M. E. Abramyan

LECTURES ON INTEGRAL CALCULUS OF FUNCTIONS OF ONE VARIABLE AND SERIES THEORY

ЮЖНЫЙ ФЕДЕРАЛЬНЫЙ УНИВЕРСИТЕТ
ЮФУ
РОСТОВ-НА-ДОНУ·ТАГАНРОГ

MINISTRY OF SCIENCE AND HIGHER EDUCATION
OF THE RUSSIAN FEDERATION
SOUTHERN FEDERAL UNIVERSITY

Mikhail E. Abramyan

LECTURES ON INTEGRAL CALCULUS OF FUNCTIONS OF ONE VARIABLE AND SERIES THEORY

For students of science and engineering

Rostov-on-Don – Taganrog
Southern Federal University Press
2021

UDC 517.4(075.8)
BBC 22.162я73
A164

Published by decision of the Educational-Methodical Commission of the I. I. Vorovich Institute of Mathematics, Mechanics, and Computer Science of the Southern Federal University (minutes No. 5 dated April 12, 2021)

Reviewers:

doctor of Physical and Mathematical Sciences, Professor of the Department of Applied Mathematics of the South Russian State Polytechnic University, Honorary official of higher professional education of the Russian Federation, Professor *A. E. Pasenchuk*;
candidate of Physical and Mathematical Sciences, Associate Professor of the Department of Algebra and Discrete Mathematics of the I. I. Vorovich Institute of Mathematics, Mechanics, and Computer Science of the Southern Federal University, Docent *A. V. Kozak*

Abramyan, M. E.
A164 Lectures on integral calculus of functions of one variable and series theory / M. E. Abramyan ; Southern Federal University. – Rostov-on-Don ; Taganrog : Southern Federal University Press, 2021. – 252 p.
ISBN 978-5-9275-3829-4

The textbook contains lecture material for the second part of the course on math-ematical analysis and includes the following topics: indefinite integral, definite inte-gral and its geometric applications, improper integral, numerical series, functional sequences and series, power series, Fourier series. A useful feature of the book is the possibility of studying the course material at the same time as viewing video lectures recorded by the author and available on youtube.com. Sections and subsections of the textbook are provided with information about the lecture number, the start time of the corresponding fragment and the duration of this fragment. In the electronic version of the textbook, this information is presented as hyperlinks, allowing reader to immediately view the required fragment of the lecture.

The textbook is intended for students specializing in science and engineering.

UDC 517.4(075.8)
ISBN 978-5-9275-3829-4 BBC 22.162я73

Contents

In memory of Professor
Vladimir Stavrovich Pilidi
(1946–2021)

Preface

The book is a continuation of the textbook [1] and contains lecture material of the second part of the course on mathematical analysis, which was read by the author for several years at the I. I. Vorovich Institute of Mathematics, Mechanics, and Computer Science of the Southern Federal University (specialization 01.03.02 – "Applied Mathematics and Computer Science"). The book includes the following topics: indefinite integral, definite integral and its geometric applications, improper integral, numerical series, functional sequences and series, power series, Fourier series.

Beyond the scope of the course material presented in [1] and this book, there are topics related to the differential and integral calculus of functions of many variables.

This book, like the book [1], can be attributed to the category of "short textbooks", covering only the material that can usually be given in lectures. In this respect, it is similar to books [10, 16] and differs from the "detailed textbooks" that cover the subject with much greater completeness. In particular, topics related to the integral calculus of functions of one variable are described in detail in textbooks [4, 6, 8, 11, 14, 18, 19], and topics related to series theory are included in textbooks [4, 6, 7, 9, 12, 14, 18–20]; moreover, the theory of Fourier series is often presented separately (see [5, 13, 15]).

Most of the statements in the book are provided with detailed proof; for a few auxiliary facts taken without proof, references are given to textbooks in which these facts are proved (the textbook [18] was chosen as the main source for such references).

Like the book [1], the proposed book has two main features: relationship with the set of video lectures and the presence of two versions: in Russian and English (the Russian version of the book [1] is [2]). The noted features and the additional advantages for the reader resulting from them are described in detail in the preface to [1]. Books [3, 10, 17] can be mentioned as additional sources in English that are closest to Russian textbooks.

The index to the book is composed on the same principles as the index to [1]: it contains all definitions and theorems; all references to theorems include their detailed descriptions grouped in the section “Theorem”. In addition, all theorems and other concepts containing surnames in their titles are given in the positions corresponding to these surnames. In the electronic version of the book, page numbers in the index, as well as in the table of contents, are hyperlinks allowing to go directly to this page.

