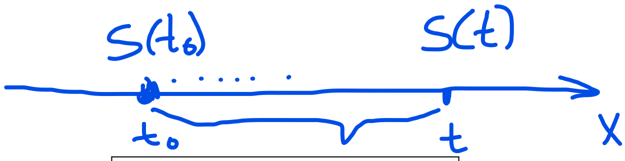
f(x) is diff. at x_0 $\quad f'(x_0)=\lim_{x\to x_0}\frac{\overbrace{f(x)-f(x_0)}^{\Delta f}}{\underbrace{x-x_0}_{\Delta x}}=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}$

19. Physical sense and geometric sense of the derivative

$S(t)=\underline{V}\cdot(t-t_0)+\underline{S(t_0)}$



Physical sense of the derivative

17B/16:45 (08:23)

For simplicity, we will consider *one-dimensional motion*, that is, displacement along the OX axis. Let the *law of motion* be defined by the function $S(t)$, where t is the time, and the value $S(t)$ determines the position of the point on the axis OX at the time t. Let us choose some initial moment of time t_0.

The simplest type of motion is *uniform motion*, that is, motion with a constant velocity V. In this case, the law of motion has the form of a linear function: $S(t) = V(t-t_0) + S(t_0)$ and therefore, to find the velocity V it's enough to divide the distance traveled over a period of time from t_0 to t by the value of this time period:

$$V = \frac{S(t)-S(t_0)}{t-t_0}. \quad = \cancel{const} \qquad \frac{m}{s} \quad \frac{km}{h} \ldots$$

The expression on the right-hand side is the ratio of the increment of the function to the increment of the argument, that is, it is an expression whose limit (if it exists) is the derivative of the function $S(t)$. However, in this simplest case, it is not necessary to pass to the limit, since the expression on the right-hand side, as well as on the left-hand side, is a constant V.

Now assume that the law of motion $S(t)$ is not linear. In this case, we cannot talk about a constant velocity of the motion, but we can find the *average velocity* $V_{avr}(t_0,t)$ over a period of time from t_0 to t:

$t_0 \leftarrow t$

(t_0,t)

$$V_{avr}(t_0,t) = \frac{S(t)-S(t_0)}{t-t_0}. \qquad \lim_{t\to t_0}$$

This formula shows the velocity of uniform motion that allows us to move from the point $S(t_0)$ to the point $S(t)$ over a period of time from t_0 to t.

If we move the value of t closer to t_0 then we will obtain the average velocity over a decreasing time interval (t_0,t), and if there exists a limit as $t\to t_0$, then it is natural to call such a limit $V(t_0)$ the *instantaneous velocity* at the time t_0:

$$V(t_0) = \lim_{t \to t_0} \frac{S(t) - S(t_0)}{t - t_0}. = S'(t_0)$$

The word "~~instantaneous~~" can be omitted and we can simply speak of the velocity $V(t_0)$ at the time t_0. Thus, we can determine the velocity at the time t_0 if the law of motion $S(t)$ is a differentiable function at the point t_0. So, the velocity at a given time is equal to the derivative of the function S:

$$\boxed{V(t_0) = S'(t_0).}$$

Therefore, the physical sense of the derivative is that *the derivative of a certain quantity determines the rate of change of this quantity.*

The rate of change of velocity is called the *acceleration*. To find the acceleration $a(t)$ for a given law of motion $S(t)$, we must differentiate the function $V(t) = S'(t)$, that is, find the *second* derivative of the function $S(t)$ at the given point (the higher-order derivatives will be considered in detail in the next chapter):

$$a(t_0) = V'(t_0) = (S'(t_0))'.$$

$a(t_0) = S''(t_0)$

$\boxed{F(t) = m.S''(t)}$

$S(t)$

Newton's second law allows us to relate the force $F(t)$ acting on the body at a given time t and the acceleration $a(t)$ with which the body moves under the action of this force: $F(t) = ma(t)$, where m is the mass of the body. Thus, if we know the force acting on the body, then we know its acceleration, and if we know the acceleration then, by performing the inverse operation for differentiation, we can find the velocity of motion $V(t)$. Then, applying the inverse operation for differentiation once again to the velocity $V(t)$, we can find the motion law $S(t)$ according to which the body moves, that is, we can completely determine how the body behaves under the action of a given force. Note that the inverse operation for differentiation is called *integration*. Integration, like differentiation, is the subject of study of calculus.

Thus, thanks to differential and integral calculus, it is possible to solve the problem of describing the motion of a body if the forces acting on it are known. For this reason, differential and integral calculus plays such an important role in various branches of physics.

