisfied for $A = 1$ and, by virtue of the sufficient part of the previous criterion, we obtain that $\lim_{x\to 0} f(x) = 1$.

This example demonstrates a situation where the limit of a function at a point is not equal to its value at that point.

4. Consider the function $f(x) = \sin \frac{1}{x}$, $E = \mathbb{R} \setminus \{0\}$. We will show that the function has no limit at the point 0.

To do this, we consider two sequences that do not take the value 0 and converge to 0:

$$x'_n = \frac{1}{2\pi n}, \quad x''_n = \frac{1}{\frac{\pi}{2} + 2\pi n}.$$

Due to the properties of the function sin,

$$\lim_{n\to\infty} f(x'_n) = \lim_{n\to\infty} \sin\frac{1}{\frac{1}{2\pi n}} = \lim_{n\to\infty} \sin(2\pi n) = \lim_{n\to\infty} 0 = 0,$$

$$\lim_{n\to\infty} f(x''_n) = \lim_{n\to\infty} \sin\frac{1}{\frac{1}{\frac{\pi}{2}+2\pi n}} = \lim_{n\to\infty} \sin\Bigl(\frac{\pi}{2} + 2\pi n\Bigr) = \lim_{n\to\infty} 1 = 1.$$

Thus, $\lim_{n\to\infty} f(x'_n) \ne \lim_{n\to\infty} f(x''_n)$. Since condition (3) is not satisfied, we conclude, by virtue of the necessary part of the previous criterion, that the function $\sin\frac{1}{x}$ has no limit at the point 0.

Limits at the points at infinity and infinite limits

8B/33:52 (08:22)

Recall the definitions of *neighborhoods of the points at infinity*:

$$U^R_{+\infty} = \{x \in \mathbb{R} : x > R\},$$

$$U^R_{-\infty} = \{x \in \mathbb{R} : x < R\},$$

$$U^R_{\infty} = \{x \in \mathbb{R} : |x| > R\}.$$

Definition 1 of the limit at the point at infinity (in the language of neighborhoods).

Let the function f act from E to $\mathbb{R}$. If the set E is not bounded from above, then the point $+\infty$ is its limit point. In this case, it is said that *the limit of the function $f(x)$, as x approaching $+\infty$, is A* ($\lim_{x\to+\infty} f(x) = A$) if

$$\forall\, U_A \quad \exists\, V^R_{+\infty} \quad \forall\, x \in E \cap V^R_{+\infty} \quad f(x) \in U_A.$$

If the set E is not bounded from below, then the point $-\infty$ is its limit point. In this case, it is said that *the limit of the function $f(x)$, as x approaching $-\infty$, is A* ($\lim_{x\to-\infty} f(x) = A$) if

$$\forall\, U_A \quad \exists\, V^R_{-\infty} \quad \forall\, x \in E \cap V^R_{-\infty} \quad f(x) \in U_A.$$

If the set E is unbounded, then the point ∞ is its limit point. In this case, it is said that *the limit of the function $f(x)$, as x approaching ∞, is A* ($\lim_{x\to\infty} f(x) = A$) if

$$\forall\, U_A \quad \exists\, V^R_{\infty} \quad \forall\, x \in E \cap V^R_{\infty} \quad f(x) \in U_A.$$

Definition 2 of the limit at the point at infinity (in the language ε–R).

$$\lim_{x\to+\infty} f(x) = A:$$

$$\forall\, \varepsilon > 0 \quad \exists\, R \in \mathbb{R} \quad \forall\, x \in E, x > R, \quad |f(x) - A| < \varepsilon.$$

$\lim\limits_{x\to-\infty} f(x) = A :$

$$\forall\, \varepsilon > 0 \quad \exists\, R \in \mathbb{R} \quad \forall\, x \in E, x < R, \quad |f(x) - A| < \varepsilon.$$

$\lim\limits_{x\to\infty} f(x) = A :$

$$\forall\, \varepsilon > 0 \quad \exists\, R > 0 \quad \forall\, x \in E, |x| > R, \quad |f(x) - A| < \varepsilon.$$

Definitions 1 and 2 of the limit at the point at infinity are equivalent.

Definition 1 of infinite limit equal to $+\infty$ (in the language of neighborhoods).

Let the function f act from E to $\mathbb{R}$, let the point a be the limit point of E. It is said that *the limit of the function $f(x)$, as x approaching a, is $+\infty$* ($\lim_{x\to a} f(x) = +\infty$) if

$$\forall\, U^R_{+\infty} \quad \exists\, \overset{\circ}{V}_a \quad \forall\, x \in E \cap \overset{\circ}{V}_a \quad f(x) \in U^R_{+\infty}.$$

Definition 2 of infinite limit equal to $+\infty$ (in the language R–δ).

$\lim\limits_{x\to a} f(x) = +\infty :$

$$\forall\, R \in \mathbb{R} \quad \exists\, \delta > 0 \quad \forall\, x \in E, 0 < |x - a| < \delta, \quad f(x) > R.$$

Definitions 1 and 2 of infinite limit equal to $+\infty$ are equivalent.

Similarly, infinite limits equal to $-\infty$ and ∞ are defined, as well as infinite limits at the points at infinity.

Remark.

For the limit of a function at points at infinity and for infinite limits, one can prove a criterion for the existence of a limit in terms of sequences.

9. Properties of the limit of a function

Limit of a function and arithmetic operations

9A/00:00 (19:33)

THEOREM (ON ARITHMETIC PROPERTIES OF THE FUNCTION LIMIT).

Let $f, g : E \to \mathbb{R}$ be functions, let a be the limit point of E. Let $\lim_{x\to a} f(x) = A \in \mathbb{R}$, $\lim_{x\to a} g(x) = B \in \mathbb{R}$. Then

1) $\lim\limits_{x\to a}\big(f(x) + g(x)\big) = A + B$,

2) $\lim\limits_{x\to a}\big(f(x)g(x)\big) = AB$,

3) $\lim\limits_{x\to a}\Big(\dfrac{f(x)}{g(x)}\Big) = \dfrac{A}{B}$ if the following additional conditions hold:

$$\forall\, x \in E \setminus \{a\} \quad g(x) \neq 0, \quad B \neq 0.$$

PROOF.

Let $\{x_n\}$ be an arbitrary sequence of points from $E \setminus \{a\}$ converging to a. Since by condition of the theorem $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$, we obtain, by virtue of the necessary condition of the criterion for the existence of the function limit in terms of sequences:

$$\lim_{n\to\infty} f(x_n) = A, \quad \lim_{n\to\infty} g(x_n) = B. \tag{1}$$

Taking into account relation (1) and the theorem on arithmetic properties of the limit of a sequence, we obtain:

$$\lim_{n\to\infty}\big(f(x_n) + g(x_n)\big) = A + B,$$

$$\lim_{n\to\infty}\big(f(x_n)g(x_n)\big) = AB.$$

If the additional conditions specified in the item 3 of the theorem are satisfied, then $\forall\, n \in \mathbb{N}$ $g(x_n) \neq 0$, as well as $B \neq 0$, therefore

$$\lim_{n\to\infty}\Big(\frac{f(x_n)}{g(x_n)}\Big) = \frac{A}{B}.$$

Since the relations obtained are satisfied for an arbitrary sequence $\{x_n\}$ with the indicated properties, we obtain, by virtue of a sufficient condition of the same criterion, that the limits of the functions $f(x) + g(x)$, $f(x)g(x)$, $\frac{f(x)}{g(x)}$, as $x \to a$, exist and are equal, respectively, $A + B$, AB, $\frac{A}{B}$. $\square$

Passing to the limit of a function in inequalities

9A/19:33 (15:26)

Theorem 1 (first theorem on passing to the limit in inequalities for functions).

Let $f, g : E \to \mathbb{R}$ be functions, let $a \in \mathbb{R}$ be the limit point of E.

Let $\lim_{x\to a} f(x) = A \in \mathbb{R}$, $\lim_{x\to a} g(x) = B \in \mathbb{R}$.

Let $\forall\, x \in E \setminus \{a\}$ $f(x) \le g(x)$. Then $A \le B$.

Proof.

Let $\{x_n\}$ be an arbitrary sequence of points from $E \setminus \{a\}$ converging to a. Since by condition of the theorem $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$, we obtain, by virtue of the necessary condition of the criterion for the existence of the function limit in terms of sequences:

$$\lim_{n\to\infty} f(x_n) = A, \quad \lim_{n\to\infty} g(x_n) = B.$$

Further, by the condition, $\forall\, n \in \mathbb{N}$ $f(x_n) \le g(x_n)$. Thus, for the sequences $\{f(x_n)\}$ and $\{g(x_n)\}$, all the conditions of the first theorem on passing to the limit in the inequalities for sequences are satisfied. By virtue of this theorem, $A \le B$. □

Theorem 2 (second theorem on passing to the limit in inequalities for functions).

Let $f, g, h : E \to \mathbb{R}$ be functions, let $a \in \mathbb{R}$ be the limit point of E.

Let $\forall\, x \in E \setminus \{a\}$ $f(x) \le g(x) \le h(x)$.

Let $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = A \in \mathbb{R}$. Then $\lim_{x\to a} g(x) = A$.

Proof.

Let $\{x_n\}$ be an arbitrary sequence of points from $E \setminus \{a\}$ converging to a. Since by condition of the theorem $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = A$, we obtain, by the necessary condition of the criterion for the existence of the function limit in terms of sequences:

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = A.$$

Further, by the condition, $\forall\, n \in \mathbb{N}$ $f(x_n) \le g(x_n) \le h(x_n)$. Thus, for the sequences $\{f(x_n)\}$, $\{g(x_n)\}$, and $\{h(x_n)\}$, all the conditions of the second theorem on passing to the limit in inequalities for sequences are satisfied, therefore, the sequence $\{g(x_n)\}$ has a limit equal to A.

We have found that for an arbitrary sequence $\{x_n\}$ with the indicated properties, the condition $\lim_{n\to\infty} g(x_n) = A$ is fulfilled. Therefore, by virtue of a sufficient condition of the same criterion, the limit of the function $g(x)$, as $x \to a$, exists and is equal to A. □

Remark.

In a similar way, one can prove theorems on passing to the limit in inequalities for functions in the case when a is the point at infinity.

The theorem on the limit of superposition of functions

Formulation and proof

9B/00:00 (23:05)

Theorem (on the limit of superposition of functions).

Let $f : E \to G$, $g : G \to \mathbb{R}$ be functions, let a be the limit point of E. Let three conditions be fulfilled:

1) $\lim\limits_{x\to a} f(x) = b$ and b is the limit point of G;
2) $\lim\limits_{y\to b} g(y) = A$;
3) if $x \neq a$, then $f(x) \neq b$.

Then there exists the limit of superposition $g \circ f$, as $x \to a$, and this limit is A:

$$\lim_{x\to a}(g \circ f)(x) = \lim_{y\to b} g(y) = A.$$

A brief formulation of the limit superposition theorem.

The limit of superposition (if the corresponding conditions are fulfilled) is equal to the limit of the external function.

Proof.

Let $\{x_n\}$ be an arbitrary sequence of points from $E \setminus \{a\}$ converging to a: $\lim_{n\to\infty} x_n = a$.

Consider the sequence $\{y_n\}$, $y_n = f(x_n)$. By condition 3, $y_n \in G \setminus \{b\}$. By condition 1 and the necessary condition of the criterion for the existence of the function limit in terms of sequences, $\lim_{n\to\infty} y_n = b$.

Thus, for the sequence $\{y_n\}$, we have: $y_n \in G \setminus \{b\}$ and $\lim_{n\to\infty} y_n = b$. Taking into account condition 2 and again applying the necessary condition of the same criterion, we obtain:

$$\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(f(x_n)) = A.$$

Thus, for an arbitrary sequence $\{x_n\}$ of points from $E \setminus \{a\}$ converging to a, the following condition is fulfilled:

$$\lim_{n\to\infty} g(f(x_n)) = \lim_{n\to\infty} (g \circ f)(x_n) = A.$$

Therefore, by virtue of the sufficient condition of the same criterion, the limit of superposition $(g \circ f)(x)$, as $x \to a$, exists and is equal to A. $\square$

Remark.

The superposition limit theorem remains valid if some of the points a, b, A are the points at infinity.

An example showing the importance of condition 3 of the superposition limit theorem

9B/23:05 (09:53)

Consider the function $h(x) = \left|\operatorname{sign}\left(x \sin \frac{1}{x}\right)\right|$ defined in $\mathbb{R} \setminus \{0\}$.

This function can be represented as a superposition as follows:

$$h(x) = (g \circ f)(x), \text{ where } g(y) = |\operatorname{sign} y|, \quad f(x) = x \sin \frac{1}{x}.$$

We are interested in the limit of the function $h(x)$ at the point $a = 0$.

We already proved that $\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin \frac{1}{x} = 0$. We also proved that $\lim_{y\to 0} g(y) = \lim_{y\to 0} |\operatorname{sign} y| = 1$. Thus, conditions 1 and 2 of the superposition limit theorem are satisfied, and the limit of the external function is 1.

However, condition 3 is not satisfied. In this case, this condition has the form: if $x \neq 0$, then $f(x) \neq 0$. This is not true: there exists an infinite set of points $x_n = \frac{1}{2\pi n}$, $n \in \mathbb{N}$, not equal to 0 and such that $f(x_n) = x_n \sin \frac{1}{x_n} = \frac{\sin(2\pi n)}{2\pi n} = 0$.

Therefore, we cannot state that the limit of superposition $(g \circ f)(x)$ at the point 0 is equal to the limit of the external function.

Indeed, the function $h(x)$ has no limit at 0. In order to prove this, it suffices to consider two auxiliary sequences:

$$x'_n = \frac{1}{\frac{\pi}{2} + 2\pi n}, \quad x''_n = \frac{1}{2\pi n}.$$

Both of these sequences belong to $\mathbb{R} \setminus \{0\}$ and converge to zero. In the same time,

$$\lim_{n\to\infty} h(x'_n) = \lim_{n\to\infty} \left|\operatorname{sign} \frac{\sin\left(\frac{\pi}{2} + 2\pi n\right)}{\frac{\pi}{2} + 2\pi n}\right| = \lim_{n\to\infty} \left|\operatorname{sign} \frac{1}{\frac{\pi}{2} + 2\pi n}\right| = $$
$$= \lim_{n\to\infty} 1 = 1,$$

$$\lim_{n\to\infty} h(x''_n) = \lim_{n\to\infty} \left|\operatorname{sign} \frac{\sin(2\pi n)}{2\pi n}\right| = \lim_{n\to\infty} |\operatorname{sign} 0| = \lim_{n\to\infty} 0 = 0.$$

Thus, $\lim_{n\to\infty} h(x'_n) \neq \lim_{n\to\infty} h(x''_n)$. Since the necessary condition of the criterion for the existence of the function limit in terms of sequences is violated, the function $h(x)$ cannot have a limit at 0.

10. One-sided limits. Some important function limits

Definition of one-sided limits of a function

10A/00:00 (07:32)

Earlier, we proved that the function $f(x) = \operatorname{sign} x$ has no limit at the point 0. At the same time, if we limit ourselves to considering the values of this function located to the right-hand side of the point 0 ($x > 0$) or values located to the left-hand side ($x < 0$), then the limit would exist and be equal, respectively, 1 or -1.

It is natural to call such limits the *one-sided limits*.

Definition.

A *right-hand neighborhood* of a point $a \in \mathbb{R}$ is a set $\{x \in \mathbb{R} : a < x < a + \delta\}$ for some $\delta > 0$. Notation: U_a^+ or $U_a^{\delta+}$.

A *left-hand neighborhood* of a point $a \in \mathbb{R}$ is a set $\{x \in \mathbb{R} : a - \delta < x < a\}$ for some $\delta > 0$. Notation: U_a^- or $U_a^{\delta-}$.

Thus, $U_a^{\delta+} = (a, a + \delta)$, $U_a^{\delta-} = (a - \delta, a)$. It should be noted that the point a itself is not included in its right-hand or left-hand neighborhood.

Definition 1 of one-sided limits (in the language of neighborhoods).

Let A be either a real number or one of the points at infinity, $f : E \to \mathbb{R}$ be a function, $a \in \mathbb{R}$ be the limit point of E.

The point A is called the *right-hand limit* of the function f, as $x \to a$, if a is the limit point of the set $E_a^+ = \{x \in E : x > a\}$ and

$$\forall\, U_A \quad \exists\, V_a^+ \quad \forall\, x \in E \cap V_a^+ \quad f(x) \in U_A.$$

Notation: $\lim\limits_{x \to a+0} f(x) = A$.

The point A is called the *left-hand limit* of the function f, as $x \to a$, if a is the limit point of the set $E_a^- = \{x \in E : x < a\}$ and

$$\forall\, U_A \quad \exists\, V_a^- \quad \forall\, x \in E \cap V_a^- \quad f(x) \in U_A.$$

Notation: $\lim\limits_{x \to a-0} f(x) = A$.

Definition 2 of finite one-sided limits (in the language ε–δ):

$$\lim_{x\to a+0} f(x) = A \in \mathbb{R}, \quad a \in \mathbb{R}:$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in E, a < x < a+\delta, \quad |f(x) - A| < \varepsilon.$$

$$\lim_{x\to a-0} f(x) = A \in \mathbb{R}, \quad a \in \mathbb{R}:$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in E, a-\delta < x < a, \quad |f(x) - A| < \varepsilon.$$

A definition similar to definition 2 can also be given for the case when A is the point at infinity.

Definitions 1 and 2 of one-sided limits are equivalent.

Remarks.

1. We can also define one-sided neighborhoods of the point ∞ assuming that any neighborhood of $+\infty$ is a right-hand neighborhood of ∞ and any neighborhood of $-\infty$ is a left-hand neighborhood of ∞. Then the limit of a function, as $x \to +\infty$, is the right-hand limit at the point ∞ and the limit of a function at $x \to -\infty$ is the left-hand limit at the point ∞.

2. For one-sided limits of functions, all the theorems considered above for usual (two-sided) limits are true. In particular, the criterion for the existence of a one-sided limit of a function in terms of sequences holds. In this criterion for the right-hand and left-hand limits, we should consider the sequences $\{x_n\}$ such that $x_n \in E$, $\{x_n\}$ converges to a, and its elements satisfy the conditions $x_n > a$ and $x_n < a$, respectively.

Criterion for the existence of the limit of a function in terms of one-sided limits

10A/07:32 (14:50)

Theorem (criterion for the existence of the limit of a function in terms of one-sided limits).

Let $f : E \to \mathbb{R}$ be a function, the point $a \in \mathbb{R} \cup \{\infty\}$ be the limit point of the sets E_a^+ and E_a^-. Then the function f has a limit, as $x \to a$, if and only if the function f has right-hand and left-hand limits at a and these limits coincide. In this case, the value of the usual limit is equal to the values of one-sided limits:

$$\Bigl(\lim_{x\to a} f(x) = A\Bigr) \Leftrightarrow \Bigl(\lim_{x\to a+0} f(x) = A, \quad \lim_{x\to a-0} f(x) = A\Bigr).$$

Proof.

1. Let us prove the necessity.

Given: there exists the usual limit $\lim_{x\to a} f(x) = A$ (A is either a real number or one of the points at infinity).

Prove: $\lim_{x\to a+0} f(x) = \lim_{x\to a-0} f(x) = A$.

By definition of the usual limit in the language of neighborhoods, we have:

$$\forall\, U_A \quad \exists\, \overset{\circ}{V}_a \quad \forall\, x \in E \cap \overset{\circ}{V}_a \quad f(x) \in U_A. \tag{1}$$

Any punctured neighborhood $\overset{\circ}{V}_a$ can be represented as the union of two one-sided neighborhoods: $\overset{\circ}{V}_a = V_a^+ \cup V_a^-$. Moreover, if $x \in E$ belongs to V_a^+ or V_a^-, then it belongs to $\overset{\circ}{V}_a$, and the condition $f(x) \in U_A$ is satisfied for it. Thus, we obtain:

$$\forall\, U_A \quad \exists\, V_a^+ \quad \forall\, x \in E \cap V_a^+ \quad f(x) \in U_A, \tag{2}$$

$$\forall\, U_A \quad \exists\, V_a^- \quad \forall\, x \in E \cap V_a^- \quad f(x) \in U_A. \tag{3}$$

Relations (2) and (3) are the definitions of the right-hand and left-hand limit A of the function f at the point a. The necessity is proven.

2. Now we prove sufficiency.

Given: there exist one-sided limits, and their values are equal to A (A is either a real number or one of the points at infinity):

$$\lim_{x\to a+0} f(x) = \lim_{x\to a-0} f(x) = A.$$

Prove: $\lim_{x\to a} f(x) = A$.

By the definition of one-sided limits in the language of neighborhoods, relations (2) and (3) hold for any neighborhood U_A.

Let us define the punctured neighborhood $\overset{\circ}{V}_a$ as the union of the one-sided neighborhoods V_a^+ and V_a^- indicated in (2) and (3). Then, if $x \in E$ belongs to the neighborhood $\overset{\circ}{V}_a$ then it belongs to either V_a^+ or V_a^- and, by virtue of relations (2) and (3), the condition $f(x) \in U_A$ is fulfilled. Thus, for any neighborhood of U_A, relation (1) holds. This means that there exists the usual limit A of the function f at the point a. The sufficiency is proven. □

Remark.

The approval of the criterion holds only when the point a is simultaneously the limit point of the sets E_a^+ and E_a^-. If the point a is not the limit point of E_a^+ and at the same time is the limit point of E_a^- then the definitions of the usual and left-hand limit of the function f at a coincide. If the point a is not the limit point of E_a^- and at the same time is the limit point of E_a^+ then the definitions of the usual and right-hand limit of the function f at a coincide.

The first remarkable limit

10A/22:22 (20:00)

THEOREM (THE FIRST REMARKABLE LIMIT).

The following limit, called the *first remarkable limit*, holds:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

PROOF.

Firstly, consider the case $x \in \left(0, \frac{\pi}{2}\right)$. On the coordinate plane, we construct a ray whose angle with the axis OX is equal to x radians and we denote by C its intersection point with the unit circle. The point C has the coordinates $(\cos x, \sin x)$. From the point B with coordinates $(\cos x, 0)$, we draw an arc of radius $\cos x$ to the intersection with the ray OC and we denote the intersection point by A. Also we denote the point with coordinates $(1, 0)$ by D (Fig. 5).

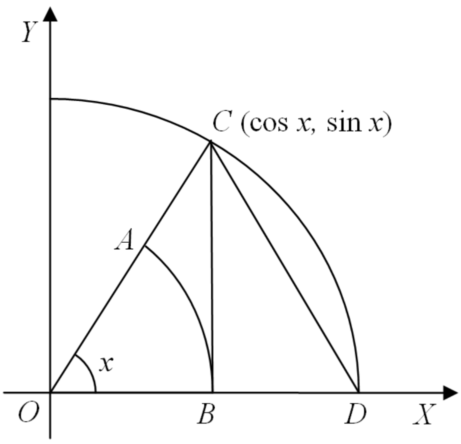

Fig. 5. Illustration of the proof of the first remarkable limit

We get three nested geometric objects: the OAB sector, the OCD triangle, and the OCD sector.

Recall the sector area formula: $\frac{\alpha r^2}{2}$, where α is the angle of the sector in radians and r is its radius.

The OAB sector has an angle of x radians and a radius of $\cos x$, so its area S_{OAB} is $\frac{x\cos^2 x}{2}$. The OCD sector has an angle of x radians and a radius of 1, so its area S_{OCD} is $\frac{x}{2}$. The OCD triangle has a base of 1 and a height of $\sin x$, so its area S_Δ is $\frac{\sin x}{2}$.

Since the OCD triangle contains the OAB sector and is contained in the OCD sector, we obtain that the following inequality holds for the areas:

$$S_{OAB} < S_\Delta < S_{OCD},$$

$$\frac{x\cos^2 x}{2} < \frac{\sin x}{2} < \frac{x}{2}.$$

Multiply all parts of the resulting double inequality by 2 and divide by x. Since $x > 0$, the signs of inequality will not change:

$$\cos^2 x < \frac{\sin x}{x} < 1. \tag{4}$$

Let us show that $\lim_{x\to+0} \cos^2 x = 1$. Indeed, for $x \in \left(0, \frac{\pi}{2}\right)$ we have the double estimate $0 < \sin x < x$. When passing to the limit as $x \to +0$, the boundary terms of this estimate approach 0, therefore the middle term also approaces 0, due to the second theorem on the passing to the limit in inequalities for functions (recall that this theorem holds for both usual and one-sided limits). We have proved that $\lim_{x\to+0} \sin x = 0$. But then, using the arithmetic properties of the limit of the function (which also hold for both usual and one-sided limits), we obtain:

$$\lim_{x\to+0} \cos^2 x = \lim_{x\to+0} (1 - \sin^2 x) = \lim_{x\to+0} 1 - \lim_{x\to+0} \sin^2 x = 1 - 0 = 1.$$

Thus, the boundary terms of double estimate (4) approach 1 as $x \to +0$. Therefore, by the same theorem on the passing to the limit in inequalities, the middle term of double estimate (4) also has a limit equal to 1:

$$\lim_{x\to+0} \frac{\sin x}{x} = 1. \tag{5}$$

Now consider the situation $x < 0$ and find the limit of the function $\frac{\sin x}{x}$ as $x \to -0$. We represent the function $\frac{\sin x}{x}$ as a superposition $\frac{\sin(-y)}{-y} \circ (-x)$. Its internal function is the function $f(x) = -x$, and its external function is the function $g(y) = \frac{\sin(-y)}{-y}$. Taking into account that the function sin is odd, we can transform the function $g(y)$ as follows:

$$g(y) = \frac{\sin(-y)}{-y} = \frac{-\sin y}{-y} = \frac{\sin y}{y}.$$

The function $f(x) = -x$ approaches $+0$ when x approaches -0. Therefore, the limit of superposition, by virtue of the theorem on the limit of superposition, will be equal to the limit of the external function $g(y)$ as $y \to +0$:

$$\lim_{x\to-0} \frac{\sin x}{x} = \lim_{y\to+0} g(y) = \lim_{y\to+0} \frac{\sin y}{y} = 1. \tag{6}$$

We proved that for the function $\frac{\sin x}{x}$ there exist right-hand and left-hand limits at the point 0 and they are equal to 1 (see (5) and (6)). Therefore, by virtue of the criterion for the existence of the limit in terms of one-sided limits, for this function there also exists a usual limit at the point 0, which is equal to 1. □

Definition.

If $\lim_{x\to a}\frac{f(x)}{g(x)}=1$, then they say that the *functions f and g are equivalent as $x\to a$*, and this fact is written as follows: $f(x)\sim g(x)$, $x\to a$.

Thus, we have proved the following equivalence of functions:

$$\sin x\sim x,\quad x\to 0. \tag{7}$$

Using this equivalence and the theorem on the limit of superposition, it is easy to prove more general result:

$$\sin f(x)\sim f(x),\quad x\to a,\quad \text{if } \lim_{x\to a} f(x)=0. \tag{8}$$

Further, in the final section of Chapter 15, we prove that the limit of the product does not change if one of its *factors* is replaced by an equivalent function. This fact simplifies the calculation of limits. It should be emphasized that one can replace functions with equivalent ones only in *products*, but not in sums.

Example.

Consider the limit $\lim_{x\to 0}\frac{\sin 3x}{x}$. Since $3x\to 0$ as $x\to 0$, we obtain that $\sin 3x\sim 3x$ as $x\to 0$. Replacing the numerator in the original expression with an equivalent function, we finally get:

$$\lim_{x\to 0}\frac{\sin 3x}{x}=\lim_{x\to 0}\frac{3x}{x}=\lim_{x\to 0}3=3.$$

Remark.

When calculating the limit in the above example, we could use the superposition limit theorem, but this would require more complicated transformations:

$$\lim_{x\to 0}\frac{\sin 3x}{x}=\lim_{x\to 0}\frac{3\sin 3x}{3x}=3\lim_{x\to 0}\left(\frac{\sin y}{y}\circ 3x\right)=3\lim_{y\to 0}\frac{\sin y}{y}=3.$$

Based on equivalence (7), it is easy to prove the following useful equivalences for trigonometric functions at the point 0:

$$\tan x\sim x,\quad x\to 0,$$

$$1-\cos x\sim\frac{x^2}{2},\quad x\to 0.$$

For these equivalences, one can also obtain generalizations similar to (8).

The second remarkable limit

10B/00:00 (25:30)

Before formulating and proving the theorem on the second remarkable limit, we consider two simpler limits, for which we use the same technique based on the superposition limit theorem.

Example 1.

Let us show that $\lim_{x\to+\infty}\frac{x}{a^x}=0$ if $a>1$.

If we use the function $[x]$ – the integer part of the number x, then the function $\frac{x}{a^x}$ can be estimated from above and from below as follows:

$$0<\frac{x}{a^x}<\frac{[x]+1}{a^{[x]}}. \tag{9}$$

In this estimate, we used the following property of the integer part:

$$\forall\, x\in\mathbb{R}\quad [x]\le x<[x]+1.$$

The limit of the function $\frac{[x]+1}{a^{[x]}}$ can be found using the well-known limit for the sequence $\lim_{n\to\infty}\frac{n}{a^n}=0$ (see the final section of Chapter 5) and the superposition limit theorem. As an internal function, we will use the function $n(x)=[x]$, which takes integer values:

$$\lim_{x\to+\infty}\frac{[x]+1}{a^{[x]}}=\lim_{x\to+\infty}\left(\frac{n+1}{a^n}\circ[x]\right).$$

Since $\lim_{x\to+\infty}[x]=+\infty$, we finally obtain:

$$\lim_{x\to+\infty}\left(\frac{n+1}{a^n}\circ[x]\right)=\lim_{n\to\infty}\frac{n+1}{a^n}=\lim_{n\to\infty}\frac{n}{a^n}+\lim_{n\to\infty}\frac{1}{a^n}=0.$$

Recall that we write $n\to\infty$ without specifying a "+" sign for limits of sequences.

Thus, when passing to the limit, as $x\to+\infty$, in the double inequality (9), its boundary terms approach 0. Then, by the second theorem on passing to the limit in inequalities for functions, the middle term of inequality (9) will also have a limit equal to 0.

Example 2.

Let us show that $\lim_{x\to+\infty}\frac{\log_a x}{x}=0$ if $a>1$.

We represent the function $\frac{\log_a x}{x}$ in the form of a superposition using the fact that $x=a^{\log_a x}$, and then we apply the theorem on the limit of superposition:

$$\lim_{x\to+\infty}\frac{\log_a x}{x}=\lim_{x\to+\infty}\left(\frac{y}{a^y}\circ\log_a x\right)=\lim_{y\to+\infty}\frac{y}{a^y}.$$

It is proved in example 1 that the last limit is 0.

Remark.

The above examples show that the exponential function a^x (for $a>1$) grows most rapidly on $+\infty$ of all the considered functions, the linear function x grows more slowly, and the logarithmic function $\log_a x$ has the slowest growth (for $a>1$). The result can be generalized by proving that any power function x^α ($\alpha>0$) grows at $+\infty$ slower than any exponential function and faster than any logarithmic function.

Theorem (the second remarkable limit).

The following limit, called the *second remarkable limit*, holds:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Proof.

Earlier, we already studied a similar limit for the sequence (see the final section of Chapter 5):

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

The difference is that now the argument x is real and the limit is two-sided (whereas for a sequence, the number n can only approach $+\infty$, although this is not indicated explicitly in the formula).

As with the study of the first remarkable limit, we will divide the proof into two stages. Firstly, we prove the existence of the limit, as $x \to +\infty$. To do this, we evaluate the initial function from above and from below as follows:

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{[x]}\right)^{[x]+1}.$$

Given the known limit for the sequence $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and applying the superposition limit theorem in the same way as in Example 1, it is easy to prove that the boundary terms of this double inequality approach e when x approaches $+\infty$. Therefore, by the second theorem on passing to the limit in inequalities for functions, the middle term of the inequality also has the limit e:

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e.$$

At the second stage of proof, consider the case $x \to -\infty$. Let us represent the original expression as a superposition with the internal function $f(x) = -x$ and transform the external function $g(y)$ so that the previously obtained result can be used for the transformed expression:

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(\left(1+\frac{1}{-y}\right)^{-y}\circ(-x)\right) =$$

$$= \lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y} = \lim_{y\to+\infty}\left(\frac{y-1}{y}\right)^{-y} =$$

$$= \lim_{y\to+\infty}\left(\frac{y}{y-1}\right)^{y} = \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y} =$$

$$= \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right) = e\cdot 1 = e.$$

We have proved that the one-sided limits of the function $\left(1 + \frac{1}{x}\right)^x$, as $x \to +\infty$ and $x \to -\infty$, exist and are equal to e. Therefore, by virtue of the criterion for the existence of the limit in terms of one-sided limits, there exists a limit for this function, as $x \to \infty$, which is also equal to e. □

From the second remarkable limit, a number of useful equivalences can be obtained:

$$e^x - 1 \sim x, \quad x \to 0,$$

$$\ln(1 + x) \sim x, \quad x \to 0,$$

$$(1 + x)^\alpha \sim 1 + \alpha x, \quad x \to 0.$$

We will prove these equivalences later (see the final section of Chapter 12).

11. The limits of monotone bounded functions. Cauchy criterion for functions

Monotone and bounded functions

Monotone functions and bounded functions: definitions

10B/25:30 (10:13)

Let f be a function acting from E to $\mathbb{R}$.

DEFINITION.

The function f is called *increasing* if

$$\forall\, x_1, x_2 \in E, x_1 < x_2, \quad f(x_1) < f(x_2).$$

A function f is called *non-decreasing* if

$$\forall\, x_1, x_2 \in E, x_1 < x_2, \quad f(x_1) \le f(x_2).$$

A function f is called *decreasing* if

$$\forall\, x_1, x_2 \in E, x_1 < x_2, \quad f(x_1) > f(x_2).$$

A function f is called *non-increasing* if

$$\forall\, x_1, x_2 \in E, x_1 < x_2, \quad f(x_1) \ge f(x_2).$$

Non-increasing and non-decreasing functions are called *monotonic*, and increasing and decreasing functions are called *strictly monotonic*.

DEFINITION.

A function f is called *bounded from above*, or *upper-bounded*, if

$$\exists\, M \in \mathbb{R} \quad \forall\, x \in E \quad f(x) \le M.$$

A function f is called *bounded from below*, or *lower-bounded*, if

$$\exists\, m \in \mathbb{R} \quad \forall\, x \in E \quad f(x) \ge m.$$

A function f is called *bounded* if

$$\exists\, M > 0 \quad \forall\, x \in E \quad |f(x)| \le M.$$

The boundedness of a function is equivalent to the boundedness of its image $f(E) = \{y \in \mathbb{R} : (\exists\, x \in E \quad f(x) = y)\}$.

For a bounded function, we assume by definition:

$$\sup_{x \in E} f(x) \overset{\text{def}}{=} \sup f(E), \qquad \inf_{x \in E} f(x) \overset{\text{def}}{=} \inf f(E).$$

Theorems on the limits of monotone bounded functions

10B/35:43 (02:35), 11A/00:00 (16:34)

Theorem 1 (on the limit of monotone upper-bounded function).

Let $f : E \to \mathbb{R}$ be a non-decreasing upper-bounded function. If the set E is bounded from above, then we denote $a = \sup E$, and we additionally require that a be the limit point of E. If the set E is not bounded from above, then we denote $a = +\infty$. Then there exists a finite limit of the function f, as $x \to a$, which is equal to $\sup_{x \in E \setminus \{a\}} f(x)$:

$$\lim_{x \to a} f(x) = \sup_{x \in E \setminus \{a\}} f(x) \in \mathbb{R}.$$

Remarks.

1. The point $a = \sup E$ is the limit point of the set E if and only if $a = \sup(E \setminus \{a\})$.

2. In fact, the theorem deals with the one-sided limit, since the elements of the set E can approach the point a only from the left.

Proof.

Let us prove the theorem for the case when the set E is bounded. The case of an unbounded set E is considered in a similar way (compare with the proof of the existence of a limit for monotone bounded sequences).

We denote $A = \sup_{x \in E \setminus \{a\}} f(x)$ and show that $\lim_{x \to a} f(x) = A$.

By definition of the least upper bound,

$$\forall\, x \in E \setminus \{a\} \quad f(x) \le A, \tag{1}$$

$$\forall\, \varepsilon > 0 \quad \exists\, x_\varepsilon \in E \setminus \{a\} \quad f(x_\varepsilon) > A - \varepsilon. \tag{2}$$

Since $x_\varepsilon \in E \setminus \{a\}$ and $a = \sup E$, we obtain $x_\varepsilon < a$.

Denote $\delta = a - x_\varepsilon > 0$ and consider the punctured neighborhood

$$\overset{\circ}{V}{}_a^\delta = (a - \delta, a + \delta) \setminus \{a\}.$$

Let $x \in E \cap \overset{\circ}{V}{}_a^\delta$. Then $x > a - \delta = a - (a - x_\varepsilon) = x_\varepsilon$, and, since f is non-decreasing, the inequality $x > x_\varepsilon$ implies $f(x) \ge f(x_\varepsilon)$. In view of estimate (2), we get $f(x) > A - \varepsilon$.

On the other hand, for any x from $E \setminus \{a\}$, in particular, for all $x \in E \cap \overset{\circ}{V}{}_a^\delta$, estimate (1) holds: $f(x) \le A$.

Thus, for all $x \in E \cap \overset{\circ}{V}{}_a^\delta$ we obtain a chain of inequalities $A - \varepsilon < f(x) \le A < A + \varepsilon$, which implies that $f(x) \in U_A^\varepsilon$.

We have proved that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in E \cap \overset{\circ}{V}{}_a^\delta$ the condition $f(x) \in U_A^\varepsilon$ holds. This means, by the definition of the limit of a function in the language of symmetric neighborhoods, that the function $f(x)$ has a limit at a equal to A. □

The following theorem can be proved in a similar way.

Theorem 2 (on the limit of monotone lower-bounded function).

Let $f : E \to \mathbb{R}$ be a non-increasing lower-bounded function. If the set E is bounded from above, then we denote $a = \sup E$, and we additionally require that a be the limit point of E. If the set E is not bounded from above, then we denote $a = +\infty$. Then there exists a finite limit of the function f, as $x \to a$, which is equal to $\inf_{x \in E \setminus \{a\}} f(x)$:

$$\lim_{x \to a} f(x) = \inf_{x \in E \setminus \{a\}} f(x) \in \mathbb{R}.$$

Remark.

Similar theorems on the limits of monotone bounded functions are also valid for the left endpoint b of their domain E: $b = -\infty$ if the set E is not bounded from below, $b = \inf E$ if E is bounded from below (the point b must be the limit point of E).

In particular, if f is non-decreasing and lower-bounded, then there exists a finite limit, as $x \to b$, which is equal to $\inf_{x \in E \setminus \{b\}} f(x)$.

To prove these theorems, it suffices to consider the auxiliary function $g(x) = f(-x)$ and apply theorems 1 and 2 to it.

Cauchy criterion for the existence of the function limit

Formulation of the criterion and proof of necessity

11A/16:34 (09:39)

Theorem (Cauchy criterion for the existence of the limit of a function).

Let $f : E \to \mathbb{R}$ be a function, a be a limit point of E. In order for the function $f(x)$ to have a finite limit, as $x \to a$, it is necessary and sufficient that the following condition is satisfied:

$$\forall\, \varepsilon > 0 \quad \exists\, \overset{\circ}{V}_a \quad \forall\, x', x'' \in E \cap \overset{\circ}{V}_a \quad |f(x') - f(x'')| < \varepsilon. \tag{3}$$

Proof.

1. Let us prove the necessity. Given: there exists $\lim_{x \to a} f(x) = A \in \mathbb{R}$. Prove: condition (3) is satisfied.

By the definition of the limit of a function in the language ε–δ, we have:

$$\forall\,\varepsilon > 0 \quad \exists\,\delta > 0 \quad \forall\,x \in E, 0 < |x - a| < \delta, \quad |f(x) - A| < \frac{\varepsilon}{2}.$$

Conditions $x \in E$, $0 < |x - a| < \delta$ are equivalent to the condition $x \in E \cap \overset{\circ}{V}{}_a^{\delta}$, which uses a punctured symmetric neighborhood $\overset{\circ}{V}{}_a^{\delta}$ of the point a.

Let us choose arbitrary $x', x'' \in E \cap \overset{\circ}{V}{}_a^{\delta}$. Then the following estimates are fulfilled for them: $|f(x') - A| < \frac{\varepsilon}{2}$, $|f(x'') - A| < \frac{\varepsilon}{2}$, and we obtain:

$$|f(x') - f(x'')| = |f(x') - A + A - f(x'')| \le$$
$$\le |f(x') - A| + |A - f(x'')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have obtained that condition (3) is satisfied for any ε and the neighborhood $\overset{\circ}{V}{}_a^{\delta}$ depending on it. The necessity is proven.

Proof of sufficiency 11A/26:13 (12:19), 11B/00:00 (12:04)

2. Let us prove sufficiency. Given: condition (3) is satisfied. Prove: there exists a finite limit of the function f at the point a.

2.1. Let $\{x_n\}$ be an arbitrary sequence satisfying the conditions

$$x_n \in E \setminus \{a\}, \qquad \lim_{n\to\infty} x_n = a.$$

By the criterion for the existence of the function limit in terms of sequences, it suffices to prove that the sequence $\{f(x_n)\}$ is convergent and its limit does not depend on the sequence $\{x_n\}$.

Let us show that the sequence $\{f(x_n)\}$ is fundamental.

By condition, $x_n \to a$, as $n \to \infty$, and $x_n \in E \setminus \{a\}$. This means that for any punctured neighborhood $\overset{\circ}{V}_a$, there exists a natural N such that for all $n > N$ the condition $x_n \in \overset{\circ}{V}_a$ is satisfied:

$$\forall\,\overset{\circ}{V}_a \quad \exists\,N \in \mathbb{N} \quad \forall\,n > N \quad x_n \in \overset{\circ}{V}_a. \tag{4}$$

Let us use condition (3), choose an arbitrary $\varepsilon > 0$ and construct from it a neighborhood $\overset{\circ}{V}_a$ such that $\forall\,x', x'' \in E \cap \overset{\circ}{V}_a$ the estimate $|f(x') - f(x'')| < \varepsilon$ holds. For this neighborhood $\overset{\circ}{V}_a$, by virtue of (4), there exists N such that $\forall\,n > N$ $x_n \in \overset{\circ}{V}_a$. If we choose arbitrary numbers $m > N$, $n > N$, then $x_m, x_n \in \overset{\circ}{V}_a$. In addition, $x_m, x_n \in E$. Therefore, $x_m, x_n \in E \cap \overset{\circ}{V}_a$ and, by virtue of condition (3), we obtain that $|f(x_m) - f(x_n)| < \varepsilon$.

Removing from this chain of reasoning the mention of the neighborhood $\overset{\circ}{V}_a$, we get:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N, n > N \quad |f(x_m) - f(x_n)| < \varepsilon.$$

This means that the sequence $\{f(x_n)\}$ is fundamental and, by virtue of the Cauchy criterion for sequences, has a limit.

We have shown that for any sequence $\{x_n\}$ such that $x_n \in E \setminus \{a\}$ and $\lim_{n\to\infty} x_n = a$, the sequence $\{f(x_n)\}$ has a limit.

2.2. It remains for us to prove that the limit of the sequence $\{f(x_n)\}$ does not depend on the choice of the sequence $\{x_n\}$.

Let us prove this statement by contradiction.

Suppose that there exist two sequences $\{x'_n\}$, $\{x''_n\}$ that satisfy the same conditions and such that $\lim_{n\to\infty} f(x'_n) = A'$, $\lim_{n\to\infty} f(x''_n) = A''$ and, moreover, $A' \neq A''$.

Let us construct the sequence $\{x_n\}$, which contains alternating elements of the sequences $\{x'_n\}$ and $\{x''_n\}$:

$$\{x_n\} = \{x'_1, x''_1, x'_2, x''_2, x'_3, x''_3, \ldots, x'_k, x''_k, \ldots\}.$$

Elements of the sequence $\{x_n\}$ can be defined as follows:

$$x_n = \begin{cases} x'_k, & n = 2k - 1, \\ x''_k, & n = 2k. \end{cases}$$

By construction, $x_n \in E \setminus \{a\}$. In addition, this sequence converges to a. Indeed, since $x'_n \to a$ and $x''_n \to a$, we obtain that for any neighborhood U_a

$$\exists\, N' \in \mathbb{N} \quad \forall\, n > N' \quad x'_n \in U_a,$$

$$\exists\, N'' \in \mathbb{N} \quad \forall\, n > N'' \quad x''_n \in U_a.$$

This means that for the sequence $\{x_n\}$, all its elements, starting from some number depending on N' and N'', belong to U_a, that is, $\lim_{n\to\infty} x_n = a$.

Since $x_n \in E\setminus\{a\}$ and $\lim_{n\to\infty} x_n = a$, we obtain, by the result established in stage 2.1 of the proof, that there exists a limit $\lim_{n\to\infty} f(x_n)$ equal to some value A.

But then the sequences $\{f(x'_n)\}$ and $\{f(x''_n)\}$ should also converge to A as subsequences of a convergent sequence. Thus, $A' = A'' = A$, which contradicts our assumption that $A' \neq A''$.

The obtained contradiction means that for any sequences $\{x_n\}$ such that $x_n \in E \setminus \{a\}$ and $\lim_{n\to\infty} x_n = a$, the sequences $\{f(x_n)\}$ converge to the same limit A. Therefore, by virtue of the criterion for the existence of the function limit in terms of sequences, there exists a limit of the function f at the point a. ☐

12. Continuity of function at a point

Definition of a continuous function at a point

11B/12:04 (09:27)

DEFINITION.

Let $f: E \to \mathbb{R}$ be a function, $x_0 \in E$. The function f is called *continuous at the point* x_0 if for any neighborhood $U_{f(x_0)}$ there exists a neighborhood V_{x_0} such that for any $x \in V_{x_0} \cap E$ the value of $f(x)$ belongs to $U_{f(x_0)}$:

$$\forall\, U_{f(x_0)} \quad \exists\, V_{x_0} \quad \forall\, x \in V_{x_0} \cap E \quad f(x) \in U_{f(x_0)}. \tag{1}$$

This definition is very similar to the definition of the limit of the function f at the point x_0 (in the language of neighborhoods), if we additionally assume that x_0 is a limit point of E and the limit is $f(x_0)$:

$$\forall\, U_{f(x_0)} \quad \exists\, \overset{\circ}{V}_{x_0} \quad \forall\, x \in \overset{\circ}{V}_{x_0} \cap E \quad f(x) \in U_{f(x_0)}.$$

Moreover, instead of the punctured neighborhood $\overset{\circ}{V}_{x_0}$, we can consider the usual neighborhood V_{x_0}, since the condition $f(x) \in U_{f(x_0)}$ also remains valid when $x = x_0$.

Thus, if x_0 is a limit point of E, then the continuity of the function f at a given point is equivalent to the fact that there exists a limit of this function as $x \to x_0$ and this limit is equal to $f(x_0)$:

$$\big(f \text{ is continuous at the limit point } x_0\big) \Leftrightarrow \big(\lim_{x \to x_0} f(x) = f(x_0)\big).$$

If x_0 is not a limit point of the set E, then it is called an *isolated point* of the set E. In this case, there exists a neighborhood V_{x_0} such that it does not contain any point of the set E, except for the point x_0 itself. Then, choosing this neighborhood V_{x_0} for any neighborhood $U_{f(x_0)}$, we can ensure the validity of condition (1), since the point x_0 will be the only point in the set $V_{x_0} \cap E$, and for it the condition $f(x_0) \in U_{f(x_0)}$ is always satisfied. Thus, *any function is continuous at any isolated point of its domain of definition.*

The case of an isolated point is not interesting, therefore, as a rule, in what follows, we will assume that the point, at which the continuity of a function is studied, is always a limit point of the domain of definition of this function.

Remark.

The definition of the continuity of the function f at the point x_0 can also be formulated in the language ε–δ:

$$\forall\,\varepsilon > 0 \quad \exists\,\delta > 0 \quad \forall\,x \in E, |x - x_0| < \delta, \quad |f(x) - f(x_0)| < \varepsilon. \qquad (2)$$

Examples of continuous functions

11B/21:31 (09:55)

To prove continuity, we will use the definition in the language ε–δ (see (2)).

1. Constant function: $f(x) = c$, $x \in \mathbb{R}$.

To prove the continuity of this function at any point $x_0 \in \mathbb{R}$, it suffices to note that for any points $x_0, x \in \mathbb{R}$ $|f(x) - f(x_0)| = 0$.

2. Linear function: $f(x) = x$, $x \in \mathbb{R}$.

Let us choose an arbitrary point $x_0 \in \mathbb{R}$ and an arbitrary number $\varepsilon > 0$. Let $\delta = \varepsilon$. Then the inequality $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| = |x - x_0| < \delta = \varepsilon.$$

Therefore, condition (2) is satisfied. So, the linear function is continuous at any point.

3. $f(x) = \sin x$.

Let us choose an arbitrary point $x_0 \in \mathbb{R}$ and an arbitrary number $\varepsilon > 0$. Let $\delta = \varepsilon$. Using the estimate $|\sin x| \le |x|$, which is valid for all $x \in \mathbb{R}$, and assuming that the inequality $|x - x_0| < \delta$ holds, we obtain:

$$|\sin x - \sin x_0| = \left|2\sin\frac{x - x_0}{2}\cos\frac{x + x_0}{2}\right| \le \left|2\sin\frac{x - x_0}{2}\right| \le$$

$$\le \left|2 \cdot \frac{x - x_0}{2}\right| = |x - x_0| < \delta = \varepsilon.$$

Therefore, condition (2) is satisfied, and the sine function is continuous at any point.

Similarly, using the identity $\cos x - \cos x_0 = -2\sin\frac{x-x_0}{2}\sin\frac{x+x_0}{2}$, the continuity of the cosine function can be proved.

4. Exponential function: $f(x) = a^x$, $a > 0$.

It is easy to prove that $\lim_{x\to\infty} a^{1/x} = 1$ (compare with the proof of the second remarkable limit). This relation differs from the previously proved relation for the sequence $\lim_{n\to\infty} a^{1/n} = 1$ in that now the argument x is not a positive integer, but a real number, and x approaches ∞, that is, it can take both positive and negative values.

Recall the proof scheme. First, the limit as $x \to +\infty$ is considered. In the case $a > 1$, the double relation $a^{\frac{1}{[x]+1}} \le a^{\frac{1}{x}} \le a^{\frac{1}{[x]}}$ is used and then we apply the superposition limit theorem and the second theorem on the transition to the limit in inequalities for functions. Similarly, we can consider the limit at the same point $+\infty$ for the case $a \in (0, 1)$. To prove the existence of a limit at the point $-\infty$, it suffices to use the relation $a^{1/x} = \left(\frac{1}{a}\right)^{-1/x}$. The existence of a limit as $x \to \infty$ follows from the criterion for the existence of a function limit in terms of one-sided limits.

From the relation $\lim_{x\to\infty} a^{1/x} = 1$, using the superposition limit theorem, we can obtain the relation $\lim_{x\to 0} a^x = 1$:

$$\lim_{x\to 0} a^x = \lim_{x\to 0}\left(a^{1/y} \circ \frac{1}{x}\right) = \lim_{y\to\infty} a^{1/y} = 1.$$

Then, for arbitrary x and x_0, we have:

$$\lim_{x\to x_0} (a^x - a^{x_0}) = \lim_{x\to x_0} a^{x_0}(a^{x-x_0} - 1) = 0.$$

Therefore, $\lim_{x\to x_0} a^x = a^{x_0}$, which is equivalent to the continuity of the function a^x at the point x_0.

The continuity of other elementary functions will be established later, after studying the additional properties of continuous functions. In particular, in the final section of Chapter 14, we will describe a method for proving that the function $\log_a x$ $(a > 0,\ a \neq 1)$ is continuous at any point $x \in (0, +\infty)$.

Simplest properties of continuous functions

11B/31:26 (08:09)

Theorem (on the simplest properties of continuous functions).

Let the function $f : E \to \mathbb{R}$ be continuous at the point x_0. Assume, in addition, $f(x_0) \neq 0$. Then the following two statements are true:

1) $\exists\, V_{x_0} \quad \forall\, x \in V_{x_0} \cap E \quad f(x) \neq 0,$

2) $\exists\, V_{x_0} \quad \forall\, x \in V_{x_0} \cap E \quad \operatorname{sign} f(x) = \operatorname{sign} f(x_0).$

In other words, if a continuous function at a point x_0 is non-zero at this point then it is non-zero in some neighborhood of this point, and, moreover, it retains its sign in this neighborhood.

Proof.

We will use the definition of continuity in the language ε–δ:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in E, |x - x_0| < \delta \quad |f(x) - f(x_0)| < \varepsilon.$$

The last estimate can be rewritten as a double inequality:

$$f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon.$$

By assumption, $f(x_0) = A \neq 0$. We first consider the case $A > 0$.

Let $\varepsilon = \frac{A}{2}$. Then there exists δ such that for all $x \in E$, $|x - x_0| < \delta$, the left-hand side of the above double inequality takes the form:

$$f(x) > f(x_0) - \varepsilon = A - \frac{A}{2} = \frac{A}{2} > 0.$$

Thus, for a symmetric neighborhood $V_{x_0}^{\delta}$, we have:

$$\forall\, x \in V_{x_0}^{\delta} \cap E \quad f(x) > 0.$$

In the case of $A < 0$, it suffices to choose $\varepsilon = \frac{|A|}{2}$. Given the definition of the absolute value, we obtain: $\varepsilon = -\frac{A}{2}$. Then there exists δ such that the right-hand side of the double inequality takes the form

$$f(x) < f(x_0) + \varepsilon = A - \frac{A}{2} = \frac{A}{2} < 0.$$

In this case, for a symmetric neighborhood $V_{x_0}^{\delta}$, we have:

$$\forall\, x \in V_{x_0}^{\delta} \cap E \quad f(x) < 0.$$

We simultaneously proved the first and second parts of the theorem, since in both cases the sign of $f(x)$ coincides with the sign of $f(x_0)$. □

Arithmetic properties of continuous functions

11B/39:35 (06:02)

Theorem (on arithmetic properties of continuous functions).

Let the functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be continuous at the point x_0. Then

1) the sum of the functions $f + g$ is continuous at x_0,

2) the product of the functions fg is continuous at x_0,

3) under the additional condition $g(x_0) \neq 0$, the quotient of the functions $\frac{f}{g}$ is continuous at x_0.

Proof.

Since any functions are continuous at isolated points, it suffices to consider the case when x_0 is a limit point of the set E. In this case, from the continuity of the functions f and g at the point x_0 it follows that there exist limits

$$\lim_{x \to x_0} f(x) = f(x_0), \quad \lim_{x \to x_0} g(x) = g(x_0).$$

Then, using the arithmetic properties of the limit of functions, we obtain:

$$\lim_{x\to x_0}(f+g)(x) = \lim_{x\to x_0}\bigl(f(x)+g(x)\bigr) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) =$$
$$= f(x_0)+g(x_0) = (f+g)(x_0),$$
$$\lim_{x\to x_0}(fg)(x) = \lim_{x\to x_0}\bigl(f(x)g(x)\bigr) = \lim_{x\to x_0} f(x)\cdot\lim_{x\to x_0} g(x) =$$
$$= f(x_0)g(x_0) = (fg)(x_0).$$

Since the limits of the functions $f+g$ and fg as $x \to x_0$ exist and are equal to the value of these functions at the point x_0, we obtain that the functions $f+g$ and fg are continuous at a given point.

To apply the arithmetic property on the quotient limit, it is necessary that the limit of the function g at the point x_0 be non-zero and that there exists a neighborhood of the point x_0 (generally speaking, a punctured one) at which the function g is not equal to 0.

The limit g at the point x_0 is equal to the value of $g(x_0)$, which is not equal to zero by the assumption of the theorem. The existence of the required neighborhood of the point x_0 follows from the previously proved simple properties of continuous functions. Thus, all the conditions for applying the arithmetic property on the limit of the quotient are fulfilled:

$$\lim_{x\to x_0}\left(\frac{f}{g}\right)(x) = \lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{f(x_0)}{g(x_0)} = \left(\frac{f}{g}\right)(x_0).$$

Since the limit of the function $\frac{f}{g}$ as $x \to x_0$ exists and is equal to the value of this function at the point x_0, we obtain that the function $\frac{f}{g}$ is continuous at a given point. $\square$

Corollaries.

1. The polynomial $P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ is a continuous function for all $x \in \mathbb{R}$. Indeed, we previously proved that a constant and a linear function are continuous; therefore, each term of the form a_kx^{n-k}, $k = 0, \dots, n$, is continuous as the product of continuous functions, and their sum is continuous as the sum of continuous functions.

2. The rational function $R(x) = \frac{P_n(x)}{Q_m(x)}$, where $P_n(x)$ and $Q_m(x)$ are polynomials, is continuous at all points of its domain of definition $\{x \in \mathbb{R} : Q_m(x) \neq 0\}$. This follows from corollary 1 and the theorem on the continuity of the quotient of two continuous functions.

3. The tangent function $\tan x = \frac{\sin x}{\cos x}$ is continuous at all points of its domain of definition. This follows from the continuity of the sine and cosine functions and the theorem on the continuity of the quotient of two continuous functions.

Superposition of continuous functions

The superposition limit theorem in the case when the external function is continuous

12A/00:00 (17:50)

THEOREM (THE SUPERPOSITION LIMIT THEOREM IN THE CASE WHEN THE EXTERNAL FUNCTION IS CONTINUOUS).

Let $f : E \to G$, $g : G \to \mathbb{R}$ be functions, a be a limit point of E. Let two conditions be satisfied:

1) $\lim_{x\to a} f(x) = b \in G$, b is a limit point of G;

2) the function g is continuous at the point b.

Then

$$\lim_{x\to a}(g \circ f)(x) = g(b) = g\Bigl(\lim_{x\to a} f(x)\Bigr).$$

BRIEF VERBAL FORMULATION OF THE THEOREM.

If the external function is continuous, then the limit sign can be moved under the sign of the external function, as well as out of its sign.

PROOF.

We will use the criterion for the existence of a function limit in terms of sequences.

Let $\{x_n\}$ be some sequence such that $x_n \in E \setminus \{a\}$ and $x_n \to a$ as $n \to \infty$.

Then, by virtue of the necessary part of the indicated criterion, we obtain from condition 1 of the theorem that $f(x_n) \to b$ as $n \to \infty$. Denote $y_n = f(x_n)$:

$$\lim_{n\to\infty} g\bigl(f(x_n)\bigr) = \lim_{n\to\infty} g(y_n).$$

For the sequence $\{y_n\}$, we have: $y_n \in G$, $y_n \to b$ as $n \to \infty$. By condition 2, the function $g(y)$ is continuous at the limit point b, therefore

$$\lim_{y\to b} g(y) = g(b).$$

By virtue of the necessary part of the indicated criterion, we obtain for the sequence $\{y_n\}$:

$$\lim_{n\to\infty} g(y_n) = g(b).$$

In this case, it is quite acceptable that some elements of the sequence $\{y_n\}$ take the value b, since the value $g(y_n)$ will be equal to the limit $g(b)$ for such elements.

So, we obtain that for an arbitrary sequence $\{x_n\}$ satisfying the above conditions, the limit of the sequence $\bigl\{g\bigl(f(x_n)\bigr)\bigr\}$ exists and is equal to $g(b)$.

Therefore, by virtue of a sufficient part of the indicated criterion, the limit of the superposition $g\big(f(x)\big)$ at the point a exists and is equal to $g(b)$. □

The continuity of the superposition of continuous functions

12A/17:50 (05:29)

THEOREM (ON THE CONTINUITY OF THE SUPERPOSITION OF CONTINUOUS FUNCTIONS).

Let $f : E \to G$, $g : G \to \mathbb{R}$ be functions, let f be continuous at a, which is a limit point of E, and g be continuous at $b = f(a)$, which is a limit point of G. Then the superposition $g \circ f$ is continuous at the point a.

PROOF.

Since a is a limit point of the set E, the continuity of the function f at the point a implies that there exists a limit $\lim_{x\to a} f(x) = f(a)$. By assumption, b is the limit point of G and g is continuous at the point b.

Thus, all the conditions of the previous theorem are satisfied, therefore,

$$\lim_{x\to a}(g \circ f)(x) = g(b) = g\big(f(a)\big) = (g \circ f)(a).$$

The limit of superposition $g \circ f$ at the point a is equal to its value at this point. Therefore, the superposition $g \circ f$ is continuous at the point a. □

REMARK.

The statement of the theorem also remains valid if the points a or b are isolated points of the corresponding sets.

Proof of some equivalences

12A/23:19 (17:34), 12B/00:00 (08:27)

1. If $a > 0$, $a \ne 1$, then $\log_a(1 + x) \sim x \log_a e$, $x \to 0$.

PROOF.

We need to prove the following:

$$\lim_{x\to 0} \frac{\log_a(1 + x)}{x} = \log_a e. \tag{3}$$

Let us use the second remarkable limit: $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$. Applying the superposition limit theorem, it can be rewritten in the form as follows:

$$\lim_{x\to 0}(1 + x)^{1/x} = \lim_{x\to 0}\left(1 + \frac{1}{y}\right)^y \circ \frac{1}{x} = \lim_{y\to\infty}\left(1 + \frac{1}{y}\right)^y = e.$$

Let us act on boundary sides of the last relation with the function $\log_a$:

$$\log_a\left(\lim_{x\to 0}(1 + x)^{1/x}\right) = \log_a e.$$

As noted earlier (without proof), the function $\log_a x$ is continuous in its domain of definition. Therefore, by the just proved theorem on the limit of superposition in the case when the external function is continuous, we can move the sign of the logarithm under the limit sign:

$$\lim_{x\to 0} \log_a(1+x)^{1/x} = \log_a e.$$

It remains to transform the left-hand side of the equality using the property of the logarithm:

$$\lim_{x\to 0} \log_a(1+x)^{1/x} = \lim_{x\to 0} \frac{\log_a(1+x)}{x}. \ \square$$

COROLLARY.

If $a = e$, then the proved equivalence takes the form

$$\ln(1+x) \sim x, \quad x \to 0.$$

2. If $a > 0$, $a \neq 1$, then $a^x - 1 \sim x \ln a$, $x \to 0$.

PROOF.

We need to prove the following:

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a.$$

Let $y = a^x - 1$, then $x = \log_a(y+1)$. Therefore, the function $\frac{a^x-1}{x}$ can be represented as the following superposition:

$$\frac{a^x - 1}{x} = \frac{y}{\log_a(y+1)} \circ (a^x - 1).$$

Earlier we proved that $\lim_{x\to 0}(a^x - 1) = 0$. Thus, all the conditions of the superposition limit theorem are satisfied, and, using the relation (3) already proved, we obtain:

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to 0} \frac{y}{\log_a(y+1)} \circ (a^x - 1) =$$

$$= \lim_{y\to 0} \frac{y}{\log_a(y+1)} = \frac{1}{\log_a e} = \ln a. \ \square$$

COROLLARY.

If $a = e$, then the proved equivalence takes the form

$$e^x - 1 \sim x, \quad x \to 0.$$

3. If $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then $(1+x)^\alpha - 1 \sim \alpha x$, $x \to 0$.

PROOF.

We need to prove the following:

$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha.$$

Let us represent the power function $(1+x)^\alpha$ in the form $e^{\alpha\ln(1+x)}$ and transform the function under the limit sign as follows:

$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\lim_{x\to 0}\left(\frac{e^{\alpha\ln(1+x)}}{\alpha\ln(1+x)}\cdot\frac{\alpha\ln(1+x)}{x}\right)=$$

$$=\alpha\lim_{x\to 0}\frac{e^{\alpha\ln(1+x)}}{\alpha\ln(1+x)}\lim_{x\to 0}\frac{\ln(1+x)}{x}. \tag{4}$$

The second limit on the right-hand side of (4) is 1, by the corollary of the first equivalence.

The first limit on the right-hand side of (4) can be represented as the following superposition:

$$\lim_{x\to 0}\frac{e^{\alpha\ln(1+x)}}{\alpha\ln(1+x)}=\lim_{x\to 0}\frac{e^y}{y}\circ\alpha\ln(1+x).$$

Using the first equivalence, it is easy to prove that $\lim_{x\to 0}\alpha\ln(1+x)=0$. Therefore, applying the superposition limit theorem and the corollary of the second equivalence, we obtain:

$$\lim_{x\to 0}\frac{e^{\alpha\ln(1+x)}}{\alpha\ln(1+x)}=\lim_{y\to 0}\frac{e^y}{y}=1.$$

Thus, both limits on the right-hand side of (4) are equal to 1. Therefore, the limit on the left-hand side is α. □

4. If $f(x)\to 1$ as $x\to x_0$, $g(x)\to\infty$ as $x\to x_0$, then

$$\lim_{x\to x_0}f(x)^{g(x)}=e^{\lim_{x\to x_0}g(x)(f(x)-1)}.$$

This formula reduces the indeterminate form 1^∞ to the indeterminate form $0\cdot\infty$, which, as a rule, is easier to study.

Proof.

Let us transform the original limit as follows:

$$\lim_{x\to x_0}f(x)^{g(x)}=\lim_{x\to x_0}e^{g(x)\ln f(x)}.$$

Since the function e^x is continuous on the entire real axis, we can move a limit sign under the sign of this function:

$$\lim_{x\to x_0}e^{g(x)\ln f(x)}=e^{\lim_{x\to x_0}g(x)\ln f(x)}.$$

It remains for us to represent $\ln f(x)$ in the form $\ln\big(f(x)-1+1\big)$ and notice that the last expression, by the first equivalence, is equivalent to the expression $\big(f(x)-1\big)$ since $\big(f(x)-1\big)\to 0$ as $x\to x_0$. □

13. Continuity of a function on a set

Intermediate value theorem

A function continuous on a set: definition   12B/08:27 (02:50)

DEFINITION.

Let f map E to $\mathbb{R}$. A function f is said to be *continuous on* E if f is continuous at any point of the set E.

Intermediate value theorem: formulation and proof   12B/11:17 (19:22)

THEOREM (INTERMEDIATE VALUE THEOREM).

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, with $f(a)f(b) < 0$. Then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

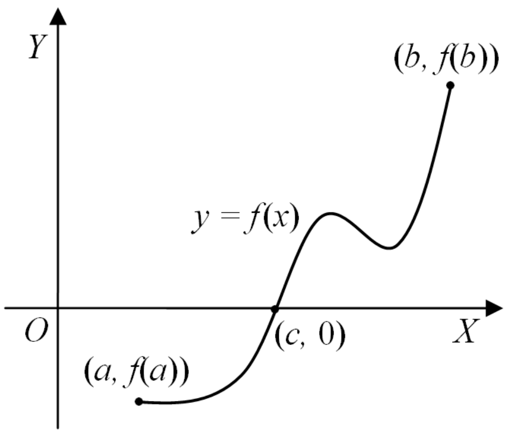

Fig. 6. A geometric interpretation of the intermediate value theorem

A GEOMETRIC INTERPRETATION OF THE INTERMEDIATE VALUE THEOREM.

The graph of the function $f(x)$, $x \in [a, b]$, necessarily intersects the axis OX at least at one point (see Fig. 6).

Proof[5].

We will not only prove that the point c exists, but also describe a process that allows us to obtain its value with any degree of accuracy. This kind of proof is called a *constructive* one.

First, assume that $f(a) < 0$, $f(b) > 0$.

Divide the original segment $[a, b]$ in half with the point $d = \frac{a+b}{2}$. Then two cases are possible:

1) $f(d) = 0$, and we have found the point c, that is, $c = d$;

2) $f(d) \neq 0$.

In the second case, we choose from two segments $[a, d]$ and $[d, b]$ the one satisfying the following condition: the function f takes values of different signs at endpoints of this segment. Denote the selected segment by $[a_1, b_1]$: if $f(d) < 0$, then $a_1 = d$, $b_1 = b$, and if $f(d) > 0$, then $a_1 = a$, $b_1 = d$. For this segment, we have: $f(a_1) < 0$, $f(b_1) > 0$, moreover, $[a_1, b_1] \subset [a, b]$ and $b_1 - a_1 = \frac{b-a}{2}$.

Repeat the described process for the segment $[a_1, b_1]$: divide it in half with the point $d_1 = \frac{a_1+b_1}{2}$.

If $f(d_1) = 0$, then the point c is found: $c = d_1$. Otherwise, we choose from two segments $[a_1, d_1]$ and $[d_1, b_1]$ the one at the endpoints of which the function f takes values of different signs, and denote it by $[a_2, b_2]$. For this segment, we have: $f(a_2) < 0$, $f(b_2) > 0$, $[a_2, b_2] \subset [a_1, b_1]$ and $b_2 - a_2 = \frac{b_1-a_1}{2} = \frac{b-a}{2^2}$.

Continuing the described process, we either obtain, at some step, a point $d_n \in (a, b)$ for which $f(d_n) = 0$, and thereby immediately prove the theorem, or construct a sequence of segments $\{[a_n, b_n]\}$ satisfying the condition $f(a_n) < 0$, $f(b_n) > 0$ for any $n \in \mathbb{N}$.

The sequence $\{[a_n, b_n]\}$ is a sequence of contracting segments, since each next segment is nested in the previous one and, in addition,

$$\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^n} = 0.$$

By the nested segments theorem, there exists a point $c \in [a, b]$ belonging to all segments $[a_n, b_n]$. We will show that the equality $f(c) = 0$ holds for this point.

Since the segments $[a_n, b_n]$ are contracted to the point c, the following relations hold:

$$\lim_{n\to\infty} a_n = c, \qquad \lim_{n\to\infty} b_n = c. \tag{1}$$

[5] In video lectures, a more complicated method of proof is given.

To prove, for example, the first of these relations, it is enough to consider the double inequality $a_n \le c \le b_n$, subtract a_n from each of its terms and pass to the limit as $n \to \infty$ in the resulting double inequality $0 \le c - a_n \le b_n - a_n$. Since the boundary terms of this inequality approach 0, we conclude, by virtue of the second theorem on the passing to the limit in inequalities for sequences, that $c - a_n \to 0$ as $n \to \infty$, i. e., $\lim_{n\to\infty} a_n = c$.

The point c belongs to the segment $[a, b]$. Therefore, by the condition of the theorem, the function f is continuous at this point:

$$\lim_{x\to c} f(x) = f(c). \tag{2}$$

Then, taking into account (1) and (2), by virtue of the necessary part of the criterion for the existence of the limit of a function in terms of sequences, we obtain:

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(c). \tag{3}$$

In addition, by constructing the sequences $\{a_n\}$ and $\{b_n\}$, we have:

$$\forall\, n \in \mathbb{N} \quad f(a_n) < 0, \quad f(b_n) > 0.$$

Using the first theorem on passing to limit in inequalities for sequences and taking into account (3), we obtain:

$$f(c) = \lim_{n\to\infty} f(a_n) \le 0, \quad f(c) = \lim_{n\to\infty} f(b_n) \ge 0.$$

The inequalities $f(c) \le 0$ and $f(c) \ge 0$ can only be satisfied simultaneously if $f(c) = 0$.

Note that $c \in (a, b)$, since, by condition, the function f does not turn into 0 at the endpoints of the original segment.

We proved the theorem for the case $f(a) < 0$, $f(b) > 0$.

It remains to consider the case $f(a) > 0$, $f(b) < 0$. For this case, we can perform similar constructions. □

Remarks.

1. If the statement of the theorem for the case $f(a) < 0$, $f(b) > 0$ has already been proved, then the statement for the case $f(a) > 0$, $f(b) < 0$ can be proved in the following, more simple way.

Consider an auxiliary function $g(x) = -f(x)$. The function g is continuous on the segment $[a, b]$ and the estimates $g(a) < 0$, $g(b) > 0$ hold for it. Therefore, by the proved statement, there exists a point $c \in (a, b)$ such that $g(c) = 0$. The same equality holds for the function f: $f(c) = -g(c) = 0$.

2. The method for finding the root c of the equation $f(x) = 0$ described in the proof of the theorem allows us to find *only one* of the possible roots lying on the interval (a, b).

Corollary

12B/30:39 (07:00)

Theorem (corollary of the intermediate value theorem).

Let the function f be defined and continuous on the segment $[a, b]$ and $f(a) \neq f(b)$. Let the point d be between the points $f(a)$ and $f(b)$. Then there exists a point $c \in (a, b)$ such that $f(c) = d$.

Proof.

Introduce the auxiliary function $g(x) = f(x) - d$.

Suppose that $f(a) < f(b)$. Then the double inequality $f(a) < d < f(b)$ holds for d.

Subtracting the value of d from each term of this inequality, we obtain:

$$f(a) - d < d - d < f(b) - d, \qquad g(a) < 0 < g(b).$$

Thus, the function $g(x)$ takes values of different signs at the endpoints of the segment $[a, b]$: $g(a)g(b) < 0$. A similar statement is also true if $f(a) > f(b)$ (since in this case the double inequality $g(a) > 0 > g(b)$ holds).

So, for the function g, the condition $g(a)g(b) < 0$ is satisfied. In addition, the function $g(x)$ is continuous on $[a, b]$ as the difference of continuous functions. Therefore, by virtue of the intermediate value theorem, there exists a point $c \in (a, b)$ for which $g(c) = 0$.

Given the definition of the function $g(x)$, we obtain

$$f(c) - d = 0, \quad f(c) = d. \ \square$$

Weierstrass theorems on the properties of functions continuous on a segment

The first Weierstrass theorem

12B/37:39 (03:58), 13A/00:00 (14:57)

Theorem 1 (first Weierstrass theorem).

Let the function f be continuous on the segment $[a, b]$. Then it is bounded on this segment.

Proof.

We need to prove the following statement:

$$\exists\, M > 0 \quad \forall\, x \in [a, b] \quad |f(x)| \leq M. \tag{4}$$

Let us prove the theorem by contradiction. Suppose that the function f is not bounded on the segment $[a, b]$. This means that the negation of statement (4) is fulfilled:

$$\forall M > 0 \quad \exists x_M \in [a, b] \quad |f(x_M)| > M.$$

Choosing all natural numbers $(1, 2, \dots, n, \dots)$ as M, we construct the sequence $\{x_1, x_2, \dots, x_n, \dots\}$, whose elements satisfy two conditions: $x_n \in [a, b]$ and $|f(x_n)| > n$.