Geometric sense of the derivative

17B/25:08 (08:08)

If the function $f(x)$ is differentiable at the point x_0 then the value of the derivative $f'(x_0)$ is an important characteristic of the graph of the function at this point.

$f(x_1) = y_1$

$f(x_0) = y_0$

$\alpha = \arctan k$

$\tan \alpha$

$\parallel$

$\boxed{y = k \cdot x + b}$

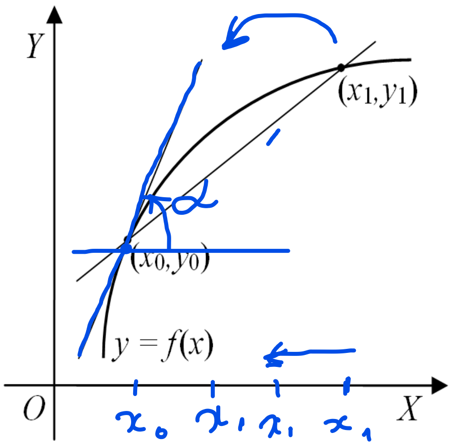


Fig. 9. Secant and tangent to the graph of a function

Let the function $f(x)$ be differentiable at the point x_0, let $y_0 = f(x_0)$. Let us choose some other point $x_1 \neq x_0$, denote $y_1 = f(x_1)$, and draw a *secant line*, i. e., a straight line passing through the points (x_0, y_0) and (x_1, y_1) (see Fig. 9).

The secant equation can be represented as follows:

$x_1 \to x_0$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).$$

$0 = 0 \qquad y_1 - y_0 = y_1 - y_0$

Indeed, this is a linear equation with respect to x, therefore it determines a straight line, and it turns into an identity when the points (x_0, y_0) and (x_1, y_1) are substituted into the equation.

If we will unlimitedly move the point x_1 to the point x_0 remaining all the time on the graph graph (i. e., assuming that $y_1 = f(x_1)$), then the secant will change its position. As a result, we get a line called the *tangent line* to the graph of the function $y = f(x)$ at the point x_0 (Fig. 9).

The tangent equation can be obtained by passing to the limit, as $x_1 \to x_0$, on the right-hand side of the secant equation:

$$y - y_0 = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).$$

This limit exists because, by condition, the function f is differentiable at the point x_0:

$$\lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0).$$

Thus, if the function f is differentiable at the point x_0, then its graph at the point (x_0, y_0), where $y_0 = f(x_0)$, has a tangent line whose equation is as follows:

$$y - y_0 = f'(x_0)(x - x_0).$$

$$y = \underbrace{\boxed{f'(x_0)}}_{\substack{\parallel \\ k \\ \text{slope}}} \cdot x + \underbrace{f(x_0) - f'(x_0) \cdot x_0}_{\substack{\parallel \\ b}}$$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$y = b$ $\quad$ $y = kx + b$ $\quad$ $k = \infty$

$k = 0$

$\tan\alpha = 0$ $\quad$ $\alpha = \frac{\pi}{2}$

$\alpha = 0$

The derivative $f'(x_0)$ is the *slope* of the equation of the tangent line to the graph of the function f at the point x_0. Since the slope is equal to the tangent of the angle of inclination of the straight line, we obtain that, knowing the derivative $f'(x_0)$, we can find the *angle of inclination of the tangent line* at x_0. To do this, we simply should find $\arctan f'(x_0)$. It is natural to call this value the *angle of inclination of the graph* of the function f at the point x_0.

$f'(x_0) = 0$

In particular, if $f'(x_0) = 0$, then this means that at this point the angle of inclination of the graph is also 0, that is, the tangent line to it is horizontal. If the derivative is equal to infinity then, given that $\lim_{y\to\pm\infty} \arctan y = \pm\frac{\pi}{2}$, we obtain that in this case the tangent line to the graph is vertical.

$\tan\alpha = \infty$

$\alpha = \frac{\pi}{2}$

$f'(x_0) \to \infty$

Thus, the geometric sense of the derivative is that *the derivative at a given point is equal to the tangent of the angle of inclination of the tangent line to the graph of the given function at this point.*

tan

Note that it is possible to talk about the tangent line to the graph of a function at a given point only if the function is differentiable at this point (that is, it has a finite derivative) or if its derivative has an infinite value at this point (in this case, the tangent line is vertical). If there is no finite or infinite derivative, then the graph has no tangent line. For example, there is no tangent line to the graph of the function $\operatorname{sign} x$ at the point $x = 0$.

$f(x) = \sin x$

$f'(x) = \cos x$

$f'(0) = \cos 0 = 1$

$\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4} = 45^\circ$