Since $\forall n \in \mathbb{N}\ x_n \in [a, b]$, the sequence $\{x_n\}$ is bounded. Therefore, by virtue of the Bolzano–Weierstrass theorem, a convergent subsequence $\{x_{n_k}\}$ can be extracted from it:

$$\lim_{k\to\infty} x_{n_k} = c. \tag{5}$$

Since for the elements of the subsequence $\{x_{n_k}\}$, the double inequality $a \le x_{n_k} \le b$ holds, we obtain, by virtue of the first theorem on passing to the limit in the inequalities for sequences, that $a \le c \le b$.

Thus, $c \in [a, b]$ and, therefore, the function f is continuous at the point c:

$$\lim_{x\to c} f(x) = f(c). \tag{6}$$

Given (5) and (6), we obtain, by virtue of the necessary part of the criterion for the existence of a function limit in terms of sequences,

$$\lim_{k\to\infty} f(x_{n_k}) = f(c).$$

Thus, we have obtained that the sequence $\{f(x_{n_k})\}$ has a finite limit.

On the other hand, since $\forall n \in \mathbb{N}\ |f(x_n)| > n$, we obtain that

$$\forall k \in \mathbb{N} \quad |f(x_{n_k})| > n_k.$$

Moreover, $n_k \to \infty$ as $n \to \infty$, therefore the sequence $\{|f(x_{n_k})|\}$ increases unlimitedly. But this contradicts the fact that this sequence has a finite limit $|f(c)|$. The obtained contradiction means that our assumption was incorrect, and the function f is bounded on the segment $[a, b]$. □

Remark.

If a function is continuous on an interval or half-interval, it may be unbounded. As an example, we can indicate the function $f(x) = \frac{1}{x}$ considered on the half-interval $(0, 1]$. The function f is continuous on this half-interval, however, it increases unlimitedly, as x approaches 0. Thus, the continuity of a function *precisely on a segment* is an important condition of the theorem.

The second Weierstrass theorem

13A/14:57 (18:25), 13B/00:00 (07:09)

Theorem 2 (second Weierstrass theorem).

Let the function f be continuous on the segment $[a, b]$. Then it attains its maximum and minimum value on this segment:

$$\exists\, x' \in [a,b] \quad f(x') = \max_{x\in[a,b]} f(x),$$

$$\exists\, x'' \in [a,b] \quad f(x'') = \min_{x\in[a,b]} f(x).$$

PROOF[6].

Let us prove the existence of a maximum value.

By the first Weierstrass theorem, the image $f([a,b])$ of the function f is bounded. Since the set $f([a,b])$ is bounded, it has the least upper bound M:

$$M = \sup f([a,b]) = \sup_{x\in[a,b]} f(x).$$

To prove the theorem, it suffices to show that the function attains its least upper bound, i. e., that there exists a point $x' \in [a,b]$ such that $f(x') = M$.

Let us prove the existence of the point x' by contradiction. Suppose that such a point does not exist. It means that

$$\forall\, x \in [a,b] \quad f(x) < M. \tag{7}$$

Consider the auxiliary function $g(x) = \frac{1}{M - f(x)}$. By condition (7), this function is defined on the entire segment $[a,b]$ and is continuous on this segment as the ratio of continuous functions for which the denominator does not turn into 0: $M - f(x) \neq 0$ for $x \in [a,b]$.

By the first Weierstrass theorem, the function $g(x)$ is bounded, i. e.,

$$\exists\, B > 0 \quad \forall\, x \in [a,b] \quad |g(x)| \le B. \tag{8}$$

On the other hand, by definition of the least upper bound, we obtain:

$$\forall\, \varepsilon > 0 \quad \exists\, x_\varepsilon \in [a,b] \quad f(x_\varepsilon) > M - \varepsilon.$$

Choosing $\varepsilon = \frac{1}{n}$ for all positive integers n, we can construct the sequence $\{x_n\}$, whose elements satisfy two conditions: $x_n \in [a,b]$, $f(x_n) > M - \frac{1}{n}$.

We can transform the inequality $f(x_n) > M - \frac{1}{n}$ as follows:

$$M - f(x_n) < \frac{1}{n}, \qquad \frac{1}{M - f(x_n)} > n.$$

So we obtain that

$$\forall\, n \in \mathbb{N} \quad g(x_n) > n.$$

This contradicts condition (8), since for any $B > 0$ we can choose a positive integer $n > B$ for which $x_n \in [a,b]$ and $g(x_n) > n > B$.

The obtained contradiction means that our assumption is false, and there exists a point x' at which the function f attains its maximum value M.

The existence of a minimum value is proved similarly. □

[6] In video lectures, a more complicated method of proof is given.

Remarks.

1. If the statement of the theorem regarding the maximum value has already been proved, then the statement for the minimum value can be proved in the following, more simple way.

Consider the auxiliary function $h(x) = -f(x)$. Obviously, if the function f is continuous on $[a, b]$ then the function h is also continuous on this segment. By virtue of the proved part of the theorem, we can state that the maximum value is attained for the function h, that is, there exists a point $x'' \in [a, b]$ for which the following condition holds:

$$\forall\, x \in [a, b] \quad h(x'') \geq h(x).$$

This condition means that for all $x \in [a, b]$, $-f(x'') \geq -f(x)$ holds, or $f(x'') \leq f(x)$, and this, in turn, means that the point x'' is the minimum point for the original function f.

2. If a function is continuous on an interval or half-interval, it may not attain its maximum or minimum value, even if it is bounded. As an example, we can indicate the function $f(x) = x$, considered on the half-interval $(0, 1]$. The function f is continuous on this half-interval, the set of its values has the greatest lower bound equal to 0, however it does not attain the greatest lower bound 0, since $\forall\, x \in (0, 1]$ $f(x) > 0$. Thus, the continuity of a function *precisely on a segment* is an important condition of this theorem.

Uniform continuity

A function uniformly continuous on a set X: definition, continuity of a uniformly continuous function 13B/07:09 (08:45)

Let the function f be defined on the set X. Using the definition of the continuity of a function at a point $x_0 \in X$ in the language ε–δ, the definition of the continuity of a function f on a set X can be written as follows:

$$\forall\, x_0 \in X \quad \forall\, \varepsilon > 0 \quad \exists\, \delta > 0$$
$$\forall\, x \in X, |x - x_0| < \delta, \quad |f(x) - f(x_0)| < \varepsilon.$$

This definition can also be rewritten in the form using the implication sign “⇒”:

$$\forall\, x_0 \in X \quad \forall\, \varepsilon > 0 \quad \exists\, \delta > 0$$
$$\forall\, x \in X \quad (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon). \tag{9}$$

It should be noted that in this definition the value of δ depends on both ε and x_0.

If we will require that the value δ can be selected depending *only* on ε, that is, for all $x_0 \in X$ at the same time, we obtain a definition of a function uniformly continuous on the set X.

DEFINITION.

A function f is called *uniformly continuous* on the set X if the following condition is satisfied:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0$$
$$\forall\, x', x'' \in X \quad (|x' - x''| < \delta \Rightarrow |f(x') - f(x'')| < \varepsilon). \tag{10}$$

This condition can be obtained from condition (9) by replacing the points x and x_0 with x' and x'' in (9), respectively, and moving the fragment "$\forall\, x'' \in X$" from the initial position of the condition to the position after the fragment "$\exists\, \delta > 0$".

Obviously, if condition (10) is satisfied, then condition (9) is also satisfied. Indeed, in condition (10), we choose the required value δ only by the value ε, therefore the value δ chosen in (10) can be used in condition (9) for the same ε and any $x_0 \in X$.

Thus, *if a function is uniformly continuous on the set X, then it is continuous on X.*

The converse is, generally speaking, not true. A function can be continuous on X, but not be uniformly continuous on this set.

An example of a continuous function that is not uniformly continuous 13B/15:54 (09:41)

Consider the function $f(x) = \sin\frac{1}{x}$ defined on the half-interval $X = (0, 1]$. This function is continuous at any point in the set X, but it is not uniformly continuous on this set.

In order to prove this, we write down the negation of condition (10):

$$\exists\, \varepsilon > 0 \quad \forall\, \delta > 0$$
$$\exists\, x', x'' \in X \quad (|x' - x''| < \delta \ \wedge\ |f(x') - f(x'')| \ge \varepsilon). \tag{11}$$

When constructing the negation, we took into account that the implication $A \Rightarrow B$ is equivalent to the expression $\overline{A} \vee B$, and the negation of the last expression is $A \wedge \overline{B}$.

We show that condition (11) is satisfied, for example, for $\varepsilon = \frac{1}{2}$.

Consider two sequences of points belonging to the half-interval $(0, 1]$:

$$x'_n = \frac{1}{2\pi n}, \quad x''_n = \frac{1}{\frac{\pi}{2} + 2\pi n}.$$

For sufficiently large $n \in \mathbb{N}$, the difference $|x'_n - x''_n|$ can be made arbitrarily small, since

$$|x'_n - x''_n| = \left|\frac{1}{2\pi n} - \frac{1}{\frac{\pi}{2} + 2\pi n}\right| = \frac{\frac{\pi}{2}}{2\pi n(\frac{\pi}{2} + 2\pi n)}.$$

The last expression approaches 0 as $n \to \infty$. Thus, choosing an arbitrary $\delta > 0$, we can choose the number $n \in \mathbb{N}$ for which $|x'_n - x''_n| < \delta$. At the same time, $|f(x'_n) - f(x''_n)| \geq \varepsilon$:

$$|f(x'_n) - f(x''_n)| = \left|\sin\frac{1}{\frac{1}{2\pi n}} - \sin\frac{1}{\frac{1}{\frac{\pi}{2}+2\pi n}}\right| =$$

$$= \left|\sin(2\pi n) - \sin\left(\frac{\pi}{2} + 2\pi n\right)\right| = |0 - 1| = 1 > \frac{1}{2} = \varepsilon.$$

We have shown that condition (11) is satisfied for the function $\sin\frac{1}{x}$; therefore, it is not uniformly continuous on the half-interval $(0, 1]$.

Remark.

As another example, we can consider the function $f(x) = \frac{1}{x}$, which is continuous on the half-interval $(0, 1]$, but is not uniformly continuous on it (prove this fact).

Cantor's theorem

13B/25:35 (04:01), 14A/00:00 (19:57)

Theorem (Cantor's theorem on uniform continuity).

If f is continuous on the segment $[a, b]$, then it is uniformly continuous on this segment.

Remark.

Due to the property of uniform continuity, functions that are continuous on a segment have a number of "good" properties (see, in particular, two Weierstrass theorems).

Proof.

Let us prove the theorem by contradiction: suppose that the function f is not uniformly continuous on $[a, b]$. This means the following (compare with (11)):

$$\exists\, \varepsilon > 0 \quad \forall\, \delta > 0$$
$$\exists\, x', x'' \in [a, b] \quad (|x' - x''| < \delta \wedge |f(x') - f(x'')| \geq \varepsilon). \tag{12}$$

We will construct a sequence of values δ approaching 0: $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. By virtue of (12), for $\delta = \frac{1}{n}$, there exist points $x'_n, x''_n \in [a, b]$ that satisfy the conditions

$$|x'_n - x''_n| < \frac{1}{n}, \tag{13}$$

$$|f(x'_n) - f(x''_n)| \geq \varepsilon. \tag{14}$$

We have obtained the sequences $\{x'_n\}$ and $\{x''_n\}$. Obviously, these sequences are bounded, since all their elements belong to the segment $[a, b]$.

According to the Bolzano–Weierstrass theorem, a convergent subsequence can be extracted from any bounded sequence. Let us choose the converging subsequence $\{y'_k\}$ for the sequence $\{x'_n\}$:

$$y'_k = x'_{n_k}, \quad 1 \leq n_1 < n_2 < \dots < n_k < \dots,$$

$$\lim_{k\to\infty} y'_k = c. \tag{15}$$

Since $y'_k \in [a, b]$, we obtain, by virtue of the first theorem on passing to the limit in inequalities for functions, that $c \in [a, b]$.

For the sequence $\{x''_n\}$, we define the subsequence $\{y''_k\}$ by taking the elements of the sequence $\{x''_n\}$ with the same indices as for the subsequence $\{y'_n\}$:

$$y''_k = x''_{n_k}, \quad 1 \leq n_1 < n_2 < \dots < n_k < \dots$$

Let us show that the sequence $\{y''_k\}$ is also convergent, and its limit is equal to the same value c as the limit of the sequence $\{y'_k\}$. To do this, we estimate the following difference:

$$|y''_k - c| = |y''_k - y'_k + y'_k - c| \leq |y''_k - y'_k| + |y'_k - c| =$$

$$= |x''_{n_k} - x'_{n_k}| + |y'_k - c|. \tag{16}$$

By virtue of (13), the first term on the right-hand side of (16) is estimated by $\frac{1}{n_k}$:

$$|x''_{n_k} - x'_{n_k}| < \frac{1}{n_k}.$$

The fraction $\frac{1}{n_k}$ approaches 0 as $k \to \infty$, therefore $|x''_{n_k} - x'_{n_k}| \to 0$ as $k \to \infty$, by the first theorem on passing to the limit in inequalities for sequences.

Taking into account (15), we obtain that the second term on the right-hand side of (16) also approaches 0: $|y'_k - c| \to 0$ as $k \to \infty$.

Therefore, the left-hand side of (16) approaches 0: $|y''_k - c| \to 0$ as $k \to \infty$, which implies that $\lim_{k\to\infty} y''_k = c$.

Now consider the sequences $\{f(y'_k)\}$ and $\{f(y''_k)\}$. Since, by the condition of the theorem, the function f is continuous on $[a, b]$, and each of the sequences

$\{y'_k\}$ and $\{y''_k\}$ has a limit $c \in [a, b]$, we obtain, by virtue of the criterion for the existence of a function limit in terms of the limit of sequences, that

$$\lim_{k\to\infty} f(y'_k) = \lim_{k\to\infty} f(y''_k) = f(c).$$

Consequently,

$$\lim_{k\to\infty} (f(y'_k) - f(y''_k)) = c - c = 0. \qquad (17)$$

On the other hand, by constructing the sequences $\{x'_n\}$ and $\{x''_n\}$ and taking into account (14), we have:

$$|f(y'_k) - f(y''_k)| = |f(x'_{n_k}) - f(x''_{n_k})| \geq \varepsilon.$$

Passing to the limit in the last inequality as $k \to \infty$ and taking into account (17), we obtain the estimate $0 \geq \varepsilon$. The resulting estimate contradicts the estimate $\varepsilon > 0$ from (12). This contradiction means that our assumption is false, and the function f is uniformly continuous on $[a, b]$. □

Remark.

It should be noted that if the function f were continuous not on the segment $[a, b]$, but on the interval (a, b), then a contradiction could not be obtained. In this case, the limit c of the constructed sequences $\{y'_n\}$ and $\{y''_n\}$ may coincide with one of the endpoints of the interval (a, b). At these points, the function f does not have to be continuous, and therefore the limit relation (17) may not hold. Recall that earlier we gave examples of functions that are continuous on a half-interval, but are not uniformly continuous on it.

14. Points of discontinuity

Points of discontinuity of a function, their classification and examples

14A/19:57 (19:14)

Definition.

Let f be a function acting from E to $\mathbb{R}$. The point $x_0 \in E$ is called the *discontinuity point* of the function f if the function f is not continuous at this point.

Thus, all points of the domain E of the function f can be either points of continuity or points of discontinuity.

We noted earlier that if the point x is an isolated point of the set E, then the function f is continuous at it. Therefore, if the point x_0 is a discontinuity point then it is necessarily the limit point of the set E and the continuity condition at this point, $\lim_{x \to x_0} f(x) = f(x_0)$, is not fulfilled. Violation of this condition may be due to various reasons; this allows us to classify points of discontinuity.

Case 1: points of removable discontinuity.

The finite limit of the function $f(x)$, as $x \to x_0$, exists, but is not equal to $f(x_0)$. In this case, the point x_0 is called the *point of removable discontinuity.*

Example.

Consider the following function:

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The function f is defined at the point 0; therefore, we can consider the question of its continuity at this point. By the first remarkable limit, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 \neq 0 = f(0)$, therefore, the function f is not continuous at 0.

However, the limit at this point exists, although it is not equal to the value of the function at this point. Therefore, the point 0 is a point of removable discontinuity for the function f.

The point of the removable discontinuity x_0 of the function f has such a name, since it is enough to change the value of the function at this single point to obtain the function $\tilde{f}$ continuous at this point:

$$\tilde{f}(x) = \begin{cases} f(x), & x \neq x_0, \\ \lim\limits_{x \to x_0} f(x), & x = x_0. \end{cases}$$

In the above example, in order to remove the discontinuity at the point 0, it is enough to change the value of the function at this point by 1:

$$\tilde{f}(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

CASE 2: DISCONTINUITY POINTS OF THE FIRST KIND.

The limit of the function $f(x)$, as $x \to x_0$, does not exist, but there exist finite one-sided limits $\lim_{x \to x_0 - 0} f(x)$ and $\lim_{x \to x_0 + 0} f(x)$. In this case, the point x_0 is called the *discontinuity point of the first kind*, or the *point of jump discontinuity*.

If a usual limit does not exist and both one-sided limits exist, then the values of one-sided limits should be different, by virtue of the criterion for the existence of a limit in terms of one-sided limits:

$$\lim_{x \to x_0 - 0} f(x) \neq \lim_{x \to x_0 + 0} f(x).$$

Thus, the point x_0 is a discontinuity point of the first kind if there exist finite left-hand and right-hand limits at this point, but these limits are not equal.

A similar situation is possible only for points of the set E that are simultaneously the limit points both for a set $E_{x_0}^- = \{x \in E : x < x_0\}$ and for a set $E_{x_0}^+ = \{x \in E : x > x_0\}$. Points satisfying this condition will be called *two-sided limit points* of the set E.

EXAMPLE.

For the function $\operatorname{sign} x$, the point $x_0 = 0$ is a discontinuity point of the first kind, since

$$\lim_{x \to -0} \operatorname{sign} x = -1 \neq 1 = \lim_{x \to +0} \operatorname{sign} x.$$

CASE 3: DISCONTINUITY POINTS OF THE SECOND KIND.

Any discontinuity point x_0 that is not a point of removable discontinuity or a discontinuity point of the first kind is called the *discontinuity point of the second kind*, or the *point of essential discontinuity.* This case occurs if the point x_0 is the limit point of the set $E_{x_0}^-$ or $E_{x_0}^+$ and the corresponding one-sided limit does not exist or is infinite.

In particular, the point x_0 is a discontinuity point of the second kind if there exists an infinite limit at this point.

Examples.

1. Consider the following function:

$$f_1(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function is defined at the point $x_0 = 0$, and the limits $\lim_{x\to x_0-0} f_1(x)$ and $\lim_{x\to x_0+0} f_1(x)$ do not exist. Therefore, the function f_1 has a discontinuity of the second kind at the point $x_0 = 0$.

2. Consider the following function:

$$f_2(x) = \begin{cases} \left|\dfrac{1}{x}\right|, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function is defined at the point $x_0 = 0$, and the limits $\lim_{x\to x_0-0} f_2(x)$ and $\lim_{x\to x_0+0} f_2(x)$ are equal to $+\infty$. The existence of infinite one-sided limits means that the function f_2 has a discontinuity of the second kind at the point $x_0 = 0$. Note that the function has a usual limit at this point, but it is also infinite, which does not allow us to remove this discontinuity by simply redefining the function at a single point (similar to how this can be done at the point of a removable discontinuity).

Discontinuity points for monotone functions

Theorem on the points of discontinuity of a monotone function

14A/39:11 (03:11), 14B/00:00 (13:33)

Theorem (on points of discontinuity of a monotone function).

Let $f : E \to \mathbb{R}$ be a monotone function (not necessarily strictly monotone). Let $x_0 \in E$ be a two-sided limit point of the set E, which is the discontinuity point of the function f. Then x_0 is a discontinuity point of the first kind.

Proof.

For definiteness, we assume that the function f is non-decreasing (the case of a non-increasing function is considered similarly).

Let $E^- = \{x \in E : x < x_0\}$. Then $x_0 = \sup E^-$, and we can apply the theorem on the existence of a limit of a monotone bounded function. Indeed, consider the restriction of the function f to the set E^-. On this set, as on the entire set E, the function is non-decreasing and, in addition, it is bounded from above, since if $x < x_0$ then, by virtue of non-decreasing, $f(x) \le f(x_0)$.

Therefore, both conditions of the theorem on the existence of a limit of a monotone bounded function are satisfied, and, by virtue of this theorem, the function f has a finite limit on the right boundary of the set E^-, that is, as $x \to x_0 - 0$:

$$\lim_{x \to x_0 - 0} f(x) = \sup_{x \in E^-} f(x).$$

Thus, we have proved the existence of a finite left-hand limit at the point x_0.

Now let us turn to the right-hand limit and consider the restriction of the function f to the set $E^+ = \{x \in E : x > x_0\}$. This restriction is a function bounded from below, since if $x > x_0$, then, due to the non-decreasing f, $f(x) \geq f(x_0)$.

In this case, we can apply the theorem on the existence of a limit of a monotone bounded function on the it left boundary of the domain of definition, according to which the function f has a finite limit on the left boundary of the set E^+, i. e., as $x \to x_0 + 0$:

$$\lim_{x \to x_0 + 0} f(x) = \inf_{x \in E^+} f(x).$$

We have proved the existence of a finite right-hand limit at the point x_0.

So, we have established that the function f has finite one-sided limits at the point x_0. If they were equal, this would mean that $\sup_{x \in E^-} f(x) = \inf_{x \in E^+} f(x) = A$ and, due to the monotonicity of the function f, the value $f(x_0)$ should also be equal to A. But this contradicts the condition of the theorem that x_0 is a discontinuity point.

Hence, one-sided limits exist, but are different. Therefore, x_0 is a discontinuity point of the first kind. □

Remark.

If the point x_0 is not a two-sided limit point of the set E and, at the same time, it is a discontinuity point of the monotone function f, then it is easy to prove by similar reasoning that the point x_0 is a point of removable discontinuity.

Corollary 14B/13:33 (07:07)

Let us introduce the following notation for the values of the one-sided limits of the function f at the point x_0 (provided that the corresponding limits exist):

$$f(x_0 - 0) \overset{\text{def}}{=} \lim_{x \to x_0 - 0} f(x), \quad f(x_0 + 0) \overset{\text{def}}{=} \lim_{x \to x_0 + 0} f(x).$$

Theorem (corollary of the theorem on points of discontinuity of a monotone function).

Let $f : E \to \mathbb{R}$ be a monotone function, $x_0 \in E$ be a two-sided limit point of the set E. If x_0 is the discontinuity point of the function f, then at least one of the intervals with the endpoints $f(x_0 - 0)$ and $f(x_0)$ or with the endpoints $f(x_0)$ and $f(x_0 + 0)$ has a nonzero length and contains no values of the function f.

Proof.

As in the proof of the previous theorem, we assume for definiteness that the function f is non-decreasing. This means that if $x' < x_0 < x''$, then

$$f(x') \le f(x_0) \le f(x''). \tag{1}$$

In the proof of the theorem, we established that in this case

$$\lim_{x \to x_0 - 0} f(x) = \sup_{x < x_0} f(x), \qquad \lim_{x \to x_0 + 0} f(x) = \inf_{x > x_0} f(x).$$

Taking into account the existence of one-sided limits $f(x_0-0)$ and $f(x_0+0)$ and the monotonicity of the function f, the estimate (1) can be refined as follows: for any x' and x'', $x' < x_0 < x''$,

$$f(x') \le f(x_0 - 0) \le f(x_0) \le f(x_0 + 0) \le f(x'').$$

In the last estimate, the double equality $f(x_0 - 0) = f(x_0) = f(x_0 + 0)$ cannot be satisfied. Indeed, the fulfillment of double equality would mean that the one-sided limits are equal and, therefore, there exists a usual limit at the point x_0 and, moreover, this limit is equal to the value of the function at the point x_0. This contradicts the fact that, by condition, the point x_0 is a point of discontinuity.

Therefore, in the double inequality $f(x_0-0) \le f(x_0) \le f(x_0+0)$, at least one inequality sign should be a strict inequality sign “$<$”.

For definiteness, suppose that this is the first sign, i. e., that the double inequality $f(x_0 - 0) < f(x_0) \le f(x_0 + 0)$ is fulfilled. This means that the interval $\big(f(x_0 - 0), f(x_0)\big)$ has a nonzero length and does not contain any value of the function f, since for all $x' < x_0$ the inequality $f(x') \le f(x_0 - 0)$ holds, and for all $x'' \ge x_0$ the inequality $f(x_0) \le f(x'')$ holds.

If the sign of the second inequality is a strict one, then, by similar reasoning, we obtain that the interval $\big(f(x_0), f(x_0 + 0)\big)$ has a nonzero length and does not contain any value of the function f. $\square$

Criterion for the continuity of a monotone function

14B/20:40 (13:07)

Theorem (criterion for the continuity of a monotone function).

Let f be a monotone and non-constant function defined on the segment $[a, b]$. The function f is continuous on $[a, b]$ if and only if its image $f([a, b])$ is a segment.

Proof.

For definiteness, we assume that the function f is non-decreasing. In this case, the statement that $f([a, b])$ is a segment can be clarified as follows: $f([a, b]) = [f(a), f(b)]$. Moreover, the segment $[f(a), f(b)]$ has a nonzero length, since, by condition, f is not constant and therefore $f(a) < f(b)$.

1. Given: f is continuous on $[a, b]$. Prove: $f([a, b]) = [f(a), f(b)]$.

We need to prove the equality of two sets. To do this, it suffices to prove two embeddings: $f([a, b]) \subset [f(a), f(b)]$ and $[f(a), f(b)] \subset f([a, b])$. The first of these embeddings follows from the monotonicity of the function: if $a \le x \le b$, then $f(a) \le f(x) \le f(b)$, that is, for any $x \in [a, b]$ we obtain $f(x) \in [f(a), f(b)]$, therefore, $f([a, b]) \subset [f(a), f(b)]$. Note that in the proof of this embedding the continuity of the function f is not used.

The second embedding $[f(a), f(b)] \subset f([a, b])$ means that for any number d such that $f(a) < d < f(b)$ there exists a point $c \in (a, b)$ such that $f(c) = d$. This statement coincides with the statement of the corollary of the intermediate value theorem, which holds in this case, since, by condition, the function f is continuous on $[a, b]$. Note that in the proof of this embedding we do not use the monotonicity of the function f (except for the fact that $f(a) < f(b)$).

It follows from the proved embeddings that $f([a, b]) = [f(a), f(b)]$.

2. Given: $f([a, b]) = [f(a), f(b)]$. Prove: f is continuous on $[a, b]$.

Let us prove it by contradiction. We suppose that the function f is not continuous on the entire segment $[a, b]$, that is, it has at least one discontinuity point x_0 on this segment.

First, suppose that the discontinuity point x_0 belongs to the interval (a, b). Then, by the corollary of the theorem on discontinuity points of monotone functions, we obtain that at least one of the intervals $(f(x_0 - 0), f(x_0))$ or $(f(x_0), f(x_0 + 0))$ has a nonzero length and does not contain the values of the function f. Since $f(a) \le f(x_0 - 0) < f(x_0 + 0) \le f(b)$, there exists a part of the segment $[f(a), f(b)]$ that does not contain values of the function f, which contradicts the condition $f([a, b]) = [f(a), f(b)]$.

A similar reasoning leads to a contradiction also in the case when the discontinuity point x_0 coincides with a or b. In this case, it should be taken into account that this point is the point of removable discontinuity (see the remark on the theorem on the discontinuity points of monotone functions). □

Inverse function theorem

Formulation and proof

14B/33:47 (08:49)

Theorem (inverse function theorem).

Let $f : E \to \mathbb{R}$ be a strictly monotone function.

1. Then the function f, acting from E to $F = f(E)$, is one-to-one, it has the inverse function f^{-1}, acting from F to E, and this inverse function is also strictly monotonous, moreover, the type of monotonicity of the inverse function f^{-1} coincides with the type of monotonicity of the original function f.

2. If we additionally require that the function f is continuous on E and the set E is a segment, then $f(E) = F$ is also a segment and the inverse function $f^{-1} : F \to E$ is continuous on F.

Proof of the first part[7].

Recall that a function $f : E \to F$ is called one-to-one if for any element $y \in Y$ there exists a unique element $x \in X$ such that $f(x) = y$. The existence of the element x follows from the fact that the set F is equal to the image $f(E)$ of the function f. The uniqueness of the element x follows from the strict monotonicity of the function f: if $x_1, x_2 \in X$ и $x_1 \neq x_2$, then either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, that is, $f(x_1) \neq f(x_2)$.

Since the function $f : E \to f(E)$ is one-to-one, it has the inverse function f^{-1}.

It remains to show that the function f^{-1} has the same monotonicity type as f. Suppose, for definiteness, that f is increasing, and consider arbitrary points $y_1, y_2 \in f(E)$ for which the following condition holds: $y_1 < y_2$.

Denote $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. We need to prove that $x_1 < x_2$. The functions f and f^{-1} are mutually inverse, therefore $f(x_1) = y_1$, $f(x_2) = y_2$. If the inequality $x_1 < x_2$ does not hold, then we would have $x_1 \geq x_2$ and, due to the fact that the function f is increasing, it would follow that $f(x_1) \geq f(x_2)$, i. e., $y_1 \geq y_2$. But this inequality contradicts the condition

[7] In video lectures, there is no proof of the first part of the theorem.

$y_1 < y_2$. Therefore, the inequality $x_1 < x_2$ holds, and the function f^{-1} is increasing.

The case of a decreasing function f is considered similarly.

PROOF OF THE SECOND PART.

Since f is continuous on the segment E, the fact that $f(E)$ is also a segment follows from the necessary part of the criterion for the continuity of a monotone function.

So, we obtain that the inverse function f^{-1} is strictly monotone, defined on the segment $F = f(E)$ and maps it to the segment E, since $f^{-1}(F) = f^{-1}(f(E)) = E$. Therefore, by virtue of a sufficient part of the criterion for the continuity of a monotone function, the function f^{-1} is continuous on the segment F. □

Examples of application of the inverse function theorem

15A/00:00 (08:44)

The inverse function theorem makes it easy to prove the continuity of functions that are inverse to continuous functions. Let us investigate the inverse trigonometric functions for continuity (Fig. 7).

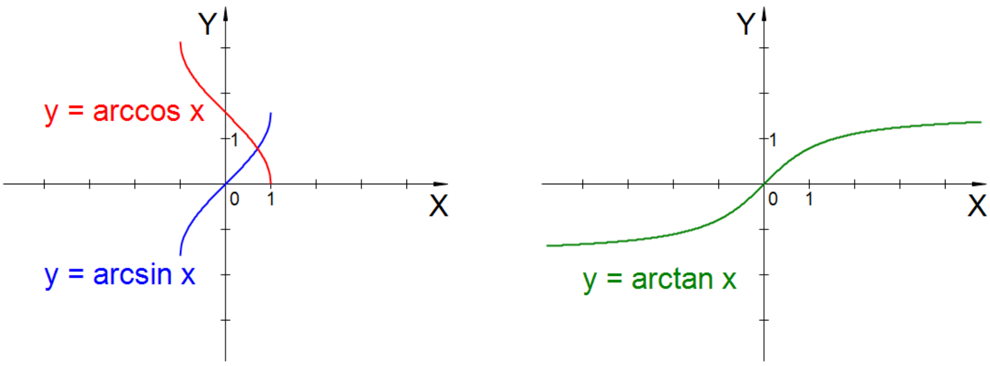

Fig. 7. Graphs of inverse trigonometric functions

1. If we consider the function $\sin x$ acting from the segment $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to the segment $[-1, 1]$, then this function will be increasing and one-to-one. In addition, this function is defined on a segment and is continuous on that segment.

Thus, all the conditions of the inverse function theorem are satisfied, and there exists an inverse function *arcsine* acting from $[-1, 1]$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is also strictly monotonous (increasing) and continuous on the segment $[-1, 1]$. Notation: $\arcsin y$.

2. In a similar way, one can prove the existence, monotonicity, and continuity of the inverse function for $\cos x$ acting from $[0, \pi]$ to $[-1, 1]$. The inverse function *arccosine* acts from $[-1, 1]$ to $[0, \pi]$ and is decreasing and continuous on $[-1, 1]$. Notation: $\arccos y$.

3. If we consider the function $\tan x$ acting from the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to the entire real axis $\mathbb{R}$, then this function will be increasing and one-to-one. Therefore, for it there exists an inverse function *arctangent* acting from $\mathbb{R}$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is also increasing. Notation: $\arctan y$.

However, in this case, we cannot immediately apply the second part of the inverse function theorem, since the original function $\tan x$ is defined not on a segment, but on an interval. Nevertheless, in this case it can be proved that the function arctan is continuous in its entire domain of definition. To do this, it suffices to prove that the function arctan is continuous at any point $y_0 \in \mathbb{R}$.

We prove this as follows.

Let $y_0 \in \mathbb{R}$. We denote $\arctan y_0 = x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and choose a segment $[a, b]$ containing the point x_0 and embedded in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If we consider the restriction of the function $\tan x$ to the segment $[a, b]$, then, for this restriction, all the conditions of the second part of the theorem will be satisfied, since the function $\tan x$ is continuous on the segment $[a, b]$. Therefore, for this restriction, there exists a continuous inverse function $\arctan y$ acting from $[c, d]$ to $[a, b]$, where $[c, d] = \tan([a, b])$. Since $x_0 \in [a, b]$, we obtain $y_0 = \tan x_0 \in [c, d]$, and the function $\arctan y$ is continuous at the point y_0 since it is continuous on the segment $[c, d]$. The choice of the point $y_0 \in \mathbb{R}$ is arbitrary, so we conclude that the function $\arctan y$ is continuous on the entire real axis $\mathbb{R}$.

Remark.

In a similar way, one can prove that the function $\log_a x$ ($a > 0$, $a \neq 1$), acting from $(0, +\infty)$ to $\mathbb{R}$, is also continuous in the entire domain of definition.

15. O-notation

Functions which are infinitesimal in comparison with other functions

15A/08:44 (14:58)

Definition 1.

Let f and g be functions acting from E to $\mathbb{R}$, x_0 (a real number or the point at infinity) be the limit point of E. The function $f(x)$ is called to be *infinitesimal in comparison with the function* $g(x)$ *as* $x \to x_0$ if there exists a punctured neighborhood $\overset{\circ}{U}_{x_0}$ of the point x_0 such that the function $f(x)$ can be represented as $\alpha(x)g(x)$ for all $x \in E \cap \overset{\circ}{U}_{x_0}$ and $\lim_{x\to x_0} \alpha(x) = 0$:

$$\exists\, \overset{\circ}{U}_{x_0} \quad \forall\, x \in E \cap \overset{\circ}{U}_{x_0} \quad f(x) = \alpha(x)g(x), \quad \lim_{x\to x_0} \alpha(x) = 0.$$

This is denoted as follows: $f(x) = o\big(g(x)\big)$, $x \to x_0$ ("$f(x)$ is *little-o* of $g(x)$ as x approaches x_0"). When using the notation "little-o", it is necessary to indicate, which point the argument of the function approaches.

If the function $g(x)$ does not equal zero in some punctured neighborhood $\overset{\circ}{U}_{x_0}$, then the equality $f(x) = o\big(g(x)\big)$, $x \to x_0$, is equivalent to the following limit relation:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0. \tag{1}$$

Example 1.

$$x^2 = o(x), \quad x \to 0.$$

If we move x from the right-hand side of the equality to the left, then we obtain $\frac{x^2}{x}$, that is, the function x, and this function approaches 0 as $x \to 0$. Thus, the function x^2 is infinitesimal in comparison with x as $x \to 0$ (that is, it approaches 0 faster).

Example 2.

$$x^2 = o(e^x), \quad x \to +\infty.$$

If we move the function e^x to the left-hand side of the equality, then we obtain $\frac{x^2}{e^x}$, and this function approaches 0 as $x \to +\infty$. Thus, the function x^2 is infinitesimal in comparison with e^x as $x \to +\infty$ (that is, it approaches $+\infty$ more slowly).

It should be noted that a similar relation $x^2 = o(e^x)$ also holds as $x \to 0$, since

$$\lim_{x\to 0} \frac{x^2}{e^x} = \frac{0}{1} = 0.$$

Example 3.

The relation $f(x) = o(1)$, $x \to x_0$, means that the function $f(x)$ is infinitesimal at the point x_0. Indeed, if we move the constant 1 to the left-hand side and use relation (1), then we obtain:

$$\lim_{x\to x_0} \frac{f(x)}{1} = \lim_{x\to x_0} f(x) = 0.$$

This relation means that $f(x)$ is infinitesimal at the point x_0. However, it should be noted that the notation $f(x) = o(1)$, $x \to x_0$, does not allow us to estimate the rate with which the function f approaches zero.

The meaning of the "little-o" notation can be clarified if we additionally know how the function g or f behaves when x approaches x_0.

Definition 2.

If the function g in definition 1 is infinitesimal at the point x_0, that is, $\lim_{x\to x_0} g(x) = 0$, then the function f is called *an infinitesimal of a higher order at the point x_0 (compared to the function $g(x)$).*

Example.

The previously given relation $x^2 = o(x)$, $x \to 0$, means that the function x^2 is an infinitesimal of a higher order at the point 0 in comparison with the function x.

In contrast to the previously considered relation $f(x) = o(1)$, the relation $f(x) = o(x)$, $x \to 0$, not only shows that the function $f(x)$ approaches 0 when x approaches 0, but it also allows us to estimate the rate with which it approaches 0: in this case, the function $f(x)$ decreases faster than x.

Definition 3.

If the function f in definition 1 is infinitely large at the point x_0, that is, $\lim_{x\to x_0} f(x) = \infty$, then the function g is called *an infinitely large of a higher order at the point x_0 (compared to the function $f(x)$).*

Example.

The previously given relation $x^2 = o(e^x)$, $x \to +\infty$, means that the function e^x is infinitely large of a higher order at the point $+\infty$ compared to function x^2. Although both functions x^2 and e^x approach ∞ as $x \to +\infty$, the function e^x grows at a faster rate, which explains the term "infinitely large function of a higher order".

REMARK.

The expression $o\big(g(x)\big)$, $x \to x_0$, can be considered as the set of all functions $f(x)$ representable as $\alpha(x)g(x)$ in some neighborhood of the point x_0, where $\alpha(x) \to 0$ as $x \to x_0$. Therefore, the notation $f(x) = o\big(g(x)\big)$, $x \to x_0$, can be interpreted as the fact that the function $f(x)$ belongs to the set described above. At the same time, when using the expression $o\big(g(x)\big)$ in formulas, it is usually assumed that in place of $o\big(g(x)\big)$ there is some specific function $f(x)$, the exact form of which is unknown, but at the same time it is known that $f(x) = o\big(g(x)\big)$, $x \to x_0$.

Functions which are bounded in comparison with other functions

15A/23:42 (05:15)

DEFINITION 4.

Let f and g be functions acting from E to $\mathbb{R}$, x_0 (a real number or the point at infinity) be the limit point of E. The function $f(x)$ is said to be *bounded in comparison with the function* $g(x)$ *as* $x \to x_0$ if there exists a punctured neighborhood $\overset{\circ}{U}_{x_0}$ of the point x_0 and the constant $C > 0$ such that for any $x \in E \cap \overset{\circ}{U}_{x_0}$ the estimate $|f(x)| \le C|g(x)|$ holds:

$$\exists\, \overset{\circ}{U}_{x_0} \quad \exists\, C > 0 \quad \forall\, x \in E \cap \overset{\circ}{U}_{x_0} \quad |f(x)| \le C|g(x)|.$$

This is denoted as follows: $f(x) = O\big(g(x)\big)$, $x \to x_0$ ("$f(x)$ is *big-O* of $g(x)$ as x approaches x_0"). When using the notation "big-O", it is necessary to indicate, which point the function argument approaches.

If the function $g(x)$ does not equal zero in some punctured neighborhood $\overset{\circ}{U}_{x_0}$, then the equality $f(x) = O\big(g(x)\big)$, $x \to x_0$, is equivalent to the following condition:

$$\forall\, x \in E \cap \overset{\circ}{U}_{x_0} \quad \left|\frac{f(x)}{g(x)}\right| \le C.$$

Definition 4 can be reformulated using the function α: it is said that $f(x) = O\big(g(x)\big)$ if in some neighborhood $\overset{\circ}{U}_{x_0}$ for all $x \in E \cap \overset{\circ}{U}_{x_0}$ the function $f(x)$ is representable in the form $\alpha(x)g(x)$ and the function $\alpha(x)$ is bounded:

$$\exists\, \overset{\circ}{U}_{x_0} \quad \exists\, C > 0 \quad \forall\, x \in E \cap \overset{\circ}{U}_{x_0} \quad f(x) = \alpha(x)g(x), \quad |\alpha(x)| \le C.$$

EXAMPLE.

$$x \sin\frac{1}{x} = O(x), \quad x \to 0.$$

If we move x from the right-hand side of the equality to the left then we obtain $\frac{x \sin \frac{1}{x}}{x}$, that is, the function $\sin \frac{1}{x}$. The function $\sin \frac{1}{x}$ is bounded in any punctured neighborhood of the point 0: $\left|\sin \frac{1}{x}\right| \le 1$, $x \ne 0$. Thus, the function $x \sin \frac{1}{x}$ is bounded in comparison with x as $x \to 0$. It should be noted that the function $x \sin \frac{1}{x}$ is not infinitesimal in comparison with x for $x \to 0$, since the function $\sin \frac{1}{x}$ has no limit at the point 0. At the same time, we can write that $x \sin \frac{1}{x} = o(1)$, since $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

Some properties related to O-notation 15A/28:57 (10:58)

1. Let f act from E to $\mathbb{R}$, x_0 be the limit point of the set E. If $f(x) = o\big(g(x)\big)$, $x \to x_0$, then $f(x) = O\big(g(x)\big)$, $x \to x_0$.

Proof.

By definition 1, there exists a neighborhood $\overset{\circ}{U}_{x_0}$ such that $\forall\, x \in \overset{\circ}{U}_{x_0} \cap E$ $f(x) = \alpha(x) g(x)$ and $\lim_{x \to x_0} \alpha(x) = 0$.

Since the function $\alpha(x)$ has a limit at x_0, it is bounded in some neighborhood of this point. Indeed, by definition of the limit,

$$\forall\, \varepsilon > 0 \quad \exists\, \overset{\circ}{V}_{x_0} \quad \forall\, x \in \overset{\circ}{V}_{x_0} \cap E \quad |\alpha(x)| \le \varepsilon.$$

Since the function $\alpha(x)$ is bounded in some neighborhood $\overset{\circ}{V}_{x_0}$ by the value ε, the relation $|f(x)| = |\alpha(x) g(x)| \le \varepsilon |g(x)|$ holds, and by definition 4, we obtain that $f(x) = O\big(g(x)\big)$, $x \to x_0$. $\square$

Remark.

The converse is not true: if $f(x) = O\big(g(x)\big)$, $x \to x_0$, then it does not follow that $f(x) = o\big(g(x)\big)$, $x \to x_0$. To show this, it is enough to give an example. Earlier, we established that $x \sin \frac{1}{x} = O(x)$, $x \to 0$, while $x \sin \frac{1}{x} \ne o(x)$, $x \to 0$.

2. The following relation holds: $o(f) O(g) = o(fg)$, $x \to x_0$.

Proof.

This relation should be understood as follows: if $h_1 = o(f)$, $h_2 = O(g)$, then $h_1 h_2 = o(fg)$, $x \to x_0$.

By definition 1, for some neighborhood $\overset{\circ}{U}{}'_{x_0}$, we have: $h_1(x) = \alpha_1(x) f(x)$ and $\alpha_1(x) \to 0$ as $x \to x_0$.

By definition 4, for some neighborhood $\overset{\circ}{U}{}''_{x_0}$, we have: $h_2(x) = \alpha_2(x) g(x)$ and $|\alpha_2(x)| \le C$ for all $x \in \overset{\circ}{U}{}''_{x_0}$.

Then for the intersection of these neighborhoods $\overset{\circ}{U}{}'_{x_0} \cap \overset{\circ}{U}{}''_{x_0}$, we have: $h_1(x) h_2(x) = \alpha(x) f(x) g(x)$, where $\alpha(x) = \alpha_1(x) \alpha_2(x)$. Since $\alpha_1(x)$ is infinitesimal as $x \to x_0$ and $\alpha_2(x)$ is bounded in a neighborhood of x_0, we obtain that $\alpha(x)$ is infinitesimal.

Therefore, for the functions $h_1(x)h_2(x)$ and $f(x)g(x)$, the condition of definition 1 is satisfied, that is, $h_1h_2 = o(fg)$, $x \to x_0$. $\square$

Equivalent functions at a point

15A/39:55 (05:28), 15B/00:00 (07:24)

DEFINITION 5.

Let f and g be functions acting from E to $\mathbb{R}$, x_0 (a real number or the point at infinity) be the limit point of E. The function $f(x)$ is called to be *equivalent to the function* $g(x)$ *as* $x \to x_0$ if there exists a punctured neighborhood $\overset{\circ}{U}_{x_0}$ of the point x_0 such that the function $f(x)$ can be represented in the form $\alpha(x)g(x)$ for all $x \in E \cap \overset{\circ}{U}_{x_0}$ and $\lim_{x\to x_0} \alpha(x) = 1$.

This is denoted as follows: $f(x) \sim g(x)$, $x \to x_0$. When using the notation "$\sim$", it is necessary to indicate, which point the function argument approaches.

If the function $g(x)$ does not equal zero in some punctured neighborhood $\overset{\circ}{U}_{x_0}$, then the relation $f(x) \sim g(x)$, $x \to x_0$ is equivalent to the following limit relation:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1.$$

THEOREM (ON EQUIVALENCE OF FUNCTIONS).

The equivalence of functions as $x \to x_0$ is an *equivalence relation*, that is, the following three properties are satisfied for it:

1) $f \sim f$, $x \to x_0$ (*reflexivity*);

2) if $f \sim g$, $x \to x_0$, then $g \sim f$, $x \to x_0$ (*symmetry*);

3) if $f \sim g$ and $g \sim h$, $x \to x_0$, then $f \sim h$, $x \to x_0$ (*transitivity*).

PROOF.

These properties are proved directly by definition. For example, transitivity is proved as follows.

Since $f \sim g$, $x \to x_0$, therefore, $f(x) = \alpha_1(x)g(x)$, and $\alpha_1(x) \to 1$ as $x \to x_0$. Since $g \sim h$, $x \to x_0$, therefore, $g(x) = \alpha_2(x)h(x)$, and $\alpha_2(x) \to 1$ as $x \to x_0$. Then $f(x) = \alpha_1(x)g(x) = \alpha_1(x)\alpha_2(x)h(x) = \alpha(x)h(x)$, where $\alpha(x) = \alpha_1(x)\alpha_2(x) \to 1$ as $x \to x_0$. Therefore, $f \sim h$, $x \to x_0$. $\square$

Theorem (on the relation between the equivalence and O-notation).

The following relations are equivalent:

$$(f \sim g,\, x \to x_0) \Leftrightarrow (f = g + o(g),\, x \to x_0).$$

Proof.

1. Given: $f = g + o(g)$, $x \to x_0$. Prove: $f \sim g$, $x \to x_0$.

The expression $f = g + o(g)$, $x \to x_0$, means that $f(x)$ can be represented as $g(x) + \alpha(x)g(x)$ and $\alpha(x) \to 0$ as $x \to x_0$.

Denote $\tilde{\alpha}(x) = 1 + \alpha(x)$. Obviously,

$$\lim_{x\to x_0} \tilde{\alpha}(x) = \lim_{x\to x_0} \bigl(1 + \alpha(x)\bigr) = 1 + \lim_{x\to x_0} \alpha(x) = 1.$$

Thus, $f(x) = g(x) + \alpha(x)g(x) = \bigl(1+\alpha(x)\bigr)g(x) = \tilde{\alpha}(x)g(x)$ and $\tilde{\alpha}(x) \to 1$ as $x \to x_0$. Therefore, by definition 5, $f \sim g$, $x \to x_0$.

2. Given: $f \sim g$, $x \to x_0$. Prove: $f = g + o(g)$, $x \to x_0$.

The expression $f \sim g$, $x \to x_0$, means that $f(x)$ can be represented as $\alpha(x)g(x)$ and $\alpha(x) \to 1$ as $x \to x_0$.

Denote $\tilde{\alpha}(x) = \alpha(x) - 1$. Obviously,

$$\lim_{x\to x_0} \tilde{\alpha}(x) = \lim_{x\to x_0} \bigl(\alpha(x) - 1\bigr) = \lim_{x\to x_0} \alpha(x) - 1 = 1 - 1 = 0.$$

Thus, $f(x) = \alpha(x)g(x) = \bigl(1 + \alpha(x) - 1\bigr)g(x) = g(x) + \tilde{\alpha}(x)g(x)$ and $\tilde{\alpha}(x) \to 0$ as $x \to x_0$. Therefore, by definition 1, $f = g + o(g)$, $x \to x_0$. □

We have already noted that when calculating limits, one can replace functions with equivalent functions (*in products*). Let us give a rigorous formulation and proof of this fact.

Theorem (on the use of equivalences in finding limits).

Let the functions f, $\tilde{f}$, and g be defined on the set E, x_0 be the limit point of E. Let $f \sim \tilde{f}$, $x \to x_0$. If there exists one of the limits $\lim_{x\to x_0} fg$ or $\lim_{x\to x_0} \tilde{f}g$, then there exists another limit and the values of these limits are equal:

$$\lim_{x\to x_0} fg = \lim_{x\to x_0} \tilde{f}g.$$

Proof.

For definiteness, suppose that there exists a limit $\lim_{x\to x_0} \tilde{f}g$.

Since by condition $f \sim \tilde{f}$, $x \to x_0$, we obtain that $f = \alpha(x)\tilde{f}(x)$, where $\alpha(x) \to 1$ as $x \to x_0$.

Then

$$\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \alpha(x)\tilde{f}(x)g(x).$$

Since the limit of the function $\alpha(x)$ exists (and equals 1) and the limit of the product $\tilde{f}(x)g(x)$ exists by condition, we obtain, by virtue of the arithmetic properties of the limit, that the limit of the product $f(x)g(x)$ also exists and

$$\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \alpha(x) \lim_{x\to x_0} \tilde{f}(x)g(x) = \lim_{x\to x_0} \tilde{f}(x)g(x). \ \square$$

Example.

Let us calculate the limit of the function $\frac{\sin x^2}{2x^2}$ at the point 0 using the equivalence $\sin f(x) \sim f(x)$, which holds for $x \to x_0$ if $\lim_{x\to x_0} f(x) = 0$:

$$\lim_{x\to 0} \frac{\sin x^2}{2x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Remark.

It should be emphasized that replacement of a function with an equivalent function can be performed *only in products.* Such a replacement *cannot be performed in sums.*

16. Differentiable functions

Preliminary remarks and basic definitions

Differentiable functions: preliminary remarks 15B/07:24 (06:23)

A function differentiable at a point behaves like a *linear function* in a neighborhood of this point. Moreover, the smaller the neighborhood, the closer to linear the behavior of this function will be. Thus, to study such an important property of the function as the *rate of its change at a given point*, we can replace the original differentiable function f with some linear function of the form $Ax + b$ and analyze the coefficient A. This coefficient A exists for any differentiable function and is called the *derivative of this function*.

Differentiability of a function at a point: definition 15B/13:47 (05:31)

Definition.

Let the function f act from E to $\mathbb{R}$ and the point $x_0 \in E$ be the limit point of the set E. The function f is called *differentiable* at the point x_0 if this function is representable in a neighborhood of the point x_0 in the following form:

$$f(x) = f(x_0) + A(x - x_0) + o(x - x_0), \quad x \to x_0. \tag{1}$$

Thus, the function f is representable as the linear part $f(x_0) + A(x - x_0)$ and the nonlinear part $o(x - x_0)$, which approaches zero faster than $x - x_0$ as $x \to x_0$. Therefore, for x close to x_0, we can assume that the function f behaves like a linear function $f(x_0) + A(x - x_0)$.

Equality (1) can be rewritten more briefly if x is represented as $x_0 + h$:

$$f(x_0 + h) = f(x_0) + Ah + o(h), \quad h \to 0. \tag{2}$$

The difference $x - x_0$ is called the *increment of the argument* at the point x_0 and is denoted by $\Delta_{x_0} x$ or simply Δx when it is clear at which point x_0 this increment is considered. The difference $f(x) - f(x_0)$ is called the *increment of the function* f at the point x_0 and is denoted by $\Delta_{x_0} f$ or simply Δf when it is clear at which point x_0 this increment is considered. Note that the notation Δx is often used to indicate a quantity approaching 0; in this case, it plays the same role as h in (2).

If we move the term $f(x_0)$ to the left-hand side in equality (1) and use the notation of the argument increment and the function increment, then we can obtain another relation valid for a differentiable function in a neighborhood of the point x_0:

$$\Delta f = A\Delta x + o(\Delta x), \quad \Delta x \to 0. \tag{3}$$

Thus, if a function is differentiable, then its increment is equal to the argument increment multiplied by some coefficient A plus some additional term, which decreases faster than the argument increment Δx as $\Delta x \to 0$.

Derivative of a function

15B/19:18 (09:32)

Definition.

Let the function f act from E to $\mathbb{R}$ and the point $x_0 \in E$ be the limit point of E. If there exists a limit of the form $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$, then it is called the *derivative* $f'(x_0)$ of the function f at the point x_0:

$$f'(x_0) \overset{\text{def}}{=} \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

So, the derivative is equal to the limit of the ratio of the increment of the function to the increment of the argument at a given point as the increment of the argument approaches zero:

$$f'(x_0) = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}.$$

A derivative can be denoted in various ways: not only $f'(x_0)$, but also $f'(x)\big|_{x=x_0}$, $\frac{df}{dx}(x_0)$, $\left.\frac{df}{dx}\right|_{x=x_0}$.

Theorem (on the equivalence of the differentiability and the existence of a derivative).

The derivative of the function f at the point x_0 exists if and only if the function f is differentiable at a given point, and the derivative is equal to the coefficient A given in the definition of differentiability (see (1), (2), (3)).

Proof.

First, suppose that the function has a derivative, that is, there exists the following limit, which we denote by the letter A:

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = A. \tag{4}$$

We introduce an auxiliary function

$$\alpha(h) = \frac{f(x_0 + h) - f(x_0)}{h} - A. \tag{5}$$

It follows from (4) that $\alpha(h) \to 0$ as $h \to 0$.

Let us transform equality (5) by multiplying it by h:

$$\alpha(h)h = f(x_0 + h) - f(x_0) - Ah. \tag{6}$$

Then we regroup the terms of the resulting equality:

$$f(x_0 + h) = f(x_0) + Ah + \alpha(h)h. \tag{7}$$

Since $\alpha(h) \to 0$ as $h \to 0$, we can replace the term $\alpha(h)h$ with $o(h)$ as $h \to 0$. As a result, we obtain relation (2), which means that the function f is differentiable at the point x_0. Moreover, the coefficient A in this relation is, by virtue of (4), the derivative of the function f at the point x_0.

To prove the statement in the opposite direction, the above transformations should be performed in the reverse order. We start with equality (2), which means that the function f is differentiable at the point x_0, and represent the term $o(h)$ in the form $\alpha(h)h$, where $\alpha(h) \to 0$ as $h \to 0$. As a result, we obtain equality (7).

Transforming equality (7) to the form (6), we then divide equality (6) by h and get equality (5).

Since $\alpha(h) \to 0$ as $h \to 0$, the limit on the right-hand side of equality (5) exists and is equal to 0, therefore equality (4) holds.

Equality (4) means that the derivative of the function f at the point x_0 exists and is equal to A, that is, the coefficient in equality (2) from the definition of a differentiable function. □

Continuity of a differentiable function 15B/28:50 (09:08)

Theorem (on the continuity of a differentiable function).

If the function f is differentiable at the point x_0, then it is continuous at this point.

Proof.

We use equality (1) from the definition of differentiability and pass to the limit in it as $x \to x_0$:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \big(f(x_0) + A(x - x_0) + o(x - x_0)\big) =$$
$$= \lim_{x \to x_0} f(x_0) + \lim_{x \to x_0} A(x - x_0) + \lim_{x \to x_0} o(x - x_0).$$

The first of the limits on the right-hand side of the equality is $f(x_0)$ and the other two are 0 (the middle limit as the product of the constant A and the infinitesimal value $x - x_0$ and the last limit as the product of two infinitesimals, since $o(x - x_0) = \alpha(x)(x - x_0)$ and $\lim_{x\to x_0} \alpha(x) = 0$).

Thus,

$$\lim_{x\to x_0} f(x) = f(x_0).$$

The obtained limit relation means that the function f is continuous at the point x_0. □

REMARK.

The converse is not true: if the function is continuous, then it is not necessarily differentiable. As an example, consider the function $f(x) = |x|$ at the point $x_0 = 0$. Obviously, this function is continuous at zero:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0 = f(0).$$

At the same time, the derivative of the function $|x|$ at zero does not exist, since the ratio of the function increment to the argument increment in this case has the form $\frac{|x|-|0|}{x-0} = \frac{|x|}{x} = \operatorname{sign} x$, and the function $\operatorname{sign} x$ has no limit at the point 0.

Differential of a function

16A/00:00 (12:50)

We previously proved that the coefficient A from the definition of the differentiability of a function f at a point x_0 is equal to the derivative of this function at a given point. Therefore, relation (1) from the definition of differentiability can be rewritten as follows:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0), \quad x \to x_0. \tag{8}$$

DEFINITION.

Let the function f be differentiable at the point x_0. The *differential* of the function f at the point x_0 is the expression $f'(x_0)(x - x_0)$. The differential is denoted by $df(x_0, x)$ or, more briefly, by $df(x_0)$ and depends on two parameters: the point x_0 at which the derivative is calculated and the point x included in the increment of the argument.

The differential is said to be a *linear part of the increment of the function*, since it is a linear function with respect to the increment of the argument $x - x_0$ and the increment of the function at the point x_0, by virtue of (8), can be represented as follows:

$$f(x) - f(x_0) = df(x_0) + o(x - x_0), \quad x \to x_0. \tag{9}$$

Thus, the increment of a differentiable function consists of two parts: linear part $df(x_0)$ and nonlinear part $o(x-x_0)$, which decreases faster than the linear part as $x \to x_0$.

Consider the simplest linear function $g(x) = x$. Obviously, this function is differentiable at any point x_0 and its derivative is equal to 1:

$$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1 = 1.$$

We can also say that for a given function, its increment is equal to the increment of the argument: $g(x) - g(x_0) = x - x_0$.

Therefore, for the function $g(x)$, we can write $dg(x_0) = x - x_0$. If we take into account that the function $g(x)$ is defined by the equality $g(x) = x$, then its differential can be rewritten as follows: $dg(x_0) = dx(x_0) = x - x_0$. In the notation $dx(x_0)$, the parameter x_0 is usually not written, so we obtain:

$$dx = x - x_0.$$

The value dx is called the *differential of the independent variable* x. Using the notation dx for the differential of the independent variable, the expression for the differential of an arbitrary differentiable function f can be rewritten as follows:

$$df(x_0) = f'(x_0)dx. \tag{10}$$

It should not be assumed that dx is a small quantity. This is just some increment of an argument of the form $x - x_0$. However, if dx is small (that is, the difference $x - x_0$ is small), then, by virtue of (9), the differential will be a good approximation for the function increment $f(x) - f(x_0)$.

If both sides of equality (10) are divided by dx, then we obtain the relation $\frac{df(x_0)}{dx} = f'(x_0)$. This explains why the derivative $f'(x_0)$ also uses the following notation: $\frac{df}{dx}(x_0)$.

Derivatives of some elementary functions

16A/12:50 (10:50)

1. The constant function α.

$$\alpha' = \lim_{x \to x_0} \frac{\alpha - \alpha}{x - x_0} = \lim_{x \to x_0} 0 = 0.$$

Thus, the derivative of a constant function at any point is 0.

2. The exponential function a^x, $a > 0$, $a \neq 1$.

$$(a^x)'|_{x=x_0} = \lim_{x \to x_0} \frac{a^x - a^{x_0}}{x - x_0} = \lim_{x \to x_0} a^{x_0} \frac{a^{x-x_0} - 1}{x - x_0}.$$

To calculate the limit, we use the equivalence $a^{x-x_0} - 1 \sim (x - x_0)\ln a$, $x \to x_0$. Finally we get:

$$(a^x)'|_{x=x_0} = \lim_{x\to x_0} a^{x_0}\frac{(x-x_0)\ln a}{x-x_0} = a^{x_0}\ln a.$$

Thus, an exponential function is differentiable at any point in its domain of definition, moreover,

$$(a^x)' = a^x \ln a.$$

This formula has the simplest form for the derivative of the function e^x:

$$(e^x)' = e^x \ln e = e^x.$$

Therefore, the *function e^x does not change during differentiation.*

3. THE LOGARITHM $\log_a x$, $a > 0$, $a \neq 1$.

$$(\log_a x)'|_{x=x_0} = \lim_{x\to x_0}\frac{\log_a x - \log_a x_0}{x - x_0} = \lim_{x\to x_0}\frac{1}{x_0}\cdot\frac{\log_a \frac{x}{x_0}}{\frac{x}{x_0} - 1}.$$

To calculate the obtained limit, we use the following equivalence: $\log_a \frac{x}{x_0} \sim \left(\frac{x}{x_0} - 1\right)\frac{1}{\ln a}$, $x \to x_0$. Finally we get:

$$(\log_a x)'|_{x=x_0} = \lim_{x\to x_0}\frac{1}{x_0}\cdot\frac{\frac{x}{x_0} - 1}{(\frac{x}{x_0} - 1)\ln a} = \lim_{x\to x_0}\frac{1}{x_0}\cdot\frac{1}{\ln a} = \frac{1}{x_0\ln a}.$$

Thus, the logarithm function is differentiable at any point in its domain of definition, moreover,

$$(\log_a x)' = \frac{1}{x\ln a}.$$

This formula has the simplest form for the derivative of the natural logarithm:

$$(\ln x)' = \frac{1}{x\ln e} = \frac{1}{x}.$$

4. THE SINE.

$$(\sin x)'|_{x=x_0} = \lim_{x\to x_0}\frac{\sin x - \sin x_0}{x - x_0} = \lim_{x\to x_0}\frac{2\sin\frac{x-x_0}{2}\cos\frac{x+x_0}{2}}{x - x_0}.$$

For the expression $\sin\frac{x-x_0}{2}$, we can use the following equivalence: $\sin\frac{x-x_0}{2} \sim \frac{x-x_0}{2}$, $x \to x_0$. Finally we get:

$$(\sin x)'|_{x=x_0} = \lim_{x\to x_0}\frac{2\cdot\frac{x-x_0}{2}\cos\frac{x+x_0}{2}}{x - x_0} = \lim_{x\to x_0}\cos\frac{x+x_0}{2} = \cos x_0.$$

Thus, the sine function is differentiable at any point in its domain of definition, moreover,

$$(\sin x)' = \cos x.$$

5. The cosine.

Using the formula $\cos x - \cos x_0 = -2\sin\frac{x-x_0}{2}\sin\frac{x+x_0}{2}$, it is easy to prove that the cosine function is differentiable in any point of its domain of definition and, moreover,

$$(\cos x)' = -\sin x.$$

Derivatives of other elementary functions will be found later, after studying additional properties of derivatives.

17. Properties of differentiable functions

Arithmetic properties of derivatives and differentials

Theorem on arithmetic properties of derivatives

16A/23:40 (14:47)

Theorem (on arithmetic properties of derivatives).

Let the functions f and g act from E to $\mathbb{R}$, $x_0 \in E$ be the limit point of E, and the functions f and g be differentiable at the point x_0. Then

1) the sum $f + g$ is differentiable at the point x_0 and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0);$$

2) the product fg is differentiable at the point x_0 and

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0);$$

3) if the condition $g(x_0) \neq 0$ is additionally satisfied, then the quotient $\frac{f}{g}$ is differentiable at the point x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

Proof.

It is sufficient for us to prove the validity of the above formulas for the derivatives, since the existence of the derivative at a given point implies the differentiability of the function at this point.

1. The formula for the derivative of the sum immediately follows from the arithmetic property for the limit of the sum of functions:

$$(f + g)'(x_0) = \lim_{x \to x_0} \frac{(f + g)(x) - (f + g)(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}.$$

Since both the obtained limits exist and are equal to the derivatives of the original functions f and g at the point x_0, the derivative of the sum $f + g$ at the point x_0 also exists and is equal to $f'(x_0) + g'(x_0)$.

2. For the derivative of the product, we perform the following transformations of the initial limit:

$$(fg)'(x_0) = \lim_{x\to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} =$$
$$= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} =$$
$$= \lim_{x\to x_0} g(x)\frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} f(x_0)\frac{g(x) - g(x_0)}{x - x_0}.$$

In the first of the obtained limits, the first factor $g(x)$ approaches $g(x_0)$ as $x \to x_0$, since by condition the function $g(x)$ is differentiable at the point x_0 and therefore it is continuous at this point. The second factor $\frac{f(x)-f(x_0)}{x-x_0}$ approaches $f'(x_0)$, since by condition the function $f(x)$ is differentiable at the point x_0. Due to the differentiability of the function $g(x)$, the second of the limits is $f(x_0)g'(x_0)$.

Since the both limits exist, we obtain that the derivative $(fg)'(x_0)$ also exists and is equal to $f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

3. For the quotient $\frac{f}{g}$, first of all, we note that it is defined in a certain neighborhood of the point x_0, since by condition the function $g(x)$ does not vanish at the point x_0 and is differentiable at this point (and therefore is continuous at this point). So, by the simplest property of continuous functions, $g(x)$ does not vanish in some neighborhood of the point x_0 and therefore the quotient $\frac{f}{g}$ is defined in this neighborhood.

To find the derivative of the quotient, it is necessary to perform a little more complicated transformations of the initial limit (compared to finding the derivative of the product):

$$\left(\frac{f}{g}\right)'(x_0) = \lim_{x\to x_0} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x_0)g(x)(x - x_0)}.$$

At the final stage of the transformations, we should add the terms $-f(x_0)g(x_0) + f(x_0)g(x_0)$ to the numerator, rearrange the terms in the numerator in the same way as in the proof of the formula for the product derivative, and take into account that the limit of the factors $g(x_0)g(x)$ in the denominator equals $g^2(x_0)$. □

Corollaries 16A/38:27 (06:46), 16B/00:00 (07:45)

1. The derivative of a function multiplied by a constant is a constant multiplied by the derivative of the original function: $\big(\alpha f(x)\big)' = \alpha f'(x)$.

This formula immediately follows from the formula for the product derivative, given that the derivative of the constant is 0.

2. The derivative of the linear combination of differentiable functions is equal to the linear combination of the derivatives of the original functions:

$$\big(\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x)\big)' =$$
$$= \alpha_1 f_1'(x) + \alpha_2 f_2'(x) + \cdots + \alpha_n f_n'(x).$$

More briefly, this relation can be written using the *summation sign* $\sum$:

$$\Big(\sum_{i=1}^{n} \alpha_i f_i(x)\Big)' = \sum_{i=1}^{n} \alpha_i f_i'(x).$$

The formula follows from the formula for the derivative of the sum of functions and the formula for the derivative of the product of a function by a constant.

3. The tangent function is differentiable at any point in its domain of definition and

$$(\tan x)' = \frac{1}{\cos^2 x}.$$

It is enough to use the formulas for the derivatives of sine and cosine, as well as the formula for the derivative of the quotient:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} =$$
$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

In a similar way, one can prove the formula for the derivative of the cotangent:

$$(\cot x)' = -\frac{1}{\sin^2 x}.$$

4. The power function x^n with the natural exponent n is differentiable for all $x \in \mathbb{R}$ and $(x^n)' = nx^{n-1}$.

We prove this statement by induction.

For $n = 1$, the statement is true, since we have already proved that $x' = 1$ and the formula being proved gives the same value: $1x^0 = 1$.

Suppose that the formula being proved is valid for $n = k$, that is, that $(x^k)' = kx^{k-1}$, and prove it for $n = k+1$. To do this, we represent x^{k+1} in the form $x^k x$ and use the inductive hypothesis and the formula for the product derivative:

$$(x^{k+1})' = (x^k x)' = (x^k)'x + x^k(x)' = kx^{k-1}x + x^k = (k+1)x^k.$$

Thus, assuming that the formula is valid for $n = k$, we proved it for $n = k + 1$. Therefore, according to the principle of mathematical induction, this formula is valid for all positive integers n.

5. A polynomial of degree n $P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ is a differentiable function and its derivative is a polynomial of degree $n-1$: $P_n'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1}$.

To prove this statement, it suffices to use corollaries 2 and 4.

6. Arithmetic properties of differentials.

Let the functions f and g act from E to $\mathbb{R}$, $x_0 \in E$ be the limit point of E, and the functions f and g be differentiable at the point x_0. Then for the differentials of the sum $f + g$, the product fg, and the quotient $\frac{f}{g}$ at the point x_0, the following formulas are valid (the formula for the quotient $\frac{f}{g}$ is valid under the additional condition $g(x_0) \neq 0$):

$$d(f+g)(x_0) = df(x_0) + dg(x_0),$$
$$d(fg)(x_0) = df(x_0)g(x_0) + f(x_0)dg(x_0),$$
$$d\Big(\frac{f}{g}\Big)(x_0) = \frac{df(x_0)g(x_0) - f(x_0)dg(x_0)}{g^2(x_0)}.$$

The validity of these formulas immediately follows from the definition of the differential and the corresponding arithmetic properties of derivatives. For the differential of the sum, we have:

$$d(f+g)(x_0) = (f+g)'(x_0)dx = \big(f'(x_0) + g'(x_0)\big)dx =$$
$$= f'(x_0)dx + g'(x_0)dx = df(x_0) + dg(x_0).$$

The formulas for the product and quotient differentials are proved similarly.

Differentiation of superposition

Theorem on the differentiation of superposition

16B/07:45 (15:16)

Theorem (on differentiation of superposition).

Let the function $f(x)$ act from E to G and be differentiable at the point $x_0 \in E$ (this means, in particular, that x_0 is the limit point of E).

Let the function $g(y)$ act from G to $\mathbb{R}$ and be differentiable at the point $y_0 = f(x_0)$ (this means, in particular, that y_0 is the limit point of G).

Then the superposition $h = g \circ f$, acting from E to $\mathbb{R}$, is differentiable at the point x_0 and for its derivative the following formula holds:

$$h'(x_0) = (g \circ f)'(x_0) = g'(y_0) f'(x_0). \tag{1}$$

PROOF.

Let us write the definitions of the differentiability of the function f at the point x_0 and the function g at the point y_0:

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \alpha(x)(x - x_0), \quad \lim_{x \to x_0} \alpha(x) = 0; \tag{2}$$

$$g(y) - g(y_0) = g'(y_0)(y - y_0) + \beta(y)(y - y_0), \quad \lim_{y \to y_0} \beta(y) = 0. \tag{3}$$

In relations (2) and (3), we replaced the terms $o(x - x_0)$ and $o(y - y_0)$ with their representations containing infinitesimal functions $\alpha(x)$ and $\beta(y)$ respectively.

We need to obtain a similar representation for increment of the superposition $h(x) - h(x_0) = g\big(f(x)\big) - g\big(f(x_0)\big)$. To obtain such an increment on the left-hand side of the equality, it suffices to substitute $f(x)$ in place of y and $f(x_0)$ in place of y_0 in relation (3):

$$g\big(f(x)\big) - g\big(f(x_0)\big) = g'(y_0)\big(f(x) - f(x_0)\big) + \beta\big(f(x)\big)\big(f(x) - f(x_0)\big).$$

We did not replace y_0 with $f(x_0)$ in the constant factor $g'(y_0)$, since we will not change it.

On the right-hand side of the resulting equality, we can replace the increments $f(x) - f(x_0)$ using relation (2):

$$\begin{aligned} g\big(f(x)\big) - g\big(f(x_0)\big) &= \\ &= g'(y_0)\big(f'(x_0)(x - x_0) + \alpha(x)(x - x_0)\big) + \\ &\quad + \beta\big(f(x)\big)\big(f'(x_0)(x - x_0) + \alpha(x)(x - x_0)\big) = \\ &= g'(y_0) f'(x_0)(x - x_0) + \alpha(x) g'(y_0)(x - x_0) + \\ &\quad + \beta\big(f(x)\big) f'(x_0)(x - x_0) + \beta\big(f(x)\big)\alpha(x)(x - x_0). \end{aligned} \tag{4}$$

If we prove that all terms on the right-hand side of the obtained equality, except the first, are $o(x - x_0)$ as $x \to x_0$, then this equality can be rewritten in the form

$$g\big(f(x)\big) - g\big(f(x_0)\big) = g'(y_0) f'(x_0)(x - x_0) + o(x - x_0).$$

This immediately implies the differentiability of the superposition $g\big(f(x)\big)$ at the point x_0, as well as validity of formula (1), since the value of the derivative is equal to the coefficient of $(x - x_0)$ on the right-hand side of the resulting equality.

So, it remains for us to prove that all terms on the right-hand side of the equality (4), except the first, are $o(x - x_0)$, that is, they can be represented in the form $\gamma(x)(x-x_0)$, where $\gamma(x) \to 0$ as $x \to x_0$. Taking the factor $(x-x_0)$ out of the brackets in these terms, we obtain that the function $\gamma(x)$ can be represented as follows:

$$\gamma(x) = \alpha(x)g'(y_0) + \beta\bigl(f(x)\bigr)f'(x_0) + \beta\bigl(f(x)\bigr)\alpha(x). \tag{5}$$

By virtue of (2), $\lim_{x\to x_0} \alpha(x) = 0$, so the first term is $o(x-x_0)$ as the product of an infinitesimal by a constant.

Let us turn to the superposition $\beta\bigl(f(x)\bigr)$. We are interested in its limit as $x \to x_0$. Since the function $f(x)$ is differentiable at the point x_0 and, therefore, continuous at it, we obtain: $\lim_{x\to x_0} f(x) = f(x_0) = y_0$. We can define the function β at the point y_0 by continuity as follows: $\beta(y_0) = 0$. For the superposition $\beta\bigl(f(x)\bigr)$, all the conditions of the superposition limit theorem in the case, when the external function is continuous, are satisfied. By virtue of this theorem,

$$\lim_{x\to x_0} \beta\bigl(f(x)\bigr) = \lim_{y\to y_0} \beta(y) = 0.$$

Thus, the two remaining terms in (5) are also $o(x-x_0)$. □

Remark.

Formula (1) of the derivative of superposition $g \circ f$ at the point x_0 can also be written as follows, without using the notation y_0:

$$(g \circ f)'(x_0) = g'(y)\big|_{y=f(x_0)}\, f'(x_0).$$

One can also specify the expression $f(x_0)$ as an argument to the function g', but in this case it is desirable to clarify that differentiation is carried out with respect to the variable y:

$$(g \circ f)'(x_0) = g'_y\bigl(f(x_0)\bigr)f'(x_0).$$

For the derivative $f'(x_0)$, such a clarification is not required, since only one kind of differentiation is possible for it, namely, with respect to the variable x.

Corollaries

16B/23:01 (15:56)

1. The power function x^α: $(0,+\infty) \to (0,+\infty)$ with an arbitrary real exponent $\alpha \neq 0$ is differentiable for all $x \in (0,+\infty)$ and

$$(x^\alpha)' = \alpha x^{\alpha-1}.$$

Remark.

Thus, we generalized the previously proved fact for a power function with a natural exponent. It should be noted, however, that in the case of an arbitrary real α, we can only consider the positive arguments x.

PROOF.

Let us represent the function x^α as a superposition:

$$x^\alpha = e^{\alpha \ln x} = (e^y) \circ (\alpha \ln x).$$

Since we have already proved the differentiability of the functions e^y and $\ln x$, we can apply the superposition differentiation theorem:

$$(x^\alpha)' = \big((e^y) \circ (\alpha \ln x)\big)' = (e^y)'\big|_{y=\alpha \ln x} (\alpha \ln x)' =$$

$$= e^y\big|_{y=\alpha \ln x} \frac{\alpha}{x} = \alpha e^{\alpha \ln x} \frac{1}{x} = \alpha x^\alpha x^{-1} = \alpha x^{\alpha-1}. \ \square$$

2. The power-exponential function $f(x)^{g(x)}$, which is defined for an arbitrary function $g(x)$ and a function $f(x)$ that takes positive values, $f(x) > 0$, is differentiable at any point of its domain of definition if the functions f and g are differentiable at this point, and for its derivative the formula holds:

$$\big(f(x)^{g(x)}\big)' = g'(x) f(x)^{g(x)} \ln f(x) + f'(x) g(x) f(x)^{g(x)-1}.$$

REMARK.

To remember this formula, it is enough to notice that its first term on the right-hand side can be obtained by differentiating an exponential function of the form $a^{g(x)}$ (that is, $(a^{g(x)})' = g'(x) a^{g(x)} \ln a$), after which the base a is replaced by $f(x)$, and the second term can be obtained by differentiating a power function of the form $f(x)^\alpha$ (that is, $(f(x)^\alpha)' = f'(x) \alpha f(x)^{\alpha-1}$), after which the exponent α is replaced by $g(x)$.

PROOF.

Let us represent a power-exponential function as $f(x)^{g(x)} = e^{g(x) \ln f(x)}$ and use the superposition differentiation theorem:

$$\big(f(x)^{g(x)}\big)' = (e^{g(x) \ln f(x)})' = (e^y) \circ \big(g(x) \ln f(x)\big) =$$

$$= (e^y)'\big|_{y=g(x) \ln f(x)} \big(g(x) \ln f(x)\big)'.$$

Since $(e^y)' = e^y$,

$$(e^y)'\big|_{y=g(x) \ln f(x)} = e^{g(x) \ln f(x)} = f(x)^{g(x)}.$$

We transform the factor $\big(g(x) \ln f(x)\big)'$ separately using the formula of the derivative of product and the formula of the derivative of superposition:

$$\big(g(x) \ln f(x)\big)' = g'(x) \ln f(x) + g(x) \big(\ln f(x)\big)' =$$

$$= g'(x) \ln f(x) + g(x)\ (\ln y)'\big|_{f(x)} f'(x) =$$

$$= g'(x) \ln f(x) + \frac{f'(x) g(x)}{f(x)}.$$

Let us multiply the resulting expressions:

$$\left(f(x)^{g(x)}\right)' = f(x)^{g(x)}\left(g'(x)\ln f(x) + \frac{f'(x)g(x)}{f(x)}\right) =$$
$$= g'(x)f(x)^{g(x)}\ln f(x) + \frac{f'(x)g(x)f(x)^{g(x)}}{f(x)}.$$

Taking into account that $\frac{f(x)^{g(x)}}{f(x)} = f(x)^{g(x)-1}$, we obtain the formula to be proved. □

Differentiation of inverse function

Theorem on the differentiation of an inverse function

16B/38:57 (04:55), 17A/00:00 (14:50)

Theorem (on the differentiation of inverse function).

Let the function f be continuous and strictly monotone on the segment $[a, b]$, be differentiable at the point $x_0 \in (a, b)$ and $f'(x_0) \neq 0$. By virtue of continuity and strict monotonicity, the function f has the inverse function $f^{-1}(y)$ defined and continuous on the segment $[c, d] = f([a, b])$, and the interval (c, d) contains the point $y_0 = f(x_0)$. Then the function f^{-1} is differentiable at the point y_0 and

$$\left(f^{-1}(y_0)\right)' = \frac{1}{f'(x_0)}. \tag{6}$$

Proof[8].

Let us prove the validity of formula (6) for the derivative of the function f^{-1} at the point y_0, which immediately implies the differentiability of the function f^{-1} at a given point.

Let us write the definition of the derivative of f^{-1} at the point y_0:

$$\left(f^{-1}(y_0)\right)' = \lim_{y\to y_0}\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{\lim_{y\to y_0}\frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)}}. \tag{7}$$

The fraction in the denominator makes sense in the neighborhood of the point y_0 due to the strict monotonicity of the function f^{-1}: if $y \neq y_0$, then $f^{-1}(y) \neq f^{-1}(y_0)$, therefore, the denominator of the fraction does not vanish.

Using the fact that $y = f\big(f^{-1}(y)\big)$ and $f(x_0) = y_0$, we represent the fraction in the denominator as a superposition:

$$\frac{1}{\lim_{y\to y_0}\frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)}} = \frac{1}{\lim_{y\to y_0}\frac{f(x)-f(x_0)}{x-x_0}\circ f^{-1}(y)}.$$

[8] This version of the proof is slightly different from the version of the video lectures.

Since $f^{-1}(y)$ is continuous at y_0, we obtain that

$$\lim_{y\to y_0} f^{-1}(y) = f^{-1}(y_0) = x_0.$$

The limit of the external function $\frac{f(x)-f(x_0)}{x-x_0}$, as $x \to x_0$, exists, since the function $f(x)$ has a derivative at the point x_0. In addition, due to the strict monotonicity of the function f^{-1}, if $y \neq y_0$, then $f^{-1}(y) \neq f^{-1}(y_0)$. Thus, all the conditions of the limit superposition theorem are satisfied, the limit of superposition exists and is equal to the limit of the external function:

$$\frac{1}{\lim_{y\to y_0} \frac{f(x)-f(x_0)}{x-x_0} \circ f^{-1}(y)} = \frac{1}{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)}.$$

Therefore, the limit (7), from which we started the transformation, also exists and is equal to $\frac{1}{f'(x_0)}$. Formula (6) and, thus, the differentiability of the inverse function $f^{-1}(y)$ at the point y_0 are proved. □

REMARKS.

1. Formula (6) of the derivative of the inverse function at the point y_0 can also be written as follows, without using the notation x_0:

$$\big(f^{-1}(y_0)\big)' = \frac{1}{f'(x)\big|_{x=f^{-1}(y_0)}}. \tag{8}$$

One can also specify the expression $f^{-1}(y_0)$ as an argument to the function f', but in this case it is desirable to clarify that differentiation is carried out with respect to the variable x:

$$\big(f^{-1}(y_0)\big)' = \frac{1}{f'_x\big(f^{-1}(y_0)\big)}.$$

2. Under the assumption that the differentiability of the inverse function has already been proved, formula (6) can be easily obtained from the superposition differentiation theorem. Consider the identity $y = f\big(f^{-1}(y)\big)$ and find the derivative of both its parts at the point y_0. On the left-hand side we obtain 1, and on the right-hand side we differentiate a superposition as follows:

$$1 = \Big(f\big(f^{-1}(y_0)\big)\Big)' = f'(x)\big|_{x=f^{-1}(y_0)} \big(f^{-1}(y_0)\big)'.$$

If we divide the left-hand and right-hand sides of the resulting equality by $f'(x)\big|_{x=f^{-1}(y_0)}$, we obtain formula (8), which is one of versions of formula (6).

Corollaries: derivatives of inverse trigonometric functions

17A/14:50 (13:12)

The inverse function differentiation theorem simplifies finding derivatives for functions that are inverse to those elementary functions for which derivative formulas are already known. As an example, we find formulas for derivatives of inverse trigonometric functions.

1. The function $\arcsin y$ acts from $[-1,1]$ to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and is monotonically increasing and continuous. According to the inverse function differentiation theorem, the derivatives for the function $\arcsin y$ at the point y and for the function $\sin x$ at the point $x=\arcsin y$ are related by the equality $(\arcsin y)'=\frac{1}{(\sin x)'}$ provided that $(\sin x)'\neq 0$.

Since $(\sin x)'=\cos x$, and the equality $\cos x=0$ holds only at $x=\pm\frac{\pi}{2}$ on the segment $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we obtain that the arcsine derivative exists at all points y such that $\arcsin y\neq\pm\frac{\pi}{2}$, that is, at all points of the interval $(-1,1)$.

Assuming $y\in(-1,1)$ we get the formula for the derivative of $\arcsin y$:

$$(\arcsin y)'=\frac{1}{(\sin x)'|_{x=\arcsin y}}=\frac{1}{\cos(\arcsin y)}.$$

For $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, the value of $\cos x$ is greater than 0, so from the Pythagorean trigonometric identity we obtain: $\cos x=\sqrt{1-\sin^2 x}$. In addition, since the functions sine and arcsine are mutually inverse, the identity $\sin(\arcsin y)=y$ holds for all $y\in(-1,1)$. Therefore,

$$\frac{1}{\cos(\arcsin y)}=\frac{1}{\sqrt{1-\sin^2(\arcsin y)}}=\frac{1}{\sqrt{1-y^2}}.$$

So, the formula for the arcsine derivative is as follows:

$$(\arcsin y)'=\frac{1}{\sqrt{1-y^2}}.$$

This formula makes sense for all $y\in(-1,1)$. The derivative $(\arcsin y)'$ approaches $+\infty$ as $y\to\pm 1$.

2. The function $\arccos y$ acts from $[-1,1]$ to $[0,\pi]$ and is monotonically decreasing and continuous. Its derivative exists for $y\in(-1,1)$ and is calculated by the formula

$$(\arccos y)'=-\frac{1}{\sqrt{1-y^2}}.$$

This formula can be proved in the same way as the formula for the arcsine derivative, given that $(\cos x)'=-\sin x$.

3. The function $\arctan y$ is defined for all $y \in \mathbb{R}$, takes values on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and is monotonically increasing and continuous. Since the function $\tan x$ has a derivative that is not equal to 0 on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $(\tan x)' = \frac{1}{\cos^2 x} \neq 0$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the function $\arctan y$ is differentiable at any point $y \in \mathbb{R}$:

$$(\arctan y)' = \frac{1}{(\tan x)'|_{x=\arctan y}} = \frac{1}{\frac{1}{\cos^2(\arctan y)}} =$$

$$= \cos^2(\arctan y) = \frac{1}{1+\tan^2(\arctan y)} = \frac{1}{1+y^2}.$$

In deriving this formula, we used the relation $\frac{1}{\cos^2 x} = 1 + \tan^2 x$, which follows from the Pythagorean trigonometric identity.

So, the formula for the arctangent derivative is as follows:

$$(\arctan y)' = \frac{1}{1+y^2}.$$

This formula makes sense for all $y \in \mathbb{R}$. The derivative $(\arctan y)'$ approaches 0 as $y \to \pm\infty$.

Remark.

It is interesting to note that the derivative of arctangent, like the derivative of the logarithm, is a rational function, although the original functions are not rational.

18. Hyperbolic and inverse hyperbolic functions

Hyperbolic functions and their properties

17A/28:02 (11:04)

Definition.

The functions *hyperbolic sine* (notation $\sinh x$) and *hyperbolic cosine* (notation $\cosh x$) are defined as follows:

$$\sinh x \stackrel{\text{def}}{=} \frac{e^x - e^{-x}}{2}, \qquad \cosh x \stackrel{\text{def}}{=} \frac{e^x + e^{-x}}{2}.$$

The *hyperbolic tangent* function (notation $\tanh x$) is the ratio of the hyperbolic sine to the hyperbolic cosine:

$$\tanh x \stackrel{\text{def}}{=} \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\ .$$

Although the definitions of the hyperbolic sine and cosine do not resemble the definitions of the "ordinary" trigonometric functions $\sin x$ and $\cos x$, many properties of hyperbolic functions are similar to the properties of trigometric functions.

Consider the basic properties of hyperbolic functions (Fig. 8).

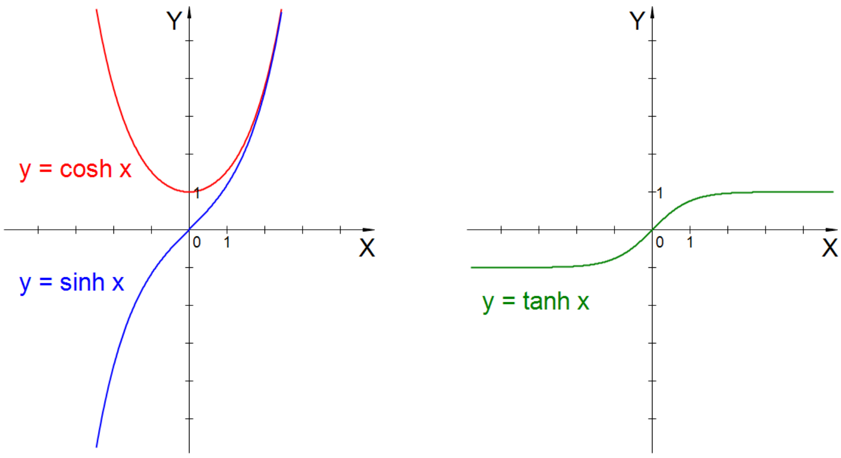

Fig. 8. Graphs of hyperbolic functions

The hyperbolic cosine function is an even function, it is 1 at the point 0: $\cosh 0 = \frac{e^0+e^0}{2} = 1$. The hyperbolic cosine approaches $+\infty$ as $x \to \pm\infty$ and it grows as an exponential function, that is, faster than any power function.

The hyperbolic sine function is an odd function, it is 0 at the point 0: $\sinh 0 = \frac{e^0 - e^0}{2} = 0$. The hyperbolic sine approaches $\pm\infty$ as $x \to \pm\infty$ and it also grows as an exponential function.

It should be noted that the difference $\cosh x - \sinh x$ is e^{-x}, therefore it is always positive and $\cosh x - \sinh x \to 0$ as $x \to +\infty$.

The hyperbolic tangent function is an odd function, it is 0 at the point 0: $\tanh 0 = \frac{\sinh 0}{\cosh 0} = 0$. The hyperbolic tangent approaches ± 1 as $x \to \pm\infty$. For example, let us prove this for $x \to +\infty$:

$$\lim_{x \to +\infty} \tanh x = \lim_{x \to +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to +\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1.$$

For hyperbolic functions, there exists an analogue of the Pythagorean trigonometric identity:

$$\cosh^2 x - \sinh^2 x = 1. \tag{1}$$

This relation can be proved directly using the definitions of the functions $\sinh x$ and $\cosh x$.

Let us find the derivatives of hyperbolic functions:

$$(\sinh x)' = \frac{(e^x - e^{-x})'}{2} = \frac{e^x + e^{-x}}{2} = \cosh x,$$

$$(\cosh x)' = \frac{(e^x + e^{-x})'}{2} = \frac{e^x - e^{-x}}{2} = \sinh x.$$

To find the derivative of the hyperbolic tangent, we use the formulas already found for the derivatives $\sinh x$ and $\cosh x$, as well as the relation (1):

$$(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{(\sinh x)' \cosh x - \sinh x (\cosh x)'}{\cosh^2 x} =$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$

Thus, the formulas of the derivatives for hyperbolic functions are very similar to the formulas of the derivatives of trigonometric functions.

Inverse hyperbolic functions and their properties

17B/00:00 (16:45)

Graphs of hyperbolic functions (see Fig. 8 in the previous section) allow us to assume that the functions $\sinh x$ and $\tanh x$ are one-to-one, and, therefore, there exist inverse functions for them. Let us derive a formula for the inverse function to the hyperbolic sine. To do this, we express the variable x through the variable y in the equation $\sinh x = y$:

$$\sinh x = y, \quad \frac{e^x - e^{-x}}{2} = y, \quad e^x - e^{-x} = 2y.$$

Move the $2y$ term to the left and multiply the resulting equality by e^x:

$$e^x - 2y - e^{-x} = 0, \quad e^{2x} - 2ye^x - 1 = 0.$$

If we make the change of variables $t = e^x$, then the last equation takes the form:

$$t^2 - 2yt - 1 = 0.$$

Let us find the roots of the obtained quadratic equation: $t_{1,2} = y \pm \sqrt{y^2+1}$. Since $t = e^x$, we are only interested in the positive root:

$$y + \sqrt{y^2+1} = e^x, \quad x = \ln\left(y + \sqrt{y^2+1}\right).$$

We have obtained the formula for the inverse function to the hyperbolic sine, which makes sense for all $y \in \mathbb{R}$. The function that is inverse to the hyperbolic sine is called the *areasine* and is denoted by $\operatorname{arsinh} y$. Thus, the function $\operatorname{arsinh} y$ acts from $\mathbb{R}$ to $\mathbb{R}$ and is expressed by the following formula:

$$\operatorname{arsinh} y = \ln\left(y + \sqrt{y^2+1}\right). \tag{2}$$

Similarly, solving the equation $\tanh x = y$ relative to x, we can obtain the formula for the inverse function to the hyperbolic tangent:

$$x = \frac{1}{2}\ln\frac{1+y}{1-y}.$$

The function inverse to the hyperbolic tangent is called the *areatangent* and denoted by $\operatorname{artanh} y$. Since the function $\tanh x$ acts from $\mathbb{R}$ to $(-1, 1)$, we obtain that the function $\operatorname{artanh} y$ acts from $(-1, 1)$ to $\mathbb{R}$ and is expressed by the following formula:

$$\operatorname{artanh} y = \frac{1}{2}\ln\frac{1+y}{1-y}. \tag{3}$$

The function $\cosh x$ does not have an inverse function on the entire numerical axis $\mathbb{R}$, since the equality $\cosh x = \cosh(-x)$ holds for all $x \neq 0$. However, the restriction of the function $\cosh x$ to the half axis $[0, +\infty)$ has the inverse function. This function is called the *areacosine*, denoted by $\operatorname{arcosh} y$, and acts from $[1, +\infty)$ to $[0, +\infty)$ by the formula, which can be obtained in the same way as the formula for the areasine:

$$\operatorname{arcosh} y = \ln\left(y + \sqrt{y^2-1}\right). \tag{4}$$

Since the derivatives of the functions $\sinh x$ and $\tanh x$ do not vanish anywhere, we obtain, by virtue of the theorem on the differentiation of the inverse function, that the functions $\operatorname{arsinh} y$ and $\operatorname{artanh} y$ are also differentiable at all points of their domain of definition. The derivative of the function $\cosh x$ vanishes at $x = 0$; therefore, the function $\operatorname{arcosh} y$ is non-differentiable at the point $y = \cosh 0 = 1$, however, it is differentiable at other points in its domain of definition, that is, $y \in (1, +\infty)$.

Derivatives of inverse hyperbolic functions can be found by differentiating formulas (2), (3), (4), which expess these functions by means of other elementary functions, and using the theorem on differentiation of superposition.

As an example, let us find the derivative $\operatorname{arsinh} y$ in this way:

$$(\operatorname{arsinh} y)' = (\ln\left(y + \sqrt{y^2+1}\right))' = (\ln t \circ \left(y + \sqrt{y^2+1}\right))' =$$
$$= \frac{1}{t}\bigg|_{y+\sqrt{y^2+1}} \left(1 + \frac{2y}{2\sqrt{y^2+1}}\right) =$$
$$= \frac{1}{y+\sqrt{y^2+1}} \frac{\sqrt{y^2+1}+y}{\sqrt{y^2+1}} = \frac{1}{\sqrt{y^2+1}}.$$

Thus,

$$(\operatorname{arsinh} y)' = \frac{1}{\sqrt{1+y^2}}. \tag{5}$$

This formula makes sense for all $y \in \mathbb{R}$ and differs from the formula for the derivative of the function arcsin only in the minus sign in the denominator.

Formula (5) can also be obtained in a simpler way, if we use the theorem on the differentiation of the inverse function, the formula for the derivative $\sinh x$, and relation (1):

$$(\operatorname{arsinh} y)' = \frac{1}{(\sinh x)'}\bigg|_{x=\operatorname{arsinh} y} = \frac{1}{\cosh x}\bigg|_{x=\operatorname{arsinh} y} =$$
$$= \frac{1}{\cosh(\operatorname{arsinh} y)} = \frac{1}{\sqrt{1+\sinh^2(\operatorname{arsinh} y)}} = \frac{1}{\sqrt{1+y^2}}.$$

When finding the derivative for the function $\operatorname{arcosh} y$ in the same way, we must take into account that this function takes non-negative values, and therefore, in the formula $\sinh x = \pm\sqrt{\cosh^2 x - 1}$, which follows from relation (1), we should take the plus sign:

$$(\operatorname{arcosh} y)' = \frac{1}{(\cosh x)'}\bigg|_{x=\operatorname{arcosh} y} = \frac{1}{\sinh x}\bigg|_{x=\operatorname{arcosh} y} =$$

$$= \frac{1}{\sinh(\operatorname{arcosh} y)} = \frac{1}{\sqrt{\cosh^2(\operatorname{arcosh} y) - 1}} = \frac{1}{\sqrt{y^2 - 1}}.$$

Thus,

$$(\operatorname{arcosh} y)' = \frac{1}{\sqrt{y^2 - 1}}.$$

This formula makes sense for all $y \in (1, +\infty)$. The derivative of the function $\operatorname{arcosh} y$ approaches infinity as $y \to 1$.

Let us also give a formula for the derivative of the function $\operatorname{artanh} y$, which can be obtained in a similar way using the derivative of the function $\tanh x$ and relation (1):

$$(\operatorname{artanh} y)' = \frac{1}{1 - y^2}.$$

This formula makes sense for all y from the scope of $(-1, 1)$ of the function $\operatorname{artanh} y$. It differs from the formula for the derivative of the function arctan only in the minus sign in the denominator.

19. Physical sense and geometric sense of the derivative

Physical sense of the derivative

17B/16:45 (08:23)

For simplicity, we will consider *one-dimensional motion*, that is, displacement along the OX axis. Let the *law of motion* be defined by the function $S(t)$, where t is the time, and the value $S(t)$ determines the position of the point on the axis OX at the time t. Let us choose some initial moment of time t_0.

The simplest type of motion is *uniform motion*, that is, motion with a constant velocity V. In this case, the law of motion has the form of a linear function: $S(t) = V(t - t_0) + S(t_0)$ and therefore, to find the velocity V it's enough to divide the distance traveled over a period of time from t_0 to t by the value of this time period:

$$V = \frac{S(t) - S(t_0)}{t - t_0}.$$

The expression on the right-hand side is the ratio of the increment of the function to the increment of the argument, that is, it is an expression whose limit (if it exists) is the derivative of the function $S(t)$. However, in this simplest case, it is not necessary to pass to the limit, since the expression on the right-hand side, as well as on the left-hand side, is a constant V.

Now assume that the law of motion $S(t)$ is not linear. In this case, we cannot talk about a constant velocity of the motion, but we can find the *average velocity* $V_{avr}(t_0, t)$ over a period of time from t_0 to t:

$$V_{avr}(t_0, t) = \frac{S(t) - S(t_0)}{t - t_0}.$$

This formula shows the velocity of uniform motion that allows us to move from the point $S(t_0)$ to the point $S(t)$ over a period of time from t_0 to t.

If we move the value of t closer to t_0 then we will obtain the average velocity over a decreasing time interval (t_0, t), and if there exists a limit as $t \to t_0$, then it is natural to call such a limit $V(t_0)$ the *instantaneous velocity* at the time t_0:

$$V(t_0) = \lim_{t \to t_0} \frac{S(t) - S(t_0)}{t - t_0}.$$

The word “instantaneous” can be omitted and we can simply speak of the velocity $V(t_0)$ at the time t_0. Thus, we can determine the velocity at the time t_0 if the law of motion $S(t)$ is a differentiable function at the point t_0. So, the velocity at a given time is equal to the derivative of the function S:

$$V(t_0) = S'(t_0).$$

Therefore, the physical sense of the derivative is that *the derivative of a certain quantity determines the rate of change of this quantity.*

The rate of change of velocity is called the *acceleration*. To find the acceleration $a(t)$ for a given law of motion $S(t)$, we must differentiate the function $V(t) = S'(t)$, that is, find the *second* derivative of the function $S(t)$ at the given point (the higher-order derivatives will be considered in detail in the next chapter):

$$a(t_0) = V'(t_0) = (S'(t_0))'.$$

Newton's second law allows us to relate the force $F(t)$ acting on the body at a given time t and the acceleration $a(t)$ with which the body moves under the action of this force: $F(t) = ma(t)$, where m is the mass of the body. Thus, if we know the force acting on the body, then we know its acceleration, and if we know the acceleration then, by performing the inverse operation for differentiation, we can find the velocity of motion $V(t)$. Then, applying the inverse operation for differentiation once again to the velocity $V(t)$, we can find the motion law $S(t)$ according to which the body moves, that is, we can completely determine how the body behaves under the action of a given force. Note that the inverse operation for differentiation is called *integration*. Integration, like differentiation, is the subject of study of calculus.

Thus, thanks to differential and integral calculus, it is possible to solve the problem of describing the motion of a body if the forces acting on it are known. For this reason, differential and integral calculus plays such an important role in various branches of physics.

Geometric sense of the derivative

17B/25:08 (08:08)

If the function $f(x)$ is differentiable at the point x_0 then the value of the derivative $f'(x_0)$ is an important characteristic of the graph of the function at this point.

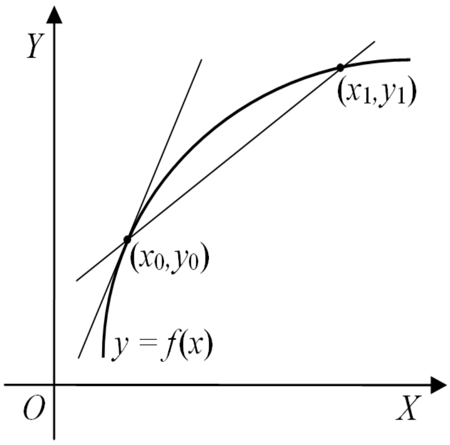

Fig. 9. Secant and tangent to the graph of a function

Let the function $f(x)$ be differentiable at the point x_0, let $y_0 = f(x_0)$. Let us choose some other point $x_1 \neq x_0$, denote $y_1 = f(x_1)$, and draw a *secant line*, i. e., a straight line passing through the points (x_0, y_0) and (x_1, y_1) (see Fig. 9).

The secant equation can be represented as follows:

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).$$

Indeed, this is a linear equation with respect to x, therefore it determines a straight line, and it turns into an identity when the points (x_0, y_0) and (x_1, y_1) are substituted into the equation.

If we will unlimitedly move the point x_1 to the point x_0 remaining all the time on the graph graph (i. e., assuming that $y_1 = f(x_1)$), then the secant will change its position. As a result, we get a line called the *tangent line* to the graph of the function $y = f(x)$ at the point x_0 (Fig. 9).

The tangent equation can be obtained by passing to the limit, as $x_1 \to x_0$, on the right-hand side of the secant equation:

$$y - y_0 = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).$$

This limit exists because, by condition, the function f is differentiable at the point x_0:

$$\lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0).$$

Thus, if the function f is differentiable at the point x_0, then its graph at the point (x_0, y_0), where $y_0 = f(x_0)$, has a tangent line whose equation is as follows:

$$y - y_0 = f'(x_0)(x - x_0).$$

The derivative $f'(x_0)$ is the *slope* of the equation of the tangent line to the graph of the function f at the point x_0. Since the slope is equal to the tangent of the angle of inclination of the straight line, we obtain that, knowing the derivative $f'(x_0)$, we can find the *angle of inclination of the tangent line* at x_0. To do this, we simply should find $\arctan f'(x_0)$. It is natural to call this value the *angle of inclination of the graph* of the function f at the point x_0.

In particular, if $f'(x_0) = 0$, then this means that at this point the angle of inclination of the graph is also 0, that is, the tangent line to it is horizontal. If the derivative is equal to infinity then, given that $\lim_{y\to\pm\infty} \arctan y = \pm\frac{\pi}{2}$, we obtain that in this case the tangent line to the graph is vertical.

Thus, the geometric sense of the derivative is that *the derivative at a given point is equal to the tangent of the angle of inclination of the tangent line to the graph of the given function at this point.*

Note that it is possible to talk about the tangent line to the graph of a function at a given point only if the function is differentiable at this point (that is, it has a finite derivative) or if its derivative has an infinite value at this point (in this case, the tangent line is vertical). If there is no finite or infinite derivative, then the graph has no tangent line. For example, there is no tangent line to the graph of the function $\operatorname{sign} x$ at the point $x = 0$.

20. Higher-order derivatives

Higher-order derivatives: definition and examples

17B/33:16 (14:19)

Definition.

Let a function f be defined and differentiable at any point in the set E (this, in particular, means that any point $x \in E$ is the limit point of this set). Then they say that the function f is *differentiable on the set* E.

In this case, the derivative $f'(x)$ of the function f is defined on the set E. If this function f', in turn, is differentiable on the set E then its derivative is called the *second derivative* of the function f on E and is denoted by $f''(x)$:

$$f''(x) \stackrel{\text{def}}{=} \big(f'(x)\big)'.$$

Derivatives of higher orders can be defined in a similar way. If there exists a derivative of a function f of order $n-1$ (denote it by $f^{(n-1)}(x)$) and if this derivative is differentiable on E, then the *derivative of a function f of order n* is, by definition, the function $\big(f^{(n-1)}(x)\big)'$:

$$f^{(n)}(x) \stackrel{\text{def}}{=} \big(f^{(n-1)}(x)\big)'.$$

In addition to the notation $f^{(n)}(x)$, the notation $d^n f/dx^n$ and $\frac{d^n f}{dx^n}(x)$ is also used. For the third-order derivative, the notation $f'''(x)$ is also used.

It is convenient to assume, by definition, that the function itself is its derivative of order 0:

$$f^{(0)}(x) \stackrel{\text{def}}{=} f(x).$$

Higher-order derivatives of some elementary functions.

1. We start by considering the function $\sin x$ and find its first four derivatives:

$$(\sin x)' = \cos x,$$

$$(\sin x)'' = (\cos x)' = -\sin x,$$

$$(\sin x)''' = (-\sin x)' = -\cos x,$$

$$(\sin x)^{(4)} = (-\cos x)' = \sin x.$$

Thus, having performed 4 differentiation operations, we get the original function. The next differentiation operations will give the same sequence of functions. It is easy to verify that the formula $(\sin x)^{(n)} = \sin(x + \frac{n\pi}{2})$ is valid for $n = 1, 2, 3, 4$, and, therefore, due the periodicity of the sine function, it is valid for all positive integers n. In addition, it remains valid for the case $n = 0$.

Thus, the derivative of order n for the sine function can be found by the formula

$$(\sin x)^{(n)} = \sin\Big(x + \frac{n\pi}{2}\Big), \quad n = 0, 1, 2, \ldots$$

A similar formula holds for derivatives of the cosine function:

$$(\cos x)^{(n)} = \cos\Big(x + \frac{n\pi}{2}\Big), \quad n = 0, 1, 2, \ldots$$

2. The function e^x has the simplest formula for derivatives of order n:

$$(e^x)^{(n)} = e^x, \quad n = 0, 1, 2, \ldots$$

Since for an exponential function with base $a > 0$, $a \neq 1$, the formula $(a^x)' = a^x \ln a$ is valid (that is, as a result of differentiation, the original function is multiplied by a constant factor $\ln a$), the derivative of order n for the function a^n can be found using a simple formula:

$$(a^x)^{(n)} = a^x (\ln a)^n, \quad n = 0, 1, 2, \ldots$$

3. Let us analyze how the power function x^α, $\alpha \neq 0$, changes during its successive differentiation:

$$(x^\alpha)' = \alpha x^{\alpha-1},$$

$$(x^\alpha)'' = \alpha(\alpha - 1)x^{\alpha-2},$$

$$(x^\alpha)''' = \alpha(\alpha - 1)(\alpha - 2)x^{\alpha-3}.$$

We see that the derivative of order n is a power function with the exponent $(\alpha - n)$ multiplied by n coefficients with values from α to $\alpha - n + 1$ (this fact can be rigorously proved using the principle of mathematical induction):

$$(x^\alpha)^{(n)} = \alpha(\alpha - 1) \ldots (\alpha - n + 1)x^{\alpha-n}, \quad n = 1, 2, 3, \ldots \tag{1}$$

The formula remains valid for $n = 0$ if we assume that 0 coefficients are indicated before the power function in this case.

If $\alpha \in \mathbb{N}$, then all derivatives, starting from the order $\alpha + 1$, vanishes, since the set of coefficients will include the coefficient $\alpha - (\alpha + 1) + 1 = 0$. If $\alpha \notin \mathbb{N}$, then none of the derivatives will vanish.

4. For the function $\ln x$, the first derivative is $\frac{1}{x}$, that is, it is a power function with the exponent -1. Therefore, starting from the second derivative, we can use the formula (1). However, in this case, a simpler formula can be obtained. To obtain it, let us analyze several initial derivatives of the function $\ln x$:

$$(\ln x)' = \frac{1}{x},$$

$$(\ln x)'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2},$$

$$(\ln x)''' = \left(-\frac{1}{x^2}\right)' = \frac{2}{x^3},$$

$$(\ln x)^{(4)} = \left(\frac{2}{x^3}\right)' = -\frac{3!}{x^4}.$$

Therefore, the derivative of a function $\ln x$ of order n is a power function with exponent $(-n)$ and coefficient $(-1)^{n-1}(n-1)!$:

$$(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}, \quad n = 1, 2, 3, \ldots$$

This formula, in contrast with all previously obtained formulas for derivatives of elementary functions of order n, is not valid for $n = 0$.

Higher-order derivatives for sum and product of functions

18A/00:00 (03:08)

The formula for the derivative of the sum of functions can be easily generalized to the case of derivatives of any order:

$$\big(f(x) + g(x)\big)' = f'(x) + g'(x),$$

$$\big(f(x) + g(x)\big)'' = (f'(x) + g'(x))' = f''(x) + g''(x),$$

$$\big(f(x) + g(x)\big)''' = (f''(x) + g''(x))' = f'''(x) + g'''(x).$$

Thus, the following formula holds for the derivative of the sum of functions of order n, which can be rigorously proved using the principle of mathematical induction:

$$\big(f(x) + g(x)\big)^{(n)} = f^{(n)}(x) + g^{(n)}(x), \quad n = 0, 1, 2, 3, \ldots$$

A similar formula is valid for the case of a linear combination of functions:

$$\Big(\sum_{i=1}^{m} c_i f_i(x)\Big)^{(n)} = \sum_{i=1}^{m} c_i f_i^{(n)}(x), \quad n = 0, 1, 2, 3, \ldots$$

For derivatives of a product of functions, the simple formula cannot be obtained, since the first derivative of a product already include a combination of two products of functions:

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x). \tag{2}$$

However, a formula for finding the derivative of order n for the product of functions exists. It is called the *Leibniz rule*. Although we will now obtain this formula, it is quite complicated and is not too convenient for practical use. Therefore, if the expression for which we want to find a derivative of higher orders contains a product, we should try to transform it so that there would be the sum of functions instead of the product. For instance, this can always be performed for products of the trigonometric functions sin and cos.

Number of combinations: definition and properties

18A/03:08 (11:56)

The Leibniz rule includes the value C_n^k which is the *number of k-combinations of n*. For the number of combinations, the notation $\binom{n}{k}$ is also used (read as "n choose k"). This value determines the number of different subsets of k elements that can be obtained from the original set of n elements. As usual, all elements of a set are considered different.

In combinatorics, the following formula is proved for C_n^k:

$$C_n^k = \frac{n!}{k!(n-k)!}, \quad n \in \mathbb{N}, \quad k = 0, \dots, n. \tag{3}$$

In the case of $k = 0$ and $k = n$, it should be assumed that $0! = 1$. Thus, $\forall\, n \in \mathbb{N}\ \ C_n^0 = C_n^n = 1$.

We will not discuss various relations valid for the number of combinations, restricting ourselves only to what we need to prove the Leibniz rule.

Lemma (property of the number of combinations).

For any $n \in \mathbb{N}$ and $k = 1, \dots, n$, the following relation is fulfilled:

$$C_n^k + C_n^{k-1} = C_{n+1}^k. \tag{4}$$

Proof, version 1.

Formula (4) can be obtained directly from formula (3):

$$C_n^k + C_n^{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}.$$

Reduce the fractions on the right-hand side of the equality to a common denominator by multiplying the first fraction by $n - k + 1$ and the second fraction by k and transform the numerator of the resulting fraction:

$$\frac{n!(n-k+1) + n!k}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n-k+1)!}.$$

Thus, the numerator of the fraction is $(n+1)!$ and the denominator can be written in the form $k!\big((n+1)-k\big)!$. Applying formula (3) again, we obtain:

$$\frac{(n+1)!}{k!\big((n+1)-k\big)!} = C_{n+1}^k. \ \square$$

Proof, version 2.

Formula (4) can be proved without using formula (3), only on the basis of definition the number of combinations C_n^k as the number of subsets of size k for a set of size n. In particular, the value of C_{n+1}^k is equal to the number of subsets of size k for a set of size $n+1$.

Denote a set of size $n+1$ by X, choose some its arbitrary element x_0, and denote the remaining n elements by $x_1, \dots, x_n$. Thus, $X = \{x_0, x_1, \dots, x_n\}$.

The subsets of k elements for the original set X can be divided into two *disjoint* parts: one (part A) consists of subsets that do not contain a special element x_0, and the other (part B) consists of subsets containing the element x_0.

All k-element subsets from the part A do not contain x_0, therefore their elements must be selected from the remaining part of the original set X containing n elements. By the definition of the number of combinations, the number of subsets included in the part A is C_n^k.

All k-element subsets from the part B contain x_0, so their remaining elements (their number is $(k-1)$) must be selected from the remaining part of the original set X containing n elements. By the definition of the number of combinations, the number of subsets included in the part B is C_n^{k-1}.

Therefore, the total number of subsets of k elements for the $(n+1)$-element set X is C_{n+1}^k, and this number is equal to the total number of subsets included in the part A and in the part B. The A part includes C_n^k subsets and the B part includes C_n^{k-1} sets. This immediately implies formula (4). $\square$

The Leibniz rule for the differentiation of a product

18A/15:04 (22:00)

Theorem (the Leibniz rule for the differentiation of a product).

Let the functions u and v be defined on the set E and have all derivatives up to the order $n \in \mathbb{N}$ on it.

Then the product uv also has all derivatives up to the order n and its derivative of the order n can be found by the formula:

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}. \tag{5}$$

Proof.

Let us prove the theorem by induction.

1. For $n = 1$, formula (5) is valid, since on the left-hand side, by virtue of (2), we have:

$$(uv)' = u'v + uv'.$$

The right-hand side of (5) for $n = 1$ contains two terms:

$$C_1^0 u^{(0)} v^{(1)} + C_1^1 u^{(1)} v^{(0)}.$$

Since $C_1^0 = C_1^1 = 1$, we obtain that the right-hand side is also equal to $u'v + uv'$.

2. Now suppose that formula (5) is valid for some selected positive integer n and prove its validity for the value $n+1$. Thus, assuming that formula (5) is valid, we must prove the following formula:

$$(uv)^{(n+1)} = \sum_{k=0}^{n+1} C_{n+1}^k u^{(k)} v^{(n+1-k)}. \tag{6}$$

By the definition of the derivative of order $n+1$, we have:

$$(uv)^{(n+1)} = \bigl((uv)^{(n)}\bigr)'.$$

Since we assume that formula (5) is valid for selected positive integer n, we obtain:

$$(uv)^{(n+1)} = \bigl((uv)^{(n)}\bigr)' = \Bigl(\sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}\Bigr)'.$$

It remains to transform the right-hand side of the obtained equality in such a way as to get the right-hand side of the equality (6). We use the fact that the derivative of a linear combination of functions is a linear combination of derivatives:

$$\Bigl(\sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}\Bigr)' = \sum_{k=0}^{n} C_n^k \left(u^{(k)} v^{(n-k)}\right)'.$$

When finding the derivative of the product $u^{(k)} v^{(n-k)}$ according to formula (2), it will be convenient for us firstly to write down the term with the differentiated function $v^{(n-k)}$ and then the term with the differentiated function $u^{(k)}$:

$$\left(u^{(k)} v^{(n-k)}\right)' = u^{(k)} v^{(n-k+1)} + u^{(k+1)} v^{(n-k)}.$$

Thus, we obtain two sums:

$$\sum_{k=0}^{n} C_n^k \left(u^{(k)} v^{(n-k)}\right)' = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k+1)} + \sum_{k=0}^{n} C_n^k u^{(k+1)} v^{(n-k)}.$$

If we analyze the sums received, we can see that almost all combinations of derivatives of one of the sums are included in another sum. An exception is the term for $k = 0$ ($C_n^0 u^{(0)} v^{(n+1)} = u^{(0)} v^{(n+1)}$) from the first sum and the term for $k = n$ ($C_n^0 u^{(n+1)} v^{(0)} = u^{(n+1)} v^{(0)}$) from the second sum; each of these terms is absent in the other sum. Extracting these terms from the sums, we obtain the following representation:

$$\sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k+1)} + \sum_{k=0}^{n} C_n^k u^{(k+1)} v^{(n-k)} =$$

$$= u^{(0)} v^{(n+1)} + \sum_{k=1}^{n} C_n^k u^{(k)} v^{(n-k+1)} + \sum_{k=0}^{n-1} C_n^k u^{(k+1)} v^{(n-k)} + u^{(n+1)} v^{(0)}.$$

In order to combine the remaining sums into one, we need to bring their summation parameters to the same range. For that purpose, we perform the following change of the summation parameter in the second sum: $m = k + 1$. Then the range of values from 0 to $n - 1$ for the summation parameter k will turn into the range of values from 1 to n for the summation parameter m, and the index in the number of combinations and orders of derivatives will change accordingly:

$$\sum_{k=0}^{n-1} C_n^k u^{(k+1)} v^{(n-k)} = \sum_{m=1}^{n} C_n^{m-1} u^{(m)} v^{(n-m+1)}.$$

After replacing the new letter of the summation parameter m with the previous letter k, it is easy to notice that the first and second sums will contain terms with the same orders of derivatives, and these terms will differ only by their coefficients. Therefore, we can combine these sums into one:

$$u^{(0)} v^{(n+1)} + \sum_{k=1}^{n} C_n^k u^{(k)} v^{(n-k+1)} + \sum_{k=1}^{n} C_n^{k-1} u^{(k)} v^{(n-k+1)} + u^{(n+1)} v^{(0)} =$$

$$= u^{(0)} v^{(n+1)} + \sum_{k=1}^{n} \left(C_n^k + C_n^{k-1}\right) u^{(k)} v^{(n-k+1)} + u^{(n+1)} v^{(0)}.$$

By the previously proved lemma (see (4)), $C_n^k + C_n^{k-1} = C_{n+1}^k$. Thus, we obtain the following formula for the derivative $(uv)^{(n+1)}$:

$$(uv)^{(n+1)} = u^{(0)}v^{(n+1)} + \sum_{k=1}^{n} C_{n+1}^{k} u^{(k)} v^{(n-k+1)} + u^{(n+1)}v^{(0)}.$$

The right-hand side of the obtained equality coincides with the right-hand side of the equality (6), since the term $u^{(0)}v^{(n+1)}$ is equal to the summand corresponding to $k = 0$ and the term $u^{(n+1)}v^{(0)}$ is equal to the term of the same sum corresponding to $k = n + 1$. Stage 2 of the induction is proved.

Therefore, by virtue of the principle of mathematical induction, formula (5) is valid for all positive integers n. $\square$

21. The basic theorems of differential calculus

Local extrema of functions. Fermat's theorem

Local extrema of functions

18A/37:04 (12:31)

Definition.

Let f act from E to $\mathbb{R}$, let the point $x_0 \in E$ be the limit point of the set E. It is said that the point x_0 is the *local minimum point* of the function f if there exists a neighborhood U_{x_0} of the point x_0 such that for any $x \in U_{x_0} \cap E$ the inequality $f(x) \geq f(x_0)$ holds:

$$\exists\, U_{x_0} \quad \forall\, x \in U_{x_0} \cap E \quad f(x) \geq f(x_0).$$

The value of the function f at the point x_0 is called the *local minimum*.

The local maximum point is defined similarly. The point x_0 is called the *local maximum point* of the function f if

$$\exists\, U_{x_0} \quad \forall\, x \in U_{x_0} \cap E \quad f(x) \leq f(x_0).$$

The value of the function f at the local maximum point is called the *local maximum*.

We also introduce the concepts of strict local maximum and minimum.

Definition.

Let f act from E to $\mathbb{R}$, let the point $x_0 \in E$ be the limit point of the set E. The point x_0 is called the *point of a strict local minimum* of the function f if there exists a punctured neighborhood $\overset{\circ}{U}_{x_0}$ of the point x_0 such that for any $x \in \overset{\circ}{U}_{x_0} \cap E$ the inequality $f(x) > f(x_0)$ holds:

$$\exists\, \overset{\circ}{U}_{x_0} \quad \forall\, x \in \overset{\circ}{U}_{x_0} \cap E \quad f(x) > f(x_0).$$

The point of a strict local maximum is defined similarly. The point x_0 is called the *point of a strict local maximum* of the function f if

$$\exists\, U_{x_0} \quad \forall\, x \in \overset{\circ}{U}_{x_0} \cap E \quad f(x) < f(x_0).$$

It should be noted that when we give definitions of the points of strict local minimum and maximum, punctured neighborhoods of these points are considered, since strict inequalities of the form $f(x) > f(x_0)$ or $f(x) < f(x_0)$ cannot be fulfilled for $x = x_0$.

The points of local minima and maxima are called the *points of local extrema*, and the values at these points are called the *local extrema* of this function. As for minima and maxima, strict and non-strict local extrema can be defined.

DEFINITION.

The point x_0 is called the *interior point* of the set E if the set E contains some neighborhood of this point:

$$\exists V_{x_0} \quad V_{x_0} \subset E.$$

DEFINITION.

Let f act from E to $\mathbb{R}$, $x_0 \in E$. It is said that the point x_0 is a *point of an interior local minimum, maximum* or *extremum* if it is a point of a local minimum, maximum or extremum and at the same time it is an interior point of the set E.

Fermat's theorem 18A/49:35 (02:40), 18B/00:00 (13:02)

THEOREM (FERMAT'S THEOREM ON INTERIOR LOCAL EXTREMA).

Let the function f be defined in some neighborhood V_{x_0} of the point x_0 and x_0 be the local extremum point of the function f. Let the function f be differentiable at the point x_0. Then $f'(x_0) = 0$.

REMARK.

Under the conditions of the theorem, it is not assumed that the point x_0 is a point of strict local extremum, but it is required that it is a point of *interior* local extremum.

PROOF[9].

Denote $f'(x_0) = A$. By definition of the derivative, we have:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = A.$$

Since the specified limit exists, we obtain, by virtue of the criterion for the existence of the limit in terms of one-sided limits, that the left-hand and right-hand limits also exist and are equal to A:

$$\lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0 + 0} \frac{f(x) - f(x_0)}{x - x_0} = A.$$

Let the point x_0 be a local minimum point. This means that there exists a neighborhood $U_{x_0} \subset V_{x_0}$ for which the following condition holds:

$$\forall x \in U_{x_0} \quad f(x) \geq f(x_0).$$

[9] In video lectures, a more complicated method of proof is given.

Then for the left-hand neighborhood $U^-_{x_0}$, we have $f(x) - f(x_0) \geq 0$, $x - x_0 < 0$, whence $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$. Passing to the limit, as $x \to x_0 - 0$, and using the first theorem on passing to the limit in inequalities for functions (its version for one-sided limits), we obtain $\lim_{x\to x_0-0} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$, i. e., $A \leq 0$.

For the right-hand neighborhood $U^+_{x_0}$, we have $f(x)-f(x_0) \geq 0$, $x-x_0 > 0$, whence $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$. Passing to the limit, as $x \to x_0 + 0$, and using the same theorem, we obtain $\lim_{x\to x_0+0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$, i. e., $A \geq 0$.

It follows from the inequalities $A \leq 0$ and $A \geq 0$ that $A = 0$, i. e., $f'(x_0) = 0$. We have proved the statement of the theorem for the case of a local minimum.

If x_0 is the local maximum point of the function f, then this point is the local minimum point of the function $-f$. Using the result already proved for a local minimum, we obtain $(-f)'(x_0) = 0$, whence $f'(x_0) = 0$. □

Rolle's theorem, Lagrange's theorem, and Cauchy's mean value theorem

Rolle's theorem

18B/13:02 (11:12)

Theorem (Rolle's theorem).

Let the function f be defined and continuous on the segment $[a, b]$ and differentiable on the interval (a, b). Let the function take the same values at the endpoints of the segment: $f(a) = f(b)$. Then there exists a point $\xi \in (a, b)$ for which $f'(\xi) = 0$.

Proof.

Since the function f is continuous on the segment, it attains its maximum M and minimum m values on this segment, by virtue of the second Weierstrass theorem:

$$\exists\, x_1 \in [a, b] \quad f(x_1) = \max_{x\in[a,b]} f(x) = M,$$

$$\exists\, x_2 \in [a, b] \quad f(x_2) = \min_{x\in[a,b]} f(x) = m.$$

Two cases are possible.

Case 1: $M = m$. This means that the function f is constant: $f(x) = M$ for all $x \in [a, b]$. Since the derivative of the constant function is 0, we can choose any point in the interval (a, b) as ξ: $f'(\xi) = 0$ for all $\xi \in (a, b)$.

Case 2: $M > m$. Then, due to the condition $f(a) = f(b)$, at least one of the points x_1 or x_2 belongs to the interval (a, b). Indeed, if points x_1 and x_2 coincide with the endpoints of the segment $[a, b]$, then $M = f(x_1) = f(x_2) = m$, which contradicts our assumption $M > m$.

Let, for definiteness, such a point be x_1: $x_1 \in (a, b)$. Then the point x_1 is the point of the interior global maximum, therefore, it is also the point of the interior local maximum. In addition, by the condition of the theorem, the function f is differentiable at the point x_1. Thus, all the conditions of Fermat's theorem are satisfied for the point x_1. Therefore, $f'(x_1) = 0$. □

Lagrange's theorem

18B/24:14 (09:45)

Theorem (Lagrange's theorem).

Let the function f be defined and continuous on the segment $[a, b]$ and differentiable on the interval (a, b). Then there exists a point $\xi \in (a, b)$ for which the following relation holds:

$$f(b) - f(a) = f'(\xi)(b - a). \tag{1}$$

Remark 1.

Lagrange's theorem is a generalization of Rolle's theorem, since in the case $f(a) = f(b)$ the left-hand side of equality (1) turns to 0, which immediately implies that $f'(\xi) = 0$.

Remark 2 (geometric sense of Lagrange's theorem).

Equality (1) can be rewritten as follows:

$$\frac{f(b) - f(a)}{b - a} = f'(\xi). \tag{2}$$

In studying the geometric sense of the derivative, we obtained the equations of the secant line and the tangent line to the graph of the function, and at the same time established that the slope of the secant line passing through the points $\bigl(x_1, f(x_1)\bigr)$ and $\bigl(x_2, f(x_2)\bigr)$ is the number $\frac{f(x_2)-f(x_1)}{x_2-x_1}$, and the slope of the tangent line at the point $\bigl(x_0, f(x_0)\bigr)$ is $f'(x_0)$.

Therefore, equality (2) means that the slope of the secant line passing through the points $\bigl(a, f(a)\bigr)$ and $\bigl(b, f(b)\bigr)$ is equal to the slope of the tangent line at some point $\bigl(\xi, f(\xi)\bigr)$, where $\xi \in (a, b)$.

Taking into account that the slope of the straight line is equal to the tangent of its angle of inclination, we obtain that there exists a tangent line to the graph of the function on the interval (a, b), whose angle of inclination coincides with the angle of inclination of the secant line passing through the endpoints of the graph of the function on the segment $[a, b]$ (Fig. 10).

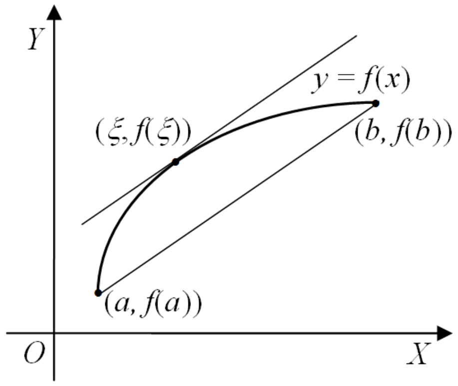

Fig. 10. Geometric sense of Lagrange's theorem

Proof.

Introduce the auxiliary function

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function F is a linear combination of the function f and the linear function; therefore, it is continuous on the segment $[a, b]$ and differentiable on the interval (a, b). In addition, the function F takes the same values at the endpoints of the segment:

$$F(a) = f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = f(a) - 0 = f(a),$$

$$F(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - \big(f(b) - f(a)\big) = f(a).$$

Thus, for the function F, all the conditions of Rolle's theorem are satisfied. By virtue of this theorem, we obtain that there exists a point $\xi \in (a, b)$ for which

$$F'(\xi) = 0. \tag{3}$$

Let us find the derivative of the function F:

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}. \tag{4}$$

From relations (3) and (4), we obtain:

$$f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0.$$

If we move the second term of the last equality to the right-hand side and multiply both sides by $(b - a)$, we get (1). □

Corollaries of Lagrange's theorem

19A/00:00 (18:01)

Corollary 1.

If the function f is defined and continuous on the segment $[a, b]$, differentiable on the interval (a, b), and $\forall\, x \in (a, b)\ f'(x) = 0$, then the function f is a constant on the segment $[a, b]$:

$$\forall\, x \in [a, b] \quad f(x) = c.$$

Remark.

The converse statement (that the derivative of the constant function vanishes) follows directly from the definition of the derivative; this statement was proved in the last section of Chapter 16.

Proof.

Let $x_1, x_2 \in [a, b]$ be arbitrarily chosen different points. For definiteness, suppose that $x_1 < x_2$.

Then the function f has the same properties on the segment $[x_1, x_2]$ as on the segment $[a, b]$, that is, it is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2) and $\forall\, x \in (x_1, x_2)\ f'(x) = 0$. Apply Lagrange's theorem for the segment $[x_1, x_2]$. By virtue of this theorem, there exists a point $\xi \in (x_1, x_2)$ for which the following relation holds:

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1). \tag{5}$$

But by condition, $f'(\xi) = 0$, therefore equality (5) implies the equality $f(x_2) - f(x_1) = 0$, or $f(x_2) = f(x_1)$.

Since x_1 and x_2 were arbitrarily selected points from the segment $[a, b]$, we obtain that the function f is a constant function on this segment. □

Corollary 2.

If the function f is defined and continuous on the segment $[a, b]$, differentiable on the interval (a, b), and $\forall\, x \in (a, b)\ f'(x) \ge 0$, then the function f is non-decreasing on the segment $[a, b]$.

Proof.

As in the proof of corollary 1, we arbitrarily choose points $x_1, x_2 \in [a, b]$, $x_1 < x_2$, and apply Lagrange's theorem for the segment $[x_1, x_2]$:

$$\exists\, \xi \in (x_1, x_2) \quad f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1). \tag{6}$$

By condition, $f'(\xi) \ge 0$, moreover, the estimate $x_2 - x_1 > 0$ holds for the difference $x_2 - x_1$, therefore the right-hand side of equality (6) is non-negative. Therefore, the left-hand side of equality (6) is also non-negative: $f(x_2) - f(x_1) \ge 0$, or $f(x_1) \le f(x_2)$. Since the points x_1 and x_2 satisfying the inequality $x_1 < x_2$ were chosen arbitrarily from the segment $[a, b]$, we obtain that the function f is non-decreasing on this segment. □

REMARK.

The following three statements can be proved in a similar way. It is assumed in each of these statements that the function f is defined and continuous on the segment $[a, b]$ and differentiable on the interval (a, b).

1. If $\forall\, x \in (a, b)\ f'(x) \le 0$, then the function f is non-increasing on the segment $[a, b]$.

2. If $\forall\, x \in (a, b)\ f'(x) > 0$, then the function f is increasing on the segment $[a, b]$.

3. If $\forall\, x \in (a, b)\ f'(x) < 0$, then the function f is decreasing on the segment $[a, b]$.

COROLLARY 3.

Let the function f be defined and continuous on the segment $[a, b]$, differentiable on the interval (a, b), its derivative does not vanish on the interval (a, b) and is continuous on this interval. Then the function f is strictly monotonous on the segment $[a, b]$ and has the inverse function f^{-1} acting from the segment $[c, d] = f([a, b])$ into the segment $[a, b]$. The inverse function is continuous on the segment $[c, d]$, differentiable on the interval (c, d), and has the same monotonicity type as the function f.

PROOF.

Since the function $f'(x)$ is continuous on (a, b) and for all $x \in (a, b)$ $f'(x) \ne 0$, we obtain, by virtue of the intermediate value theorem, that the function $f'(x)$ preserves the sign on the interval (a, b), that is, one of the following two situations is possible: either $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$. Indeed, if it turned out that at some points x_1 and x_2 from (a, b) the function f' takes values of different signs, then, by the intermediate value theorem, there would be a point $x_0 \in (x_1, x_2)$ such that $f'(x_0) = 0$, which contradicts the condition.

If any of two possible situations is fulfilled, we obtain, by virtue of the remark to corollary 2, that the function f is strictly monotonous on the segment $[a, b]$.

Therefore, if we assume that the function f acts from the segment $[a, b]$ into the image $f([a, b])$ of this segment, then it is one-to-one and, therefore, has the inverse function f^{-1}, and the function f^{-1} has the same monotonicity type as the function f (by the first part of the inverse function theorem).

Since, by condition, the function f is continuous on the segment $[a, b]$, we conclude, by virtue of the second part of the inverse function theorem, that the image $f([a,b])$ is the segment $[c,d]$ and the function f^{-1} is continuous on the segment $[c,d]$.

Since, by condition, the function f is differentiable on the interval (a,b) and $f'(x) \neq 0$ for any point $x \in (a,b)$, we conclude, by virtue of the theorem on the differentiation of an inverse function, that the function f^{-1} is differentiable on the interval (c,d). □

Cauchy's mean value theorem

19A/18:01 (13:01)

Theorem (Cauchy's mean value theorem).

Let two functions $x(t)$ and $y(t)$ be defined and continuous on the segment $[\alpha,\beta]$, differentiable on the interval (α,β), and, in addition, $x'(t) \neq 0$ for $x \in (\alpha,\beta)$. Then there exists a point $\tau \in (\alpha,\beta)$ for which the following relation holds:

$$\frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)} = \frac{y'(\tau)}{x'(\tau)}. \tag{7}$$

Remark.

This theorem is a generalization of Lagrange's theorem, since a formula similar to formula (1) for Lagrange's theorem can be obtained from formula (7) if we put $y(t) = f(t)$, $x(t) = t$ in formula (7).

Proof.

First of all, note that the denominator $x(\beta) - x(\alpha)$ in formula (7) does not turn into 0. This follows from Lagrange's theorem for the function x and the condition $x'(t) \neq 0$, which holds for all $t \in (\alpha,\beta)$, thus, for some value $\xi \in (\alpha,\beta)$, we obtain:

$$x(\beta) - x(\alpha) = x'(\xi)(\beta-\alpha) \neq 0.$$

Now, as in the proof of Lagrange's theorem, introduce the following auxiliary function:

$$F(t) = y(t) - \frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)}\big(x(t)-x(\alpha)\big).$$

This function is continuous on the segment $[\alpha,\beta]$ and differentiable on the interval (α,β). In addition, the function F takes the same values at the endpoints of the segment:

$$F(\alpha) = y(\alpha) - \frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)}\big(x(\alpha)-x(\alpha)\big) = y(\alpha) - 0 = y(\alpha),$$

$$F(\beta) = y(\beta) - \frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)}\big(x(\beta)-x(\alpha)\big) =$$

$$= y(\beta) - \big(y(\beta) - y(\alpha)\big) = y(\alpha).$$

Thus, all the conditions of Rolle's theorem are satisfied for the function F. By virtue of this theorem, we obtain that there exists a point $\tau \in (\alpha, \beta)$ for which

$$F'(\tau) = 0. \tag{8}$$

Let us find the derivative of the function F:

$$F'(t) = y'(t) - \frac{y(\beta) - y(\alpha)}{x(\beta) - x(\alpha)} x'(t). \tag{9}$$

From relations (8) and (9), we obtain:

$$y'(\tau) - \frac{y(\beta) - y(\alpha)}{x(\beta) - x(\alpha)} x'(\tau) = 0.$$

If we move the second term in the last equality to the right-hand side and divide both sides by $x'(\tau)$, we get (7). $\square$

22. Taylor's formula

Taylor's formula for polynomials and for arbitrary differentiable functions

Taylor's formula for polynomials

19A/31:02 (09:58)

Consider the polynomial $P_n(x)$ of degree $n \in \mathbb{N}$:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Let us select some point $x_0 \in \mathbb{R}$. It is known from the course of algebra that any polynomial can be expanded in powers of $(x - x_0)$; the degree of the polynomial will not change:

$$P_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n. \tag{1}$$

We want to obtain formulas for the coefficients c_k, $k = 0, \ldots, n$, using the differentiation operation.

To find the coefficient c_0, it is enough to calculate the value of the polynomial at the point x_0:

$$c_0 = P_n(x_0).$$

To find the coefficient c_1, we firstly differentiate the polynomial:

$$P_n'(x) = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \cdots + nc_n(x - x_0)^{n-1}.$$

The coefficient c_1 is a free term of the derivative $P_n'(x)$ and, therefore, to find it, it suffices to calculate the value of the derivative at the point x_0:

$$c_1 = P_n'(x_0).$$

Let us find the second derivative of the polynomial:

$$P_n''(x) = 2c_2 + 2 \cdot 3c_3(x - x_0) + \cdots + (n-1)nc_n(x - x_0)^{n-2}.$$

Substituting the value $x = x_0$ into this derivative, we obtain the formula for the coefficient c_2:

$$c_2 = \frac{P_n''(x_0)}{2}.$$

In this case, the formula contains not only the value of the derivative at the point x_0, but also the factor $\frac{1}{2}$.

A factor of $\frac{1}{6}$ will appear in the formula for the coefficient c_3. This factor is more convenient to represent as $\frac{1}{3!}$ using the factorial function $n! = 1{\cdot}2{\cdot}3 \cdots n$:

$$c_3 = \frac{P_n'''(x_0)}{3!}.$$

Continuing the process of differentiation, we finally obtain a derivative of order n, which contains a single term:

$$P_n^{(n)}(x) = 2 \cdot 3 \cdots (n-1)nc_n.$$

Thus, for the coefficient c_n, we obtain the formula

$$c_n = \frac{P_n^{(n)}(x_0)}{n!}.$$

Taking into account that $0! = 1$, all the formulas obtained for the coefficients c_k can be written in the following general form:

$$c_k = \frac{P_n^{(k)}(x_0)}{k!}, \quad k = 0, 1, \dots, n.$$

Substituting the representations for the coefficients c_k into formula (1), we obtain *Taylor's formula for the polynomial* $P_n(x)$:

$$P_n(x) = \sum_{k=0}^{n} \frac{P_n^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{2}$$

This formula allows us to obtain the expansion of the polynomial $P_n(x)$ in powers of $(x - x_0)$ using the values of the derivatives of the polynomial of order 0 up to n at the point x_0. Note that all derivatives of the polynomial $P_n(x)$ of higher orders $(n+1,\ n+2,\ \dots)$ vanish.

The version of Taylor's formula (2) when $x_0 = 0$ is also called *Maclaurin's formula.*

Deriving the binomial formula using Taylor's formula for polynomials

19A/41:00 (06:16)

Let $n \in \mathbb{N}$, $b \in \mathbb{R}$. Consider a polynomial of the form $P_n(x) = (x+b)^n$ and expand it in powers of x. To do this, we use Taylor's formula (2) with $x_0 = 0$.

Let us calculate the derivatives of the polynomial $P_n(x)$ of order 0, 1, 2 at the point 0:

$$P_n^{(0)}(x) = (x+b)^n, \quad P_n^{(0)}(0) = b^n;$$

$$P_n'(x) = n(x+b)^{n-1}, \quad P_n'(0) = nb^{n-1};$$

$$P_n''(x) = n(n-1)(x+b)^{n-2}, \quad P_n''(0) = n(n-1)b^{n-2}.$$

It is easy to see that the formula for the derivative of the polynomial $P_n(x)$ of order k at the point 0 can be written in the general form:

$$P_n^{(k)}(0) = n(n-1)\cdots(n-k+1)b^{n-k}, \quad k = 0, \ldots, n.$$

Let us transform the resulting formula by multiplying and dividing it by the product $(n-k)\cdots 3\cdot 2 = (n-k)!$:

$$P_n^{(k)}(0) = \frac{n(n-1)\cdots(n-k+1)(n-k)\cdots 3\cdot 2}{(n-k)\cdots 3\cdot 2} b^{n-k} =$$
$$= \frac{n!}{(n-k)!} b^{n-k}.$$

Substitute the found values of the derivatives into formula (2) for $x_0 = 0$:

$$(x+b)^n = \sum_{k=0}^{n} \frac{P_n^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} b^{n-k} x^k.$$

The expression found can be simplified by using the formula for the number of combinations $C_n^k = \frac{n!}{(n-k)!k!}$:

$$(x+b)^n = \sum_{k=0}^{n} C_n^k b^{n-k} x^k.$$

Replacing x by the value $a \in \mathbb{R}$, we obtain the *binomial formula*:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}.$$

Taylor's formula for arbitrary differentiable functions

19B/00:00 (07:55)

Taylor's formula found earlier for polynomials (2) is an exact equality: both left-hand and right-hand sides in this equality contain polynomials of degree n. This equality holds for all $x \in \mathbb{R}$.

Suppose that, instead of the polynomial $P_n(x)$, we consider an arbitrary function $f(x)$ defined on some interval containing the point x_0. Also suppose that the function f is differentiable at the point x_0 up to the order n.

Now we cannot write relation (2) in the form of equality, but we can introduce into consideration the quantity $r_n(x_0, x)$, by which the function $f(x)$ differs from the sum given on the right-hand side of (2):

$$r_n(x_0, x) \stackrel{\text{def}}{=} f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

The value $r_n(x_0, x)$ is called the *remainder term of Taylor's formula* for the function f.

Using the remainder term, we can write the *Taylor's formula for an arbitrary differentiable function* f as follows:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + r_n(x_0, x).$$

If for some $n \in \mathbb{N}$, $x_0 \in \mathbb{R}$, $x \in \mathbb{R}$, the value $r_n(x_0, x)$ is small, then this means that the function f can be approximated in the point x by a polynomial of degree n according to Taylor's formula, that is, we can obtain an approximation for the function f in the form of a simpler function (a polynomial).

We have reason to expect that the remainder term $r_n(x_0, x)$ will be small, at least in a situation where the point x is close to the point x_0. Indeed, taking $n = 1$, we obtain

$$r_n(x_0, x) = f(x) - \sum_{k=0}^{1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k =$$
$$= f(x) - \big(f(x_0) + f'(x_0)(x - x_0)\big).$$

Since $n = 1$ and, therefore, the function f is differentiable at the point x_0, we obtain, by the definition of differentiability, that the right-hand side of the last equality is $o(x - x_0)$ as $x \to x_0$. This means that the remainder term in this case approaches 0, as $x \to x_0$, faster than the linear function $(x - x_0)$, that is, it is small enough for x close to x_0.

We can also expect that while increasing n, that is, in a situation where the function is differentiable more times, the rate of approach to 0 of the remainder term, as $x \to x_0$, will be higher. However, in order to prove this, we need to study the properties of the remainder term.

Various representations of the remainder term in Taylor's formula

General formula for the remainder term in Taylor's formula

19B/07:55 (17:45)

Now we derive a remainder term formula, which will include some arbitrary function φ. Choosing various specific functions as this arbitrary function, we can obtain different representations of the remainder term.

In deriving the general formula, we will assume that for the function f, in addition to differentiability n times at the point x_0, the following conditions are satisfied. We select the point x, assuming for definiteness that $x > x_0$, and require that the function f is differentiable $(n+1)$ times on the segment $[x_0, x]$. Note that this implies the continuity of the first n derivatives of the function f on this segment. We do not require continuity of the derivative $f^{(n+1)}$.

Introduce the following auxiliary function F:

$$F(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k =$$
$$= f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2}(x-t)^2 -$$
$$- \frac{f'''(t)}{3!}(x-t)^3 - \dots - \frac{f^{(n)}(t)}{n!}(x-t)^n.$$

It is easy to see that the function F coincides with the remainder term $r_n(x_0, x)$ at x_0, and it is equal to zero at x:

$$F(x_0) = r_n(x_0, x), \qquad F(x) = 0. \tag{3}$$

Let us calculate the derivative of the function F. To do this, we firstly find the derivatives for the individual summands (recall that differentiation is performed with respect to the variable t):

$$\big(f(x)\big)' = 0,$$
$$\big(-f(t)\big)' = -f'(t),$$
$$\big(-f'(t)(x-t)\big)' = f'(t) - f''(t)(x-t),$$
$$\Big(-\frac{f''(t)}{2}(x-t)^2\Big)' = f''(t)(x-t) - \frac{f'''(t)}{2}(x-t)^2,$$
$$\Big(-\frac{f'''(t)}{3!}(x-t)^3\Big)' = \frac{f'''(t)}{2}(x-t)^2 - \frac{f^{(3)}(t)}{3!}(x-t)^3,$$
$$\dots$$
$$\Big(-\frac{f^{(n)}(t)}{n!}(x-t)^n\Big)' = \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

We see that as a result of differentiation of each summand (starting from the third one), a term appears that is opposite to one of the terms of the previous summand. Thus, after summing and collecting terms, we get the following formula:

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n. \tag{4}$$

Now we apply the Cauchy's mean value theorem to the function $F(t)$ and an arbitrary function $\varphi(t)$ on the segment $[x_0, x]$. The function F satisfies all the conditions of this theorem. For the function φ, it is necessary to require that it be continuous on the segment $[x_0, x]$, differentiable on the interval (x_0, x), and that $\varphi'(t) \neq 0$ for $t \in (x_0, x)$.

By virtue of the Cauchy's mean value theorem, there exists a point $\xi \in (x_0, x)$ such that the following relation holds for the functions F and φ:

$$\frac{F(x) - F(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{F'(\xi)}{\varphi'(\xi)}.$$

Substitute the values of $F(x)$, $F(x_0)$ into the resulting relation (see (3)) and use formula (4) for $F'(t)$:

$$\frac{0 - r_n(x_0, x)}{\varphi(x) - \varphi(x_0)} = \frac{-f^{(n+1)}(\xi)(x - \xi)^n}{n!\varphi'(\xi)}.$$

As a result, we obtain the following *formula for the remainder term containing an arbitrary function* $\varphi(t)$:

$$r_n(x_0, x) = \frac{f^{(n+1)}(\xi)(x - \xi)^n\big(\varphi(x) - \varphi(x_0)\big)}{n!\varphi'(\xi)}. \tag{5}$$

Representation of the remainder term in the form of Cauchy and in the form of Lagrange

19B/25:40 (12:51)

Based on formula (5), we can obtain various representations of the remainder term by choosing specific functions as the function φ.

First, put $\varphi(t) = \varphi_1(t) = x - t$. This function satisfies all the necessary conditions: it is continuous and differentiable on the segment $[x_0, x]$, and, moreover, its derivative $\varphi_1'(t)$ is equal to -1, that is, it does not vanish on interval (x_0, x).

Substituting in (5) the values $\varphi_1(x) = 0$, $\varphi_1(x_0) = x - x_0$, and $\varphi_1'(\xi) = -1$, we obtain:

$$r_n(x_0, x) = \frac{f^{(n+1)}(\xi)(x - \xi)^n(x - x_0)}{n!}. \tag{6}$$

Let us additionally transform the resulting formula by representing the value ξ in the form $\xi = x_0 + \theta(x - x_0)$. Since $\xi \in (x_0, x)$, we obtain that $\theta \in (0, 1)$. Note that this expression equals x_0 when $\theta = 0$, and it equals x when $\theta = 1$.

Replacing the value ξ in formula (6) with the expression $x_0 + \theta(x - x_0)$, we obtain:

$$r_n(x_0, x) =$$

$$= \frac{f^{(n+1)}\big(x_0 + \theta(x - x_0)\big)\Big(x - \big(x_0 + \theta(x - x_0)\big)\Big)^n (x - x_0)}{n!} =$$

$$= \frac{f^{(n+1)}\big(x_0 + \theta(x - x_0)\big)(1 - \theta)^n}{n!}(x - x_0)^{n+1}.$$

The resulting representation of the remainder term is called the *Cauchy form of the remainder term.*

Now we choose the following power function $\varphi_2(t) = (x - t)^{n+1}$ as the function $\varphi(t)$. This function also satisfies all the necessary conditions: it is continuous and differentiable on the segment $[x_0, x]$, and, moreover, its derivative $\varphi_2'(t)$ is equal to $-(n + 1)(x - t)^n$ and, therefore, does not vanish on the interval (x_0, x).

Substituting in (5) the values $\varphi_2(x) = 0$, $\varphi_2(x_0) = (x - x_0)^{n+1}$, and $\varphi_2'(\xi) = -(n + 1)(x - \xi)^n$, we obtain:

$$r_n(x_0, x) = \frac{f^{(n+1)}(\xi)(x - \xi)^n\big(0 - (x - x_0)^{n+1}\big)}{n!\big(-(n + 1)(x - \xi)^n\big)} =$$

$$= \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1}.$$

The resulting representation of the remainder term is called the *Lagrange form of the remainder term.* This representation is interesting in that it is similar to the term in Taylor's formula corresponding to $k = n + 1$, except that the derivative is found not at the point x_0, but at some point ξ from the interval (x_0, x).

Let us write Taylor's formula with the remainder term in the Lagrange form:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1}. \qquad (7)$$

Representation of the remainder term in the Peano form

20A/00:00 (20:31)

We noted earlier that for $n = 1$ the remainder term of Taylor's formula has the form $o(x - x_0)$, $x \to x_0$. It turns out that similar representations for the remainder term in the form of little-o can also be obtained for other values of n.

Theorem (on Taylor's formula with the remainder term in the Peano form).

Let the function f be n times continuously differentiable on the segment $[x_0, x]$. Then the following expansion of the function f by Taylor's formula takes place:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o\big((x - x_0)^n\big), \quad x \to x_0. \tag{8}$$

The representation of the remainder term $r_n(x_0, x) = o\big((x - x_0)^n\big)$, $x \to x_0$, used in this formula is called the *remainder term in the Peano form*. Thus, when expanding the function f by Taylor's formula up to the derivative of order n, the remainder term decreases, as $x \to x_0$, faster than the function $(x - x_0)^n$.

Remark.

The assertion of the theorem remains valid in the case when the derivative of order n is not continuous. However, the continuity condition for this derivative allows us to simplify the proof.

Proof.

The conditions of the theorem allow us to expand the function f by Taylor's formula, taking the $n-1$ term in it and representing the remainder term in the Lagrange form (see (7)):

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n. \tag{9}$$

The value of ξ, which is an argument of the function $f^{(n)}$ in the remainder term, depends on x_0 and x. The point x_0 does not change, but we can change the point x, moving it closer to x_0. When the point x changes, the point ξ will change in some way, so we can consider ξ as a function of x: $\xi = \xi(x)$. We do not know how exactly $\xi(x)$ will change when x changes, however the following double estimate will always be valid: $x_0 < \xi(x) < x$. From this double estimate, by virtue of the second theorem on passing to the limit in inequalities for functions, it follows that

$$\lim_{x \to x_0} \xi(x) = x_0. \tag{10}$$

Thus, although the properties of the function $\xi(x)$ are unknown to us, it can be stated that its limit, as $x \to x_0$, exists and is equal to x_0.

Now turn to the function $f^{(n)}(\xi)$. It can be considered as a superposition of the form $f^{(n)}\big(\xi(x)\big) = (f^{(n)} \circ \xi)(x)$, where the external function is $f^{(n)}(t)$

and the internal function is $\xi(x)$. Since, by the condition of the theorem, the function $f^{(n)}(t)$ is continuous in a neighborhood of x_0, we can calculate the limit of superposition $f^{(n)}\big(\xi(x)\big)$, as $x \to x_0$, using the theorem on the limit of superposition in the case when the external function is continuous. By virtue of this theorem, we can move the limit sign under the sign of the external function, and then use the limit relation (10):

$$\lim_{x \to x_0} f^{(n)}\big(\xi(x)\big) = f^{(n)}\left(\lim_{x \to x_0} \xi(x)\right) = f^{(n)}(x_0). \tag{11}$$

Denote $\alpha(x) = f^{(n)}\big(\xi(x)\big) - f^{(n)}(x_0)$. Then it follows from the limit relation (11) that

$$\lim_{x \to x_0} \alpha(x) = \lim_{x \to x_0} \Big(f^{(n)}\big(\xi(x)\big) - f^{(n)}(x_0)\Big) = f^{(n)}(x_0) - f^{(n)}(x_0) = 0.$$

So, we have proved that the function $f^{(n)}\big(\xi(x)\big)$ can be represented as

$$f^{(n)}\big(\xi(x)\big) = f^{(n)}(x_0) + \alpha(x), \text{ where } \alpha(x) \to 0 \text{ as } x \to x_0.$$

Substitute the obtained representation of the function $f^{(n)}\big(\xi(x)\big)$ into the remainder term from the right-hand side of relation (9):

$$\frac{f^{(n)}(\xi)}{n!}(x - x_0)^n = \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{\alpha(x)}{n!}(x - x_0)^n. \tag{12}$$

The first term on the right-hand side of equality (12) can be added to the sum of Taylor's formula (9) as the term corresponding to $k = n$. The second term can be represented as $\tilde{\alpha}(x)(x - x_0)^n$, where $\tilde{\alpha}(x) = \frac{\alpha(x)}{n!} \to 0$ as $x \to x_0$. Thus, this second term is $o\big((x - x_0)^n\big)$, $x \to x_0$.

After the indicated transformations on the right-hand side of relation (9) are performed, this relation takes the form (8). ☐

Expansions of elementary functions by Taylor's formula in a neighborhood of zero

Expansions of functions e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$

20A/20:31 (19:50)

FUNCTION e^x.

Since $(e^x)^{(n)} = e^x$, $n = 0, 1, 2, \ldots$, we obtain that the derivatives of this function of any order are equal to 1 at the point 0. Therefore, the expansion of the function e^x by Taylor's formula at the point $x_0 = 0$ with the remainder term in the Peano form will be as follows:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n) =$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n), \quad x \to 0.$$

From the obtained formula, the previously proved equivalence $e^x \sim 1+x$, $x \to 0$, follows, since for $n = 1$ the expansion takes the form $e^x = 1+x+o(x)$, $x \to 0$.

REMARK.

This and all subsequent expansions of elementary functions are valid for both positive and negative values of x belonging to the domain of definition of the function.

FUNCTION $\sin x$.

Let us sequentially find the derivatives of the function $\sin x$ at the point 0. The function $\sin x$ itself vanishes at the point 0. Its first derivative is $\cos x$, so it equals 1 at the point 0. The second derivative is $(-\sin x)$, it vanishes at the point 0. Finally, the third derivative is $(-\cos x)$, it equals -1 at the point 0. The fourth derivative coincides with the original function $\sin x$, therefore, starting from it, the set of values at the point 0 will be repeated: 0, 1, 0, -1, ...

So, we obtain that even-order derivatives vanish at the point 0, and odd-order derivatives take alternating values of 1 and -1, starting from 1. Therefore, the expansion of the function $\sin x$ by Taylor's formula at the point $x_0 = 0$ with the remainder term in the Peano form will be as follows:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) =$$

$$= \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + o(x^{2n+2}), \quad x \to 0.$$

The remainder term has the form $o(x^{2n+2})$, since we can take into account one more (zero-valued) term corresponding to $k = 2n + 2$ in the sum.

It should be noted that the expansion of the function $\sin x$ contains only odd powers of x, starting with x in the first power, and their signs alternate.

From the obtained formula, the previously proved equivalence $\sin x \sim x$, $x \to 0$, follows, since for $n = 0$ the expansion takes the form $\sin x = x+o(x^2)$, $x \to 0$.

FUNCTION $\cos x$.

With successive differentiation of the function $\cos x$, we will obtain the following functions (starting with the zero derivative): $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, $\cos x$, $-\sin x, \ldots$ At the point 0, these functions take the following values: 1, 0, -1, 0, 1, 0, ... In this case, the derivatives of odd order vanish at the point 0, and the derivatives of even order take alternating values of 1 and -1, starting from 1. Therefore, the expansion of the function $\cos x$ by Taylor's formula at the point $x_0 = 0$ with the remainder term in the Peano form will be as follows:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1}) =$$
$$= \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + o(x^{2n+1}), \quad x \to 0.$$

The remainder term has the form $o(x^{2n+1})$, since we can take into account one more (zero-valued) term corresponding to $k = 2n + 1$ in the sum.

It should be noted that the expansion of the function $\cos x$ contains only even powers of x, starting with $x^0 = 1$, and their signs alternate.

From this formula, the previously proved equivalence $\cos x \sim 1 - \frac{x^2}{2}$, $x \to 0$, follows, since for $n = 1$ the expansion takes the form $\cos x = 1 - \frac{x^2}{2} + o(x^3)$, $x \to 0$.

FUNCTIONS $\sinh x$ AND $\cosh x$.

Since $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$ and, in addition, $\sinh 0 = 0$ and $\cosh 0 = 1$, we obtain that the successive differentiation of the hyperbolic sine and cosine at the point 0 gives alternating values of 1 and 0. Moreover, for the function $\sinh x$, as for the function $\sin x$, nonzero values correspond to derivatives of odd order and for the function $\cosh x$, as for the function $\cos x$, nonzero values correspond to derivatives of even order (starting from order 0). The difference between the expansions of hyperbolic functions and the expansions of the corresponding trigonometric ones is only that the signs do not alternate in these expansions:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) =$$
$$= \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}), \quad x \to 0;$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}) =$$

$$= \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+1}), \quad x \to 0.$$

Expansions of the functions $\ln(1+x)$ **and** $(1+x)^{\alpha}$

20B/00:00 (15:52)

FUNCTION $\ln(1+x)$.

In this case, we find the expansion of the logarithm function in a neighborhood of point 1; moreover, the estimate $x > -1$ must be satisfied for x.

Let us calculate several initial derivatives of the function $\ln(1+x)$ at the point 0 and substitute them in the corresponding terms of Taylor's formula:

$$\big(\ln(1+x)\big)^{(0)}\Big|_{x=0} = \ln 0 = 0, \quad f(0) = 0;$$

$$\big(\ln(1+x)\big)'\Big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1, \quad f'(0)x = x;$$

$$\big(\ln(1+x)\big)''\Big|_{x=0} = -\frac{1}{(1+x)^2}\Big|_{x=0} = -1, \quad \frac{f''(0)x^2}{2} = -\frac{x^2}{2};$$

$$\big(\ln(1+x)\big)'''\Big|_{x=0} = \frac{2}{(1+x)^3}\Big|_{x=0} = 2, \quad \frac{f'''(0)x^3}{3!} = \frac{x^3}{3};$$

$$\big(\ln(1+x)\big)^{(4)}\Big|_{x=0} = -\frac{2\cdot 3}{(1+x)^4}\Big|_{x=0} = -3!, \quad \frac{f^{(4)}(0)x^4}{4!} = -\frac{x^4}{4}.$$

Thus, the terms have alternating signs in this expansion. In addition, in the denominator, instead of the factorial, only one factor remains, since all other factors are cancelled out with the coefficients of the corresponding derivatives:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + o(x^n) =$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k} + o(x^n), \quad x \to 0.$$

The resulting expansion of the function $\ln(1+x)$ contains all powers of x, starting from the first power, and their signs alternate. Moreover, the denominator does not have factorials.

From this formula, the previously proved equivalence $\ln(1+x) \sim x$, $x \to 0$, follows, since for $n = 1$ the expansion takes the form $\ln(1+x) = x + o(x)$, $x \to 0$.

Function $(1+x)^\alpha$, $\alpha \neq 0$.

In this case, we also find the expansion of the function in a neighborhood of the point 1; moreover, any real number, except 0, can be taken as α.

Apply the formula for the derivative of a power function of order n:

$$\big((1+x)^\alpha\big)^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}, \quad n = 0, 1, 2, \ldots$$

For derivatives at the point 0, we will sequentially obtain the values 1, α, $\alpha(\alpha-1)$, $\alpha(\alpha-1)(\alpha-2)$, ... Therefore, the expansion of the function $(1+x)^\alpha$ by Taylor's formula at the point $x_0 = 0$ with the remainder term in the Peano form will be as follows:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots + \\ + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n) = \\ = \sum_{k=0}^{n} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k + o(x^n), \quad x \to 0.$$

Note that if $\alpha \in \mathbb{N}$, then, starting from some order, all derivatives vanish, and we obtain the version of Taylor's formula for polynomials in which the remainder term equals 0.

From this formula, the previously proved equivalence $(1+x)^\alpha \sim 1 + \alpha x$, $x \to 0$, follows, since for $n = 1$ the expansion takes the form $(1+x)^\alpha = 1 + \alpha x + o(x)$, $x \to 0$.

Example of using expansions to calculate limits

20B/15:52 (05:34)

Consider the following limit: $\lim_{x\to 0} \frac{x - \sin x}{x^3}$. To calculate this limit, we cannot use the equivalence $\sin x \sim x$, $x \to 0$, since equivalences can be used only in products and quotients. Instead, we apply the expansion of the function $\sin x$ by Taylor's formula with the remainder term $o(x^3)$:

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{x - \big(x - \frac{x^3}{3!} + o(x^3)\big)}{x^3}.$$

When removing the brackets, we can omit the minus sign in front of the term $o(x^3)$, since this term simply denotes some function that decreases faster than x^3 as $x \to 0$:

$$\lim_{x\to 0} \frac{x - x + \frac{x^3}{3!} + o(x^3)}{x^3} = \lim_{x\to 0} \frac{x^3}{3!x^3} + \lim_{x\to 0} \frac{o(x^3)}{x^3}.$$

In the first limit of the right-hand side, we can reduce the factors x^3. As a result, we obtain the limit equal to $\frac{1}{6}$.

The second limit is 0, because, by the definition of the "little-o", expression $o(x^3)$ can be represented as $\alpha(x)x^3$, where $\alpha(x) \to 0$ as $x \to 0$.

Thus, the initial limit is $\frac{1}{6}$.

Note that when using the equivalence $\sin x \sim x$ we would obtain an incorrect answer equal to 0.

23. L'Hospital's rule

Formulation and proof of L'Hospital's rule

20B/21:26 (08:28)

THEOREM (L'HOSPITAL'S RULE).

Suppose that the functions f and g are defined in the punctured neighborhood $\overset{\circ}{U}_{x_0}$ of the point $x_0 \in \mathbb{R} \cup \{\infty\}$, and the following conditions are satisfied:

1) f and g are differentiable in $\overset{\circ}{U}_{x_0}$;

2) for $x \in \overset{\circ}{U}_{x_0}$, $g'(x) \neq 0$;

3) there exists a limit $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)} = A \in \mathbb{R} \cup \{\infty\}$;

4) either $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$ or $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = \infty$.

Then the limit $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ also exists and is equal to A.

PROOF[10].

Case 1. Condition 4 has the form:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0. \tag{1}$$

First, we assume that $x_0 \in \mathbb{R}$, $A \in \mathbb{R}$. Let us prove the existence of the right-hand limit: $\lim_{x \to x_0+0} \frac{f(x)}{g(x)} = A$.

We suppose that the neighborhood $\overset{\circ}{U}_{x_0}$ has the form $(a_0, x_0) \cup (x_0, b)$. By condition 1, the functions f and g are differentiable on the interval (x_0, b). Then for an arbitrary point $x \in (x_0, b)$ we obtain that the functions f and g are differentiable and, therefore, continuous on the half-interval $(x_0, x]$.

Define the functions f and g at the point x_0 as follows:

$$f(x_0) = g(x_0) = 0. \tag{2}$$

Then, taking into account (1), we obtain that the functions f and g are continuous at the point x_0. Thus, the functions f and g are continuous on the segment $[x_0, x]$, differentiable on the interval (x_0, x), and, moreover, due

[10] In video lectures, there is no proof of this theorem.

to condition 2, $g'(t) \neq 0$ for $t \in (x_0, x)$. So, all the conditions of the Cauchy's mean value theorem are satisfied on the segment $[x_0, x]$.

Applying this theorem for the segment $[x_0, x]$, we obtain that for some point $\xi \in (x_0, x)$, the following relation holds:

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}.$$

Given (2), this relation can be rewritten in the form

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}. \tag{3}$$

Let us choose some $\varepsilon > 0$. By condition 3, there exists a value $\delta > 0$ such that for all $\xi \in (x_0, x_0 + \delta)$, the following estimate holds:

$$\left| \frac{f'(\xi)}{g'(\xi)} - A \right| < \varepsilon. \tag{4}$$

Given equality (3), as well as the fact that $\xi \in (x_0, x)$, we obtain that for all $x \in (x_0, x_0 + \delta)$, the estimate holds:

$$\left| \frac{f(x)}{g(x)} - A \right| < \varepsilon.$$

So, we have proved the following statement:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in (x_0, x_0 + \delta) \quad \left| \frac{f(x)}{g(x)} - A \right| < \varepsilon.$$

This means that $\lim_{x \to x_0 + 0} \frac{f(x)}{g(x)} = A$. In exactly the same way, considering the interval (a, x_0), we can prove the existence of the left-hand limit at the point x_0 for the relation $\frac{f(x)}{g(x)}$, and this limit will also be equal to A. Therefore, by the criterion for the existence of a function limit in terms of one-sided limits, we obtain that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = A$.

If $x_0 \in \mathbb{R}$, $A = \infty$, then the proof is similar; it is only necessary to write the definition for an infinite limit instead of (4).

If $x_0 = \infty$, then introduce the auxiliary functions $\tilde{f}(y) = f\left(\frac{1}{y}\right)$ and $\tilde{g}(y) = g\left(\frac{1}{y}\right)$. For these functions, conditions 1 and 2 of the theorem will be satisfied in some neighborhood of the point $y_0 = 0$, and condition 4, taking into account (1), will take the form

$$\lim_{y \to 0} \tilde{f}(y) = \lim_{y \to 0} \tilde{g}(y) = 0.$$

In addition, due to condition 3, the limit of the ratio of the derivatives $\tilde{f}'(y)$ and $\tilde{g}'(y)$ will be equal to

$$\lim_{y\to 0}\frac{\tilde f'(y)}{\tilde g'(y)}=\lim_{y\to 0}\frac{f'(x)|_{x=\frac{1}{y}}\left(-\frac{1}{y^2}\right)}{g'(x)|_{x=\frac{1}{y}}\left(-\frac{1}{y^2}\right)}=$$

$$=\lim_{y\to 0}\frac{f'(x)|_{x=\frac{1}{y}}}{g'(x)|_{x=\frac{1}{y}}}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=A.$$

This means that all the conditions of L'Hospital's rule are satisfied for the functions $\tilde f(y)$ and $\tilde g(y)$ at the point $y_0=0$, and therefore

$$\lim_{y\to y_0}\frac{\tilde f(y)}{\tilde g(y)}=A.$$

Given the definitions of the functions $\tilde f(y)$ and $\tilde g(y)$, we obtain:

$$\lim_{y\to y_0}\frac{\tilde f(y)}{\tilde g(y)}=\lim_{y\to y_0}\frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)}=\lim_{x\to\infty}\frac{f(x)}{g(x)}=A.$$

Thus, we considered all the situations corresponding to case 1.

Case 2. Condition 4 has the form:

$$\lim_{x\to x_0}f(x)=\lim_{x\to x_0}g(x)=\infty. \tag{5}$$

First, we again assume that $x_0\in\mathbb{R}$, $A\in\mathbb{R}$. Let us prove the existence of the right-hand limit: $\lim_{x\to x_0+0}\frac{f(x)}{g(x)}=A$.

As in the analysis of case 1, we suppose that the neighborhood $\overset{\circ}{U}_{x_0}$ has the form $(a_0,x_0)\cup(x_0,b)$. We choose two arbitrary points $x,c\in(x_0,b)$ so that $x<c$. By condition 1, the functions f and g are differentiable on the interval (x_0,b). Then the functions f and g are continuous on the segment $[x,c]$, differentiable on the interval (x,c), and, moreover, by virtue of condition 2, $g'(t)\neq 0$ for $t\in(x,c)$. So, all conditions of the Cauchy's mean value theorem are satisfied on the segment $[x,c]$.

Applying this theorem for the segment $[x,c]$, we obtain that for some point $\xi\in(x,c)$, the following relation holds:

$$\frac{f(c)-f(x)}{g(c)-g(x)}=\frac{f'(\xi)}{g'(\xi)}. \tag{6}$$

Let us transform the left-hand side of relation (6) as follows:

$$\frac{f(c)-f(x)}{g(c)-g(x)}=\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f(x)}{g(x)}\cdot\frac{1-\frac{f(c)}{f(x)}}{1-\frac{g(c)}{g(x)}}\,.$$

We substitute the transformed left-hand side in (6):

$$\frac{f(x)}{g(x)} \cdot \frac{1 - \frac{f(c)}{f(x)}}{1 - \frac{g(c)}{g(x)}} = \frac{f'(\xi)}{g'(\xi)}.$$

In the resulting ratio, we move the second factor of the left-hand side to the right-hand side:

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1 - \frac{g(c)}{g(x)}}{1 - \frac{f(c)}{f(x)}}. \tag{7}$$

Let us choose some $\varepsilon > 0$. By virtue of condition 3, there exists a value $\delta' > 0$ such that for all $\xi \in (x_0, x_0 + \delta')$, the following estimate holds:

$$\left| \frac{f'(\xi)}{g'(\xi)} - A \right| < \varepsilon. \tag{8}$$

The estimate (8) can be rewritten in the form of a double inequality:

$$A - \varepsilon < \frac{f'(\xi)}{g'(\xi)} < A + \varepsilon. \tag{9}$$

Since $\xi \in (x, c)$, we obtain that relation (9) holds for $c = x_0 + \delta'$.

Let us select $c = x_0 + \delta'$. By virtue of (5), we obtain

$$\lim_{x \to x_0} \frac{1 - \frac{g(c)}{g(x)}}{1 - \frac{f(c)}{f(x)}} = 1.$$

This means that, for the previously selected $\varepsilon > 0$ and $c = x_0 + \delta'$, there exists a value $\delta'' > 0$ such that for all $x \in (x_0, x_0 + \delta'')$, the estimate holds:

$$\left| \frac{1 - \frac{g(c)}{g(x)}}{1 - \frac{f(c)}{f(x)}} - 1 \right| < \varepsilon,$$

$$1 - \varepsilon < \frac{1 - \frac{g(c)}{g(x)}}{1 - \frac{f(c)}{f(x)}} < 1 + \varepsilon. \tag{10}$$

Denoting $\delta = \min\{\delta', \delta''\}$, multiplying termwise the relations (9) and (10) and using equality (7), we obtain the following double inequality, which is valid for all $x \in (x_0, x_0 + \delta)$:

$$(A - \varepsilon)(1 - \varepsilon) < \frac{f(x)}{g(x)} < (A + \varepsilon)(1 + \varepsilon). \tag{11}$$

Taking into account that $(A-\varepsilon)(1-\varepsilon) = A-\varepsilon(A+1)+\varepsilon^2 > A-\varepsilon(A+1)-\varepsilon^2$, $(A + \varepsilon)(1 + \varepsilon) = A + \varepsilon(A + 1) + \varepsilon^2$, we obtain that the double inequality (11) implies the following estimate:

$$\left|\frac{f(x)}{g(x)} - A\right| < \varepsilon(A+1) + \varepsilon^2.$$

So, we have proved the following statement:

$$\forall\,\varepsilon > 0 \quad \exists\,\delta > 0 \quad \forall\,x \in (x_0, x_0+\delta) \qquad \left|\frac{f(x)}{g(x)} - A\right| < \varepsilon(A+1) + \varepsilon^2.$$

This means that $\lim_{x\to x_0+0} \frac{f(x)}{g(x)} = A$. In exactly the same way, considering the interval (a, x_0), we can prove the existence of the left-hand limit at the point x_0 for the relation $\frac{f(x)}{g(x)}$, and this limit will also be equal to A. Therefore, by the criterion for the existence of a limit in terms of one-sided limits, we obtain that $\lim_{x\to x_0} \frac{f(x)}{g(x)} = A$.

Special situations $A = \infty$ and $x_0 = \infty$ can be analyzed using the techniques described for case 1. □

Examples of applying L'Hospital's rule 20B/29:54 (08:59)

1. $\lim\limits_{x\to 0} \dfrac{\sin ax}{\sin bx}$.

We can use the L'Hospital rule, since all its conditions are satisfied. In particular, we have the indeterminate form $\frac{0}{0}$ for this limit, and, in addition, the limit of the ratio of derivatives can be found by simple substituting the value 0 in the calculated derivatives:

$$\lim_{x\to 0} \frac{(\sin ax)'}{(\sin bx)'} = \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a\cos 0}{b\cos 0} = \frac{a\cdot 1}{b\cdot 1} = \frac{a}{b}.$$

Thus, the limit of the relation $\frac{\sin ax}{\sin bx}$ is also equal to $\frac{a}{b}$.

When finding this limit, one can also use the equivalence $\sin x \sim x$, $x \to 0$:

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{ax}{bx} = \frac{a}{b}.$$

2. $\lim\limits_{x\to 0} \dfrac{\cosh x - \cos x}{x^2}$.

In this case, we also have the indeterminate form $\frac{0}{0}$. Let us study the limit of the ratio of derivatives:

$$\lim_{x\to 0} \frac{(\cosh x - \cos x)'}{(x^2)'} = \lim_{x\to 0} \frac{\sinh x + \sin x}{2x}. \tag{12}$$

There is also the indeterminate form $\frac{0}{0}$ for this limit, therefore, we differentiate the numerator and denominator again:

$$\lim_{x\to 0} \frac{(\sinh x + \sin x)'}{(2x)'} = \lim_{x\to 0} \frac{\cosh x + \cos x}{2} = \frac{1+1}{2} = 1.$$

The last limit exists, therefore, according to L'Hospital's rule, the limit (12) also exists and is equal to the same value. Applying L'Hospital's rule once again, we obtain that the original limit also exists and is equal to 1.

To find this limit, we can also expand the functions $\cosh x$ and $\cos x$ by Taylor's formula. In this case, it suffices to use the expansion with the remainder term $o(x^2)$:

$$\lim_{x\to 0}\frac{\cosh x-\cos x}{x^2}=\lim_{x\to 0}\frac{1+\frac{x^2}{2}+o(x^2)-\left(1-\frac{x^2}{2}+o(x^2)\right)}{x^2}.$$

After removing the brackets, the constants 1 disappear, but the difference $o(x^2)-o(x^2)$ does not disappear, because, by the definition of the "little-o", each of the expressions $o(x^2)$ can be represented as $\alpha(x)x^2$, where $\alpha(x)\to 0$ as $x\to 0$, and the functions α are, generally speaking, different for each of the expressions $o(x^2)$. Therefore, for the difference $o(x^2)-o(x^2)$, we can only say that it decreases faster than x^2, that is, $o(x^2)-o(x^2)=o(x^2)$. We obtain:

$$\lim_{x\to 0}\frac{\frac{x^2}{2}+\frac{x^2}{2}+o(x^2)}{x^2}=\lim_{x\to 0}\frac{x^2}{x^2}+\lim_{x\to 0}\frac{o(x^2)}{x^2}.$$

In the first limit of the right-hand side, we can reduce the factors x^2. As a result, we obtain the limit equal to 1. The second limit, by virtue of the definition of the "little-o", is 0. Thus, the original limit equals 1.

Supplement. An example of a differentiable function whose derivative is not continuous 21A/00:00 (17:34)

In many theorems related to Taylor's formula, it was especially required that some function f be continuously differentiable. This means that this function has a derivative on a certain interval, and also that this derivative is continuous.

Let us give an example showing that a situation is possible when a function has a derivative, but this derivative is not continuous. We consider the following function defined on the interval $(-1,1)$:

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & x\in(-1,1)\setminus\{0\},\\ 0, & x=0.\end{cases}$$

This function is continuous on the interval $(-1,1)$. For all points except 0, this follows from the continuity of elementary functions included in its definition. For the point 0, this follows from the existence of the limit $\lim_{x\to 0}x^2\sin\frac{1}{x}$ equal to 0, that is, equal to the value of the function at the point 0. This limit is 0 as the product of an infinitesimal value x^2 by a bounded value $\sin\frac{1}{x}$.

Let us show that the function f is differentiable on the interval $(-1, 1)$. For all points except 0, this follows from the differentiability of the elementary functions included in its definition. Note that to calculate the derivative of the function f at all points other than 0, it suffices to use the differentiation formulas of elementary functions and theorems on the differentiation of product and superposition:

$$f'(x) = \left(x^2 \sin\frac{1}{x}\right)' = 2x\sin\frac{1}{x} - x^2\cos\frac{1}{x}\cdot\frac{1}{x^2} = \\ = 2x\sin\frac{1}{x} - \cos\frac{1}{x}, \quad x \neq 0.$$

To show that the function is differentiable at the point 0, it suffices to prove that it has a derivative at this point. However, in this case, since the function f is defined at the point 0 in a special way, it is necessary to use the definition of the derivative:

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x^2\sin\frac{1}{x} - 0}{x} = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

The indicated limit is 0 as the product of an infinitesimal value x by a bounded value $\sin\frac{1}{x}$. So, the derivative of the function f at zero exists and is equal to 0.

We have proved that the function f is differentiable on the interval $(-1, 1)$. In addition, we found the derivative of this function in the interval:

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \in (-1, 1) \setminus \{0\}, \\ 0, & x = 0. \end{cases}$$

Obviously, the derivative $f'(x)$ is continuous at all points other than 0, as a combination of continuous functions.

However, the derivative $f'(x)$ is not continuous at the point 0, since its limit at 0 does not exist. Indeed, to find the limit of the function $f'(x)$ at zero, we must consider the following limit:

$$\lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right).$$

The first term $2x\sin\frac{1}{x}$ has a limit which equals 0. But the second term $\cos\frac{1}{x}$ has no limit. This can be proved in the same way that the previously proved (in Chapter 8) absence of a limit at the point 0 for the function $\sin\frac{1}{x}$.

So, we have given an example of a function differentiable on an interval, but not continuously differentiable on this interval, since its derivative is not continuous at one of the points of this interval. This example shows that if the continuity of a derivative is required to prove some fact, then this condition must be indicated explicitly, since the continuity of a derivative does not yet follow from its existence.

24. Application of differential calculus to the study of functions

Local extrema of functions

A necessary condition for the existence of a local extremum

20B/38:53 (06:18)

The necessary condition for the existence of an interior local extremum at the point x_0 for the function f can be obtained using previously proved Fermat's theorem.

THEOREM (A NECESSARY CONDITION FOR THE EXISTENCE OF A LOCAL EXTREMUM).

Let the function f be defined and continuous in some neighborhood U_{x_0} of the point x_0 (this, in particular, means that the point x_0 is the interior point of the domain of the function f). Let the point x_0 be the point of the interior local extremum of the function f. Then either the function f is not differentiable at the point x_0, or the function f is differentiable at the given point and $f'(x_0) = 0$.

PROOF.

The theorem is a reformulation of Fermat's theorem. □

It follows from this theorem that if the function f is differentiable at the point x_0, but its derivative at this point does not vanish, then this point cannot be a point of the interior local extremum of the function f.

Interior points at which the function is non-differentiable or the derivative vanishes are called *critical points*, or *points suspected for a local extremum.* However, a critical point may not be an extremum point.

For example, for the function $f(x) = x^3$, the point 0 is a critical point, since $f'(0) = 2x|_{x=0} = 0$, but it is not a local extremum point, since the inequality $f(x) < f(0)$ holds for all $x < 0$, and the inequality $f(x) > f(0)$ holds for all $x > 0$.

Thus, the formulated necessary condition for the existence of a local extremum is not a sufficient condition.

The first sufficient condition for the existence of a local extremum

21A/17:34 (07:51)

In the sufficient condition considered in this section, the differential properties of the function are analyzed not at the point of a local extremum, but in its neighborhood.

Theorem (first sufficient condition for the existence of a local extremum).

Let the function f be differentiable in some punctured neighborhood $\overset{\circ}{U}_{x_0}$ of the point x_0. Then the presence or absence of a local extremum at the point x_0 is determined by the signs of the derivative $f'(x)$ in the left-hand and the right-hand neighborhoods ($U^-_{x_0}$ and $U^+_{x_0}$) of the point x_0 as shown in Table 3.

Table 3

Signs of a derivative and local extrema

$U^-_{x_0}$	$U^+_{x_0}$	Local extremum
$f'(x) > 0$	$f'(x) > 0$	no extremum
$f'(x) > 0$	$f'(x) < 0$	strict local maximum
$f'(x) < 0$	$f'(x) > 0$	strict local minimum
$f'(x) < 0$	$f'(x) < 0$	no extremum

Proof.

This statement follows from the second corollary of Lagrange's theorem: a function increases on an interval on which its derivative is positive, and decreases on an interval on which its derivative is negative. Thus, if the derivative changes sign when passing through the point x_0, then this point is a point of strict local extremum, and if the sign does not change, then there is no extremum. □

The second sufficient condition for the existence of a local extremum

21A/25:25 (15:50)

In a sufficient condition considered in this section, as in the necessary condition, the differential properties of the function are analyzed at the local extremum point. However, unlike the necessary condition, it requires to analyze higher order derivatives.

Theorem (second sufficient condition for the existence of a local extremum).

Let the function f be continuously differentiable in some neighborhood of the point x_0 up to the derivative of order $n \in \mathbb{N}$, and let the first nonzero derivative at the point x_0 be the derivative of order n:

$$f'(x_0) = 0, \quad f''(x_0) = 0, \quad \ldots, \quad f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0. \quad (1)$$

If n is an odd number, then there is no local extremum at the point x_0, and if n is an even number, then in the case $f^{(n)}(x_0) > 0$, the point x_0 is a point of a strict local minimum, and in the case of $f^{(n)}(x_0) < 0$, the point x_0 is a point of a strict local maximum.

Proof.

We use Taylor's formula with the remainder term in the Peano form and expand the function f according to this formula at the point x_0 up to the term with the derivative of order n:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o\big((x - x_0)^n\big), \quad x \to x_0.$$

By virtue of conditions (1), all terms corresponding to $k = 1, 2, \ldots, n - 1$ disappear in the sum, and only terms corresponding to the function itself and its nth derivative remain:

$$f(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o\big((x - x_0)^n\big), \quad x \to x_0.$$

Transfer the term $f(x_0)$ to the left-hand side of the equality and represent the expression $o\big((x - x_0)^n\big)$ in the form $\alpha(x)(x - x_0)^n$, where $\alpha(x) \to 0$ as $x \to x_0$:

$$f(x) - f(x_0) = \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \alpha(x)(x - x_0)^n =$$

$$= \left(\frac{f^{(n)}(x_0)}{n!} + \alpha(x)\right)(x - x_0)^n. \quad (2)$$

Since $\frac{f^{(n)}(x_0)}{n!} \neq 0$ and $\alpha(x) \to 0$, we can choose a neighborhood U_{x_0} of the point x_0 in which the absolute value of function $\alpha(x)$ will be less than the absolute value of $\frac{f^{(n)}(x_0)}{n!}$. This means that the behavior of the function $\alpha(x)$ in the neighborhood U_{x_0} will not affect the sign of the factor $\big(\frac{f^{(n)}(x_0)}{n!} + \alpha(x)\big)$ on the right-hand side of (2); this sign will be determined by the sign of the derivative $f^{(n)}(x_0)$. We can also say that the factor $\big(\frac{f^{(n)}(x_0)}{n!} + \alpha(x)\big)$ preserves the sign in the neighborhood U_{x_0}.

Consider the possible cases.

Case 1: n is odd. Then the factor $(x - x_0)^n$ changes sign in the neighborhood U_{x_0}: if $x < x_0$, then this factor is negative, and if $x > x_0$, then it is positive. So, the entire right-hand side of (2) also changes sign in the neighborhood U_{x_0}. Therefore, the left-hand side of (2), that is, $f(x) - f(x_0)$, also changes sign in the neighborhood U_{x_0}. This means that there is no local extremum at the point x_0, because for some points x from U_{x_0}, the value of $f(x)$ will be greater than $f(x_0)$, and for some points x, the value of $f(x)$ will be less than $f(x_0)$.

Case 2a: n is even and $f^{(n)}(x_0) > 0$. Then both factors on the right-hand side of (2) are positive for all $x \in \mathring{U}_{x_0}$. Therefore, the difference $f(x) - f(x_0)$ is also positive for all $x \in \mathring{U}_{x_0}$. This means that $f(x) > f(x_0)$ for all $x \in \mathring{U}_{x_0}$, that is, the point x_0 is a strict local minimum point.

Case 2b: n is even and $f^{(n)}(x_0) < 0$. Then the right-hand side of (2) is negative for all $x \in \mathring{U}_{x_0}$. Therefore, the difference $f(x) - f(x_0)$ is also negative for all $x \in \mathring{U}_{x_0}$. This means that $f(x) < f(x_0)$ for all $x \in \mathring{U}_{x_0}$, that is, the point x_0 is a strict local maximum point. □

EXAMPLES.

Consider the behavior of the function $f(x) = x^n$ at the point 0 for values n of different parity.

If $n = 2$, then $f'(x) = 2x$, $f''(x) = 2$, therefore $f'(0) = 0$, $f''(0) = 2$. Thus, since 2 is an even number and $f''(0) > 0$, the point 0 is a strict local minimum point for the function x^2.

If $n = 3$, then $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, therefore $f'(0) = f''(0) = 0$, $f'''(0) = 6$. Since 3 is an odd number, there is no local extremum at the point 0 for the function x^3.

These results can be generalized as follows: for all even n $(= 2, 4, \dots)$, the function x^n has a strict local minimum at the point 0, and for all odd n $(= 1, 3, \dots)$, the function x^n does not have a local extremum at the point 0.

Convex functions

Definitions of convex functions

21B/00:00 (16:10)

DEFINITION 1 OF A CONVEX FUNCTION.

Let the function f be defined on the interval (a, b). A function f is called *convex upwards* (or *concave*) on the interval (a, b) if the graph of the secant drawn through the points $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ lies below the function graph on the interval (x_1, x_2) for any points $x_1, x_2 \in (a, b)$, $x_1 < x_2$.

Similarly, a function f is called *convex downwards* (or just *convex*) on the interval (a, b) if the graph of the secant drawn through the points $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ lies above the function graph in the interval (x_1, x_2) for any points $x_1, x_2 \in (a, b)$, $x_1 < x_2$.

The left-hand part of Fig. 11 shows an example of a function that is convex upwards, and the right-hand part of Fig. 11 shows an example of a function that is convex downwards.

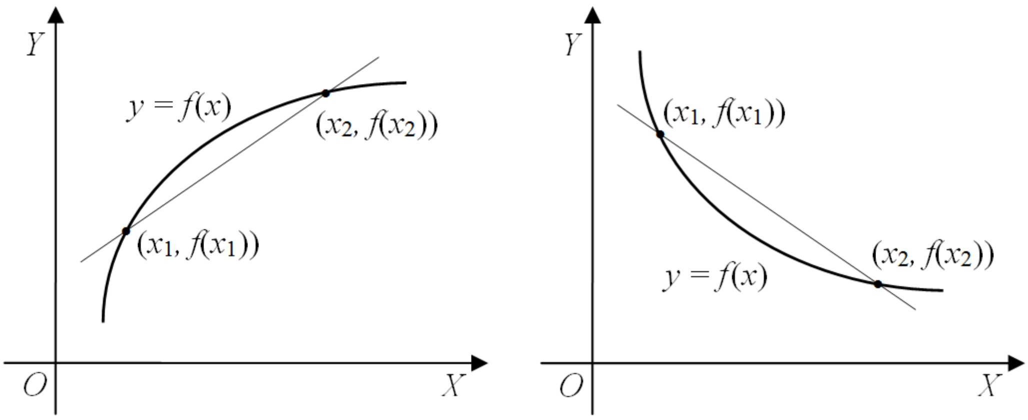

Fig. 11. Convex upwards and convex downwards functions

In order to write down convexity conditions in the form of formulas, we use the equation of a secant passing through the points $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$, $x_1 \neq x_2$. First, we write the equation of the secant in the form that was obtained when studying the geometric sense of the derivative (see Chapter 19):

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

To write down the convexity condition, it is convenient to transform this equation, leaving only the variable y on the left-hand side and moving all other terms to the right-hand side:

$$y = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1).$$

Let us transform the resulting right-hand side by presenting it in a more symmetrical form. First, we reduce both terms to a common denominator:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1) =$$
$$= \frac{\big(f(x_2) - f(x_1)\big)(x - x_1) + f(x_1)(x_2 - x_1)}{x_2 - x_1}.$$

Then we transform the numerator (for brevity, we will not rewrite the denominator):

$$\big(f(x_2) - f(x_1)\big)(x - x_1) + f(x_1)(x_2 - x_1) =$$
$$= xf(x_2) - xf(x_1) - x_1 f(x_2) + x_1 f(x_1) + x_2 f(x_1) - x_1 f(x_1) =$$

$$= xf(x_2) - xf(x_1) - x_1 f(x_2) + x_2 f(x_1) =$$
$$= f(x_1)(x_2 - x) + f(x_2)(x - x_1).$$

Thus, we have obtained the following version of the secant equation:

$$y = \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1}. \qquad (3)$$

Denote the right-hand side of equation (3) by $l_{x_1,x_2}(x)$:

$$l_{x_1,x_2}(x) \stackrel{\text{def}}{=} \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1}. \qquad (4)$$

Then the secant equation takes the form

$$y = l_{x_1,x_2}(x).$$

Since the equation of the graph of the function f has the form $y = f(x)$, we can now rewrite the definition of convexity in the language of formulas.

Definition 2 of a convex function.

A function f is called *convex upwards* on the interval (a, b) if for any points $x_1, x_2 \in (a, b)$, $x_1 < x_2$, and for any point $x \in (x_1, x_2)$, the inequality $f(x) > l_{x_1,x_2}(x)$ holds.

A function f is called *convex downwards* on the interval (a, b) if for any points $x_1, x_2 \in (a, b)$, $x_1 < x_2$, and for any point $x \in (x_1, x_2)$, the inequality $f(x) < l_{x_1,x_2}(x)$ holds.

Sufficient condition for convexity

21B/16:10 (15:42)

Theorem (a sufficient condition for convexity).

Let a function f be differentiable up to the second order on the interval (a, b). Then if $f''(x) > 0$ for any $x \in (a, b)$, then f is convex downwards on the interval (a, b), and if $f''(x) < 0$ for any $x \in (a, b)$, then f is convex upwards on the interval (a, b).

Remark.

We have previously established that the positive or negative first derivative means an increase or, accordingly, a decrease of the function. Thus, the increase and decrease of the function are associated with the properties of the first derivative, and its convexity is associated with the properties of the second derivative.

Proof.

By virtue of definition 2, to prove the theorem, it suffices to study the difference $l_{x_1,x_2}(x) - f(x)$ and show that for all $x, x_1, x_2 \in (a, b)$ satisfying the double inequality $x_1 < x < x_2$, this difference is positive in the case of a positive second derivative and negative in the case of a negative second derivative.

Let us arbitrarily choose the points $x, x_1, x_2 \in (a, b)$ that satisfy the double inequality $x_1 < x < x_2$, and transform the difference $l_{x_1,x_2}(x) - f(x)$, given the formula (4):

$$l_(x_1, x_2)(x) - f(x) = \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1} - f(x).$$

Reduce to a common denominator:

$$\frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1} - f(x) =$$
$$= \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1) - f(x)(x_2 - x_1)}{x_2 - x_1}.$$

From now on, we will only transform the numerator, without writing down the denominator.

Let us represent the difference $(x_2 - x_1)$ in the form $(x_2 - x + x - x_1)$ and rearrange the terms:

$$f(x_1)(x_2 - x) + f(x_2)(x - x_1) - f(x)(x_2 - x + x - x_1) =$$
$$= f(x_1)(x_2 - x) + f(x_2)(x - x_1) - f(x)(x_2 - x) - f(x)(x - x_1) =$$
$$= \big(f(x_1) - f(x)\big)(x_2 - x) + \big(f(x_2) - f(x)\big)(x - x_1) =$$
$$= \big(f(x_2) - f(x)\big)(x - x_1) - \big(f(x) - f(x_1)\big)(x_2 - x).$$

At the last stage, we transformed the expression so that the points were in the same order in the differences of functions and in the differences of points themselves.

Now for the differences $f(x_2) - f(x)$ and $f(x) - f(x_1)$ we apply Lagrange's theorem, all conditions of which are satisfied. By virtue of this theorem, there exist points $\xi \in (x, x_2)$ and $\eta \in (x_1, x)$ for which the following relations hold:

$$f(x_2) - f(x) = f'(\xi)(x_2 - x),$$
$$f(x) - f(x_1) = f'(\eta)(x - x_1).$$

We continue the transformation of the numerator using the obtained relations:

$$\big(f(x_2) - f(x)\big)(x - x_1) - \big(f(x) - f(x_1)\big)(x_2 - x) =$$
$$= f'(\xi)(x_2 - x)(x - x_1) - f'(\eta)(x - x_1)(x_2 - x) =$$
$$= \big(f'(\xi) - f'(\eta)\big)(x_2 - x)(x - x_1).$$

Apply Lagrange's theorem again, now for the difference of the derivatives $f'(\xi) - f'(\eta)$. By virtue of this theorem, there exists a point $\zeta \in (\eta, \xi)$ for which the following relation holds:

$$f'(\xi) - f'(\eta) = f''(\zeta)(\xi - \eta).$$

We finally get:

$$\big(f'(\xi) - f'(\eta)\big)(x_2 - x)(x - x_1) = f''(\zeta)(\xi - \eta)(x_2 - x)(x - x_1).$$

Thus, the difference $l_{x_1,x_2}(x) - f(x)$ can be represented as follows:

$$l_{x_1,x_2}(x) - f(x) = \frac{f''(\zeta)(\xi - \eta)(x_2 - x)(x - x_1)}{x_2 - x_1}. \tag{5}$$

Note that for the points included in the resulting expression, the following estimates hold: $x_1 < \eta < x < \xi < x_2$. Thus, all the differences of the points included in the right-hand side of equality (5) are positive. Therefore, the sign of the difference $l_{x_1,x_2}(x) - f(x)$ coincides with the sign of the second derivative $f''(\zeta)$ at the point $\zeta \in (\eta, \xi) \subset (a, b)$.

So, if $f''(x) > 0$ for all $x \in (a, b)$, then the difference $l_{x_1,x_2}(x) - f(x)$ is positive and, therefore, the function is convex downwards on the interval (a, b), and if $f''(x) < 0$ for all $x \in (a, b)$, then the difference $l_{x_1,x_2}(x) - f(x)$ is negative and, therefore, the function is convex upwards on the interval (a, b). □

Inflection points of a function

Definition of an inflection point

22A/00:00 (07:37)

In studying the properties of functions associated with increasing and decreasing, we introduced the notion of a local extremum point, that is, a point located between the intervals of increasing and decreasing of a function.

Similarly, we can introduce a special notion for a point located between the intervals at which the function is convex downwards and convex upwards.

Definition.

Let the function f be defined in some neighborhood of the point a and be continuous at this point. The point a is called the *inflection point* of the function f if there exist intervals (b, a) and (a, c) such that on one of them the function f is convex downwards and on the other the function f is convex upwards (Fig. 12).

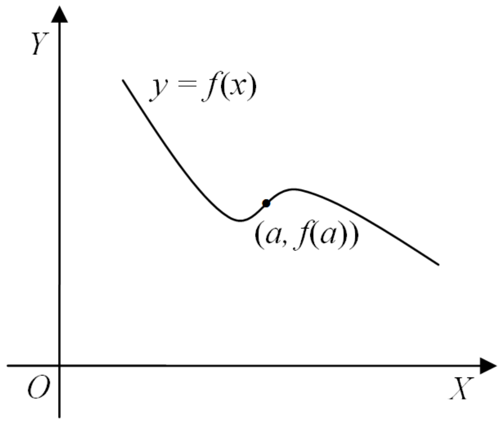

Fig. 12. Inflection point of a function

A necessary condition for the existence of an inflection point

22A/07:37 (10:20)

Theorem (a necessary condition for the existence of an inflection point).

Let a be the inflection point of the function f and let the function f be twice differentiable in some neighborhood of the point a and its second derivative be continuous at the point a. Then $f''(a) = 0$.

Proof.

Let us prove the theorem by contradiction: suppose that $f''(a) \neq 0$. For definiteness, we can assume that $f''(a) > 0$.

By condition of the theorem, the function f'' is continuous at the point a, and by our assumption $f''(a) > 0$. Then, by virtue of the theorem on the simplest properties of continuous functions, it can be stated that there exists a neighborhood U_a of the point a in which the function f'' preserves the sign, that is, in our case, $f''(x) > 0$ for $x \in U_a$.

But if the second derivative of the function is positive in some neighborhood U_a, then this means, by virtue of the previous theorem on a sufficient condition for convexity, that the function f is convex downwards in this neighborhood, which contradicts the condition that a is an inflection point. Note that if we considered the case of $f''(a) < 0$, then, by similar reasoning, we would obtain that the function f is convex upwards in some neighborhood of the point a. Therefore, our assumption is false and $f''(a) = 0$. □

Points at which the second derivative is continuous and vanishes are called *points suspected for inflection*. However, a point suspected for inflection is not necessarily an inflection point.

For example, for the functions $f_1(x) = x^3$ and $f_2(x) = x^4$, the point 0 is a point suspected for inflection, since $f_1''(0) = 6x|_{x=0} = 0$, $f_2''(0) = 12x^2|_{x=0} = 0$. However, the point 0 is an inflection point for the function x^3 and it is not an inflection point for the function x^4 (these facts will be rigorously proved later, by means of sufficient conditions for the existence of an inflection point).

Thus, the formulated necessary condition for the existence of an inflection point is not a sufficient condition.

The first sufficient condition for the existence of an inflection point

22A/17:57 (07:04)

In the sufficient condition considered in this section (as in the first sufficient condition for the existence of a local extremum), the differential properties of the function are analyzed not at the inflection point itself, but in its neighborhood.

Theorem (first sufficient condition for the existence of an inflection point).

Let the function f be continuous at the point a and twice differentiable in some punctured neighborhood $\mathring{U}_a$ of the point a. If the second derivative of the function f has different signs in the left-hand and the right-hand neighborhoods (U_a^- and U_a^+) of the point a, then the point a is an inflection point, and if the second derivative has the same signs, then the point a is not an inflection point.

Proof.

This statement immediately follows from the theorem on a sufficient condition for convexity. If, for example, the function f'' is positive in the left-hand neighborhood of the point a and negative in the right-hand neighborhood, then this means that the function f is convex downwards in the left-hand neighborhood and it is convex upwards in the right-hand neighborhood, therefore, the point a is an inflection point. A similar statement is also true if the function f'' is negative in the left-hand neighborhood and positive in the right-hand neighborhood. If the second derivative has the same signs in the left-hand and right-hand neighborhood of the point a, then the function f has the same convexity type to the left and right of the point a, therefore, the point a is not an inflection point. □

Using this sufficient condition, one can easily prove that the point 0 is an inflection point for the function x^3, but it is not an inflection point for the function x^4. Indeed, $(x^3)'' = 6x$ and, therefore, the second derivative takes values of different signs to the left and right of the point 0, while $(x^4)'' = 12x^2$ takes positive values both to the left and to the right of the point 0.

The second sufficient condition for the existence of an inflection point

22A/25:01 (12:39)

In the sufficient condition considered in this section, as in the necessary condition, the differential properties of the function are analyzed at the inflection point itself. However, unlike the necessary condition, it is required to analyze the derivative of both the second and the third order.

Theorem (second sufficient condition for the existence of an inflection point).

Let the function f be three times differentiable in some neighborhood of the point a and its third derivative be continuous at the point a. Let $f''(a) = 0$ and $f'''(a) \neq 0$. Then a is the inflection point of the function f.

Proof.

For definiteness, suppose that $f'''(a) > 0$.

Since the function f''' is continuous at the point a and $f'''(a) > 0$, we obtain, by the theorem on the simplest properties of continuous functions, that there exists a neighborhood U_a of the point a, in which the function f''' preserves the sign, that is, in our case, $f'''(x) > 0$ for $x \in U_a$. For the function f'', this means, by virtue of the second corollary of Lagrange's theorem, that it increases on the set U_a.

Since, by condition, $f''(a) = 0$, and, in addition, the function f'' increases in a neighborhood U_a, we obtain that the function f'' takes only negative values in the left-hand neighborhood U_a^-, and it takes only positive values in the right-hand neighborhood U_a^+: $f''(x) < 0$ for $x \in U_a^-$, $f''(x) > 0$ for $x \in U_a^+$. Thus, the conditions of the theorem on the first sufficient condition for the existence of an inflection point are satisfied, and the point a is an inflection point. The situation $f'''(a) < 0$ is analyzed in the same way. □

Now we can prove that the point 0 is the inflection point of the function $f(x) = x^3$ simply by calculating the values of the second and third derivatives at this point: $f''(0) = 6x|_{x=0} = 0$, $f'''(0) = 6|_{x=0} = 6 \neq 0$.

Location of the graph of a function relative to a tangent line

The theorem on the location of a tangent line in the domain of convexity of a function 22A/37:40 (02:04), 22B/00:00 (10:37)

The convexity properties of a function can also be studied by analyzing the location of a tangent line relative to the graph of a function.

It is natural to assume that if a tangent at any point x_0 of the interval (a, b) lies above the graph of the function, then the function will be convex upwards on (a, b) (see the left-hand part of Fig. 13), and if a tangent at any point $x_0 \in (a, b)$ will lie below the graph of the function, the function will be convex downwards on (a, b) (see the right-hand part of Fig. 13).

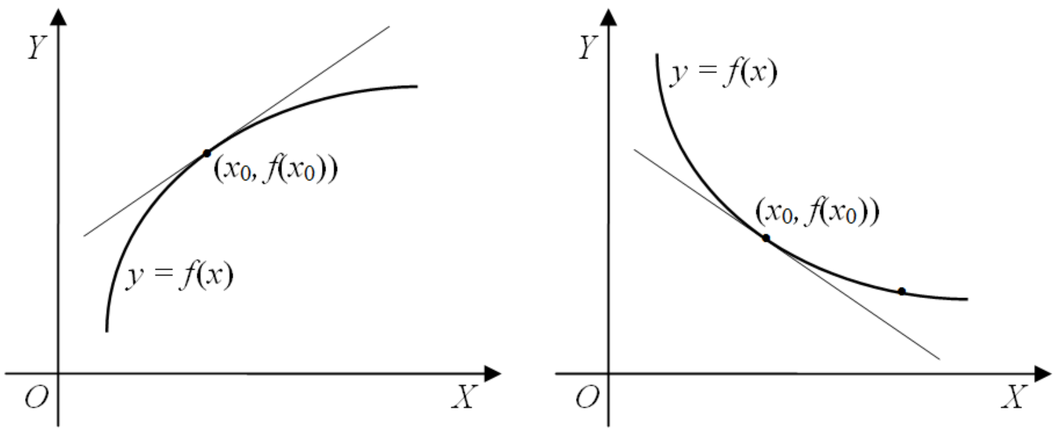

Fig. 13. Convexity property and location of tangent line

THEOREM (ON THE LOCATION OF A TANGENT IN THE DOMAIN OF CONVEXITY OF A FUNCTION).

Suppose that on the interval (a, b), the function f has a second derivative that preserves the sign: either $f''(x) > 0$ for all $x \in (a, b)$, or $f''(x) < 0$ for all $x \in (a, b)$. Then for any point $x_0 \in (a, b)$, there exists a punctured neighborhood $\overset{\circ}{U}_{x_0}$ such that the tangent at the point x_0 lies on one side of the function graph in this neighborhood.

PROOF.

Let us write the required statement in the form of a formula. To do this, we use the equation of the tangent line to the graph of the function $y = f(x)$ at the point x_0:

$$y - f(x_0) = f'(x_0)(x - x_0).$$

This equation can be written as $y = L_{x_0}(x)$, if we denote

$$L_{x_0}(x) \stackrel{\text{def}}{=} f(x_0) + f'(x_0)(x - x_0). \tag{6}$$

For definiteness, suppose that $f''(x) > 0$ for all $x \in (a, b)$, and prove that there exists a punctured neighborhood $\overset{\circ}{U}_{x_0} \subset (a, b)$ such that the estimate $f(x) - L_{x_0}(x) > 0$ holds for all $x \in \overset{\circ}{U}_{x_0}$. This estimate means that in the case of a function convex downwards on (a, b), the graph of the function lies above the graph of the tangent in $\overset{\circ}{U}_{x_0}$.

Let us choose a neighborhood $\overset{\circ}{U}_{x_0} \subset (a, b)$ and write the expansion of the function f by Taylor's formula at the point x_0 with the remainder term in the Lagrange form:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2, \quad x \in \overset{\circ}{U}_{x_0}. \tag{7}$$

Here ξ is some point lying between x_0 and x. If $x \in \overset{\circ}{U}_{x_0}$, then the point ξ will also lie in this neighborhood.

The first two terms on the right-hand side of (7) coincide with $L_{x_0}(x)$ (see (6)). Replace them with $L_{x_0}(x)$ and move them to the left-hand side. As a result, relation (7) takes the form

$$f(x) - L_{x_0}(x) = \frac{f''(\xi)}{2}(x - x_0)^2. \tag{8}$$

The factor $(x - x_0)^2$ is positive for all $x \in \overset{\circ}{U}_{x_0}$, the second derivative $f''(\xi)$ is also positive, since, by our assumption, $f''(x) > 0$ for all $x \in \overset{\circ}{U}_{x_0}$ and the point ξ belongs to $\overset{\circ}{U}_{x_0}$. Thus, the right-hand side of (8) is greater than zero for $x \in \overset{\circ}{U}_{x_0}$. Therefore, the left-hand side is also greater than zero: $f(x) - L_{x_0}(x) > 0$ for $x \in \overset{\circ}{U}_{x_0}$.

If we assume that $f''(x) < 0$ for all $x \in (a, b)$, then we can prove in the same way that $f(x) - L_{x_0}(x) < 0$ for $x \in \overset{\circ}{U}_{x_0}$, which means that in the case of a function convex upwards in (a, b), the graph of the function lies below the tangent graph in $\overset{\circ}{U}_{x_0}$. □

The theorem on the location of the tangent at an inflection point 22B/10:37 (09:46)

If there is a tangent at the inflection point, then to the left and to the right of the inflection point this tangent will lie on different sides of the function graph (Fig. 14). We prove this statement under the same assumptions as for the second sufficient condition for the existence of an inflection point.

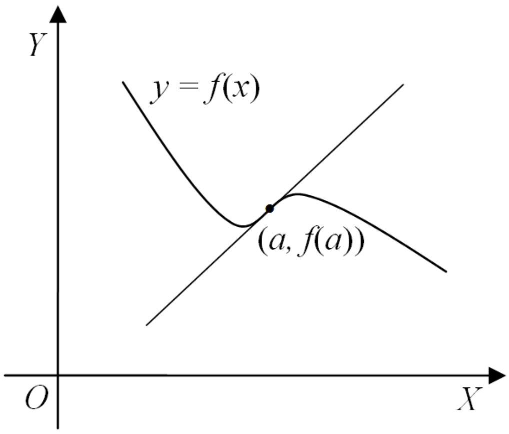

Fig. 14. Tangent line at an inflection point

Theorem (on the location of the tangent at an inflection point).

Let the function f be three times differentiable in some neighborhood of the point a and its third derivative is continuous at the point a. Let $f''(a) = 0$ and $f'''(a) \neq 0$. Then there exists a neighborhood U_a such that the tangent at the point a lies on different sides of the graph of the function on the left-hand and right-hand sides of this neighborhood.

Proof.

As in the proof of the previous theorem, we will use the equation of the tangent of the form $y = L_a(x)$, where the expression $L_a(x)$ is determined by formula (6). For definiteness, suppose that $f'''(a) > 0$ and show that in this case there exists a neighborhood U_a such that $f(x) - L_a(x) < 0$ for $x \in U_a^-$ and $f(x) - L_a(x) > 0$ for $x \in U_a^+$. This means that the tangent lies on different sides of the function graph in this neighborhood.

Since $f'''(a) > 0$ and, moreover, by condition, the third derivative is continuous at the point a, we obtain, by the theorem on the simplest properties of continuous functions, that there exists a neighborhood U_a of the point a, in which the function f''' preserves the sign, that is, in our case, $f'''(x) > 0$ for $x \in U_a$.

Let us write the expansion of the function f by Taylor's formula at the point a with the remainder term in the Lagrange form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(\xi)}{3!}(x-a)^3. \qquad (9)$$

Here ξ is some point lying between a and x. In particular, if $x \in U_a$, then the point ξ will also lie in this neighborhood.

The first two terms on the right-hand side of (9) coincide with $L_a(x)$ (see (6)). Replace them with $L_a(x)$ and move them to the left-hand side. In addition, we take into account that, by condition, $f''(a) = 0$. As a result, relation (9) takes the form:

$$f(x) - L_a(x) = \frac{f'''(\xi)}{3!}(x-a)^3. \tag{10}$$

We have chosen a neighborhood U_a in such a way that $f'''(x) > 0$ for $x \in U_a$. Since the point ξ also belongs to U_a when $x \in U_a$, we see that the factor $f'''(\xi)$ on the right-hand side of (10) is positive for all $x \in U_a$.

The difference $(x - a)$ on the right-hand side of (10) has an odd power, therefore the expression $(x-a)^3$ will be negative for $x \in U_a^-$ and it will be positive for $x \in U_a^+$. Therefore, the same estimates will be fulfilled for the left-hand side of (10): $f(x) - L_a(x) < 0$ for $x \in U_a^-$ and $f(x) - L_a(x) > 0$ for $x \in U_a^+$.

Similar reasonings allow us to prove that signs alternate for the expression $f(x) - L_a(x)$ in the case of $f'''(a) < 0$. □

Asymptotes[11]

Definition.

Let the function f be defined in a punctured neighborhood of the point x_0 (the neighborhood can be one-sided). The line $x = x_0$ is called a *vertical asymptote* of the graph of the function $y = f(x)$ if at least one of the following conditions is true:

$$\lim_{x\to x_0-0} f(x) = \infty, \qquad \lim_{x\to x_0+0} f(x) = \infty.$$

Let the function f be defined in a neighborhood of $+\infty$. The line $y = kx+b$ is called a *non-vertical asymptote* of the graph of the function $y = f(x)$, as $x \to +\infty$, if

$$\lim_{x\to+\infty}\big(f(x) - (kx+b)\big) = 0. \tag{11}$$

The non-vertical asymptote $y = kx + b$ as $x \to -\infty$ is defined similarly (provided that the function f is defined in a neighborhood of the point $-\infty$):

$$\lim_{x\to-\infty}\big(f(x) - (kx+b)\big) = 0.$$

If $k = 0$, then the non-vertical asymptote is called a *horizontal asymptote*, and if $k \neq 0$, then it is called an *oblique asymptote* or *slant asymptote*.

Examples.

The graph of the function $y = \frac{1}{x}$ has a vertical asymptote $x = 0$ and a horizontal asymptote $y = 0$ (see the left-hand part of Fig. 15). The graph

[11] This section is missing in video lectures.

of the function $y = \sqrt{x^2 + 1}$ has two oblique asymptotes: $y = x$ and $y = -x$ (see the right-hand part of Fig. 15). Note that both graphs are hyperbole branches.

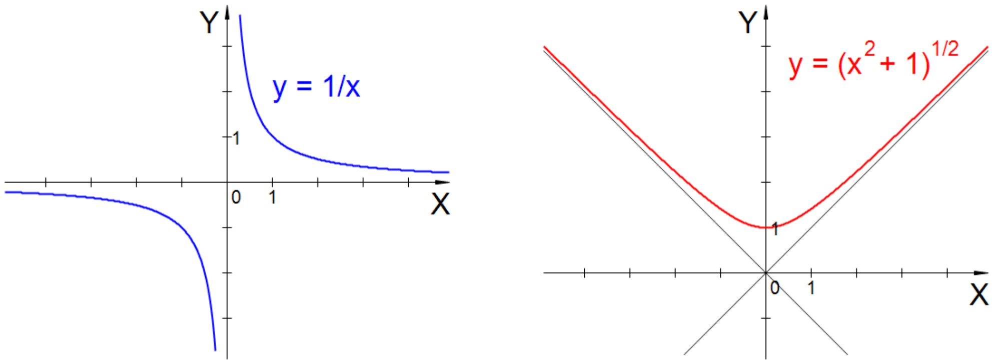

Fig. 15. Examples of asymptotes

The graph of the function $y = \tan x$ has an infinite number of vertical asymptotes of the form $y = \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$ (see Fig. 4 in the section "Preliminary information"). The graph of the function $y = \arctan x$ has two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ (see Fig. 7 in Chapter 14), and the graph of the function $y = \operatorname{artanh} x$ has two horizontal asymptotes $y = -1$ and $y = 1$ (see Fig. 8 in Chapter 18).

THEOREM (CRITERION FOR THE EXISTENCE OF A NON-VERTICAL ASYMPTOTE).

For the line $y = kx + b$ to be an asymptote of the graph of the function $y = f(x)$ as $x \to +\infty$, it is necessary and sufficient that there exist finite limits

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k, \qquad \lim_{x\to+\infty} \big(f(x) - kx\big) = b. \tag{12}$$

REMARK.

A similar criterion holds for the case $x \to -\infty$.

PROOF.

1. Necessity. Given: the line $y = kx + b$ is the asymptote of the graph of the function $y = f(x)$ for $x \to +\infty$. Prove: relations (12) holds.

Denote $\alpha(x) = f(x) - (kx + b)$. By the definition of the non-vertical asymptote (see (11)), we obtain that $\alpha(x) \to 0$ as $x \to +\infty$.

Then the function $f(x)$ can be represented as

$$f(x) = kx + b + \alpha(x). \tag{13}$$

Divide both sides of equality (13) by x:

$$\frac{f(x)}{x} = k + \frac{b}{x} + \frac{\alpha(x)}{x}.$$

Taking into account the property of the function $\alpha(x)$, we obtain that the limit of the right-hand side of the last equality, as $x \to +\infty$, is k, which implies the first of relations (12).

Now we transform equality (13) as follows:

$$f(x) - kx = b + \alpha(x).$$

The right-hand side of this equality, as $x \to +\infty$, is b, whence the second of relations (12) follows. The necessity is proven.

2. Sufficiency. Given: relations (12) holds. Prove: the line $y = kx + b$ is the asymptote of the graph of the function $y = f(x)$ for $x \to +\infty$.

Taking into account the second of relations (12), we obtain:

$$\lim_{x\to+\infty}\big(f(x) - (kx + b)\big) = \lim_{x\to+\infty}\big(f(x) - kx\big) - b = b - b = 0.$$

Thus, for the line $y = kx + b$, condition (11) from the definition of the non-vertical asymptote is satisfied. □

Example of a function study[12]

To illustrate the described methods of functions study, we apply them to the rational function $f(x) = \frac{x^2-3x-2}{x+1}$ and draw its graph.

1. Vertical asymptotes. The function $f(x)$ is defined for all real arguments, except for the point $x = -1$. At the point $x = -1$, the function has a discontinuity of the second kind, since

$$\lim_{x\to-1-1}\frac{x^2 - 3x - 2}{x + 1} = -\infty, \qquad \lim_{x\to-1+1}\frac{x^{-}3x - 2}{x + 1} = +\infty.$$

Therefore, the graph of the function has one vertical asymptote $x = -1$.

2. Non-vertical asymptotes. Let us use the criterion for the existence of non-vertical asymptotes:

$$k_\pm = \lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\frac{x^2 - 3x - 2}{x^2 + x} = 1,$$

$$b_\pm = \lim_{x\to\pm\infty}(f(x) - k_\pm x = \lim_{x\to\pm\infty}\frac{x^2 - 3x - 2}{x + 1} - x =$$

$$= \lim_{x\to\pm\infty}\frac{x^2 - 3x - 2 - x^2 - x}{x + 1} = \lim_{x\to\pm\infty}\frac{-4x - 2}{x + 1} = -4.$$

Thus, the graph of the function $f(x)$ has one non-vertical (more precisely, oblique) asymptote $y = x - 4$.

[12] This section is missing in video lectures.

3. Intersection points with coordinate axes. Since $f(0) = -2$, the graph intersects the OY axis at a single point $(0, -2)$. To find the intersection points with the OX axis, we solve the quadratic equation $x^2 - 3x - 2 = 0$:

$$x_{1,2} = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}.$$

Using the approximate value of 4.1 for $\sqrt{17}$, we get $x_1 \approx \frac{3-4.1}{2} = -0.55$, $x_2 \approx \frac{3+4.1}{2} = 3.55$. Thus, the graph intersects the OX axis at points whose approximate coordinates are $(-0.6, 0)$, $(3.6, 0)$.

4. Critical points. Find the first derivative of this function:

$$f'(x) = \Big(\frac{x^2 - 3x - 2}{x+1}\Big)' = \frac{(2x-3)(x+1) - (x^2 - 3x - 2)\cdot 1}{(x+1)^2} =$$

$$= \frac{x^2 + 2x - 1}{(x+1)^2}.$$

Now we can find the critical points of the function f, i. e., points at which the derivative f' vanishes. To do this, solve the quadratic equation $x^2 + 2x - 1 = 0$:

$$x^*_{1,2} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}.$$

Using the approximate value of 1.4 for $\sqrt{2}$, we get $x^*_1 \approx -1 - 1.4 = -2.4$, $x^*_2 \approx -1 + 1.4 = 0.4$.

5. Intervals of monotonicity and local extrema. Since in the formula for the derivative $f'(x)$ the denominator $(x+1)^2$ is non-negative, and the numerator $x^2 + 2x - 1$ takes positive values for $x \in (-\infty, x^*_1) \cup (x^*_2, +\infty)$ and negative values for $x \in (x^*_1, x^*_2)$, we obtain, by virtue of the first sufficient condition for the existence of a local extremum, that the point x^*_1 is a local maximum point (the sign of the derivative changes from "+" to "−" in a neighborhood of this point), and the point x^*_2 is a local minimum point (sign the derivative changes from "−" to "+" in a neighborhood of this point).

Let us also calculate the values of the function f at the points of local extrema:

$$f(x^*_1) = \frac{(-1-\sqrt{2})^2 - 3(-1-\sqrt{2}) - 2}{(-1-\sqrt{2}) + 1} =$$

$$= \frac{1 + 2\sqrt{2} + 2 + 3 + 3\sqrt{2} - 2}{-\sqrt{2}} = -\frac{4 + 5\sqrt{2}}{\sqrt{2}} =$$

$$= -2\sqrt{2} - 5 \approx -2 \cdot 1.4 - 5 = -7.8,$$

$$f(x_2^*) = \frac{(-1+\sqrt{2})^2 - 3(-1+\sqrt{2}) - 2}{(-1+\sqrt{2})+1} =$$
$$= \frac{1 - 2\sqrt{2} + 2 + 3 - 3\sqrt{2} - 2}{\sqrt{2}} = \frac{4 - 5\sqrt{2}}{\sqrt{2}} =$$
$$= 2\sqrt{2} - 5 \approx 2 \cdot 1.4 - 5 = -2.2.$$

So the coordinates of the local maximum and local minimum of the function f are approximately equal to $(-2.4, -7.8)$ and $(0.4, -2.2)$.

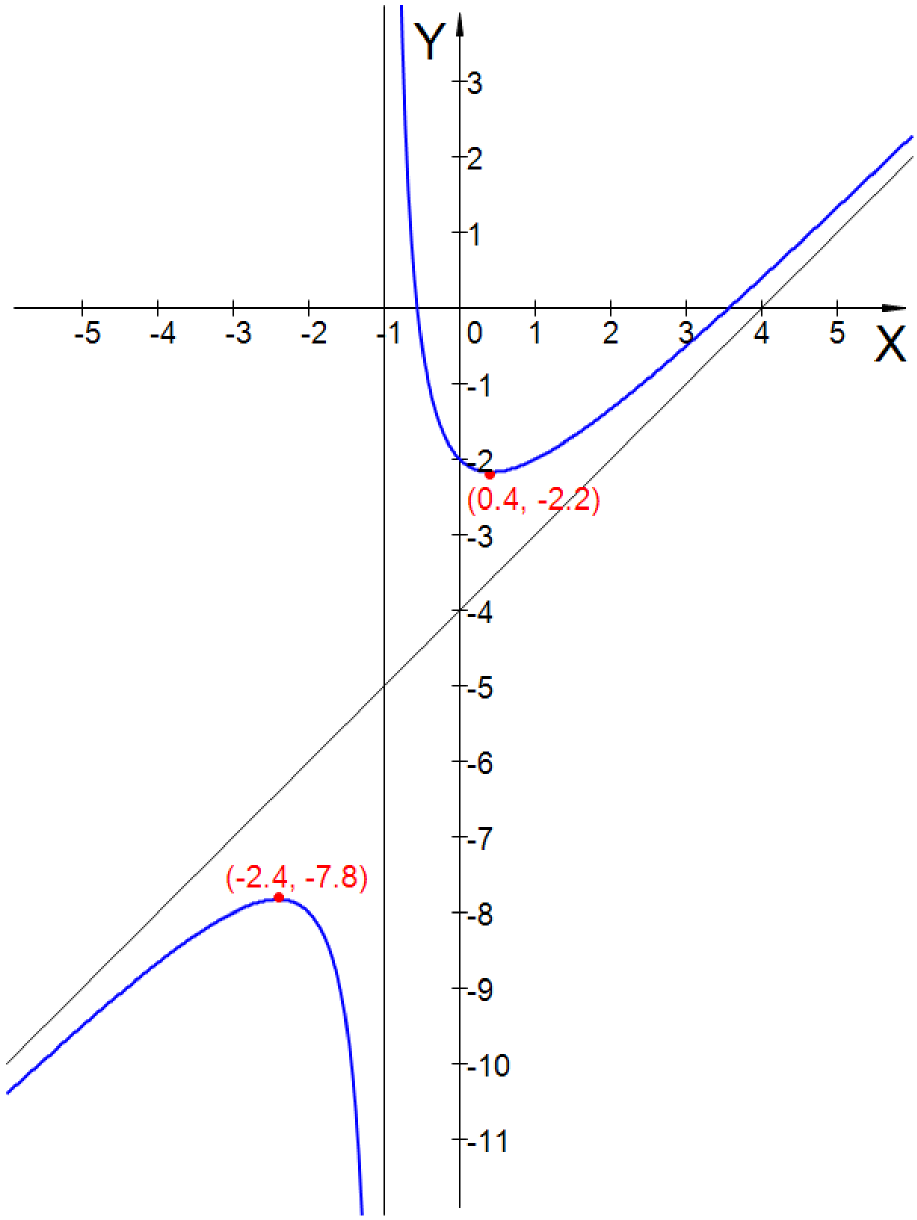

Fig. 16. Graph of the function $f(x) = \frac{x^2-3x-2}{x+1}$

6. Intervals of convexity and inflection points. To find the intervals of convexity and inflection points of the function f, we find its second derivative:

$$f''(x) = \big(f'(x)\big)' = \Big(\frac{x^2+2x-1}{(x+1)^2}\Big)' =$$
$$= \frac{(2x+2)(x+1)^2 - (x^2+2x-1)\cdot 2(x+1)}{(x+1)^4} =$$
$$= \frac{2(x+1)^2 - 2(x^2+2x-1)}{(x+1)^3} = \frac{4}{(x+1)^3}.$$

Thus, $f''(x) < 0$ for $x \in (-\infty, -1)$ and $f''(x) > 0$ for $x \in (-1, +\infty)$. By virtue of the sufficient condition for convexity, we obtain that the function f is convex upward on the interval $(-\infty, -1)$ and the function is convex downward on the interval $x \in (-1, +\infty)$. The function f does not have inflection points.

The graph of the function f is shown in Fig. 16. The figure also shows the asymptotes $x = -1$, $y = x-4$ and points of the local extremum $(-2.4, -7.8)$, $(0.4, -2.2)$.

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Index

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