The initial “Video Lectures” section provides complete information about the set of video lectures related to the book, including their Internet links. This information allows the reader to quickly access the required lecture even in the absence of an electronic version of the book.

Video lectures

If the framed text follows the title of the section or subsection, this means that a fragment of the video lecture is associated with this section or subsection. The framed text consists of three parts: the number of the video lecture, the time from which this fragment begins, and the duration of this fragment.

For example, the following text `2.1A/00:00 (16:47)` is located after the title of the first section of Chapter 1 (the section is devoted to the definition of the antiderivative and indefinite integral). It means that this topic is discussed at the very beginning of lecture 2.1A, and the corresponding fragment of the lecture lasts 16 minutes 47 seconds. The last section of Chapter 20 is the section devoted to the decreasing rate of Fourier coefficients for differentiable functions. The correspondent text is `3.19B/33:49 (06:32)`, which means that this topic is discussed in the lecture 3.19B, starting at 33:49, and the discussion lasts 6 minutes 32 seconds.

The double numbering of video lectures is due to the fact that they are taken from two sets with numbers 2 and 3 corresponding to lectures of the second and third semester; the lectures in each set are numbered starting from 1.

In the electronic version of the book, all framed texts are hyperlinks. Clicking on such text allows you to immediately play the corresponding lecture, starting from the specified time.

When using the paper version of the book, hyperlinks, of course, are not available, therefore, an additional information is provided here, which will allow you to quickly start playing the required video lecture.

All video lectures are available on `youtube.com`. The first 10 video lectures belong to set 2 and are the initial lectures of this set (with numbers from 1 to 10); the final 11 video lectures belong to the middle part of set 3 and have numbers from 9 to 19 in this set. In addition, there is a link to video lecture 2.11A, in which the topic "Curves" ends, and a link to video lecture 3.8B, in which the topic "Improper integrals" begins. All other video lectures consist of two parts: A and B. The following list of lectures contains their titles and short links to each part.

2.1. Indefinite integral
- 2.1A: https://youtu.be/66lAeLxskVA
- 2.1B: https://youtu.be/xzIopk1WCDM

2.2. Integration of rational functions
- 2.2A: https://youtu.be/aLuD104G8PI
- 2.2B: https://youtu.be/pPDP0Lv23fk

2.3. Integration of trigonometric and irrational functions
- 2.3A: https://youtu.be/_5Maq2J0eHg
- 2.3B: https://youtu.be/aSDoNpfUbAs

2.4. Definite integral. Darboux sums
- 2.4A: https://youtu.be/TRBKy1OknMM
- 2.4B: https://youtu.be/a4gf4Temgug

2.5. Classes of integrable functions
- 2.5A: https://youtu.be/oLRSzkV4FLo
- 2.5B: https://youtu.be/0XUliFTV26s

2.6. Properties of a definite integral
- 2.6A: https://youtu.be/VkS-AcA9njQ
- 2.6B: https://youtu.be/tygGvPGHTps

2.7. Newton–Leibniz formula
- 2.7A: https://youtu.be/h77yheGoE1I
- 2.7B: https://youtu.be/FPhuV0ZFZZ8

2.8. Calculation of areas
- 2.8A: https://youtu.be/Yg2rrKjorF8
- 2.8B: https://youtu.be/sX5r7CP2oR0

2.9. Calculation of volumes
- 2.9A: https://youtu.be/3Vpk5JvFLaM
- 2.9B: https://youtu.be/6VT320AFKbw

2.10. Vector functions. Calculation of a curve length
- 2.10A: https://youtu.be/Q6sxEiXVzhc
- 2.10B: https://youtu.be/xb8oN2tz4Lw

2.11. Metric spaces
- 2.11A: https://youtu.be/J29z4Sog7WE

3.8. Definition and properties of an improper integral
- 3.8B: https://youtu.be/3r3u9nmPvQI

3.9. Absolute and conditional convergence of improper integrals
- 3.9A: https://youtu.be/at_eysCbc_M

3.9B: https://youtu.be/dVh4k6yr8O8

3.10. Definition and properties of a numerical series, convergence tests
3.10A: https://youtu.be/RuNzgI_hUCk
3.10B: https://youtu.be/PcIYNHo15_Y

3.11. Convergence tests (continuation), alternating series
3.11A: https://youtu.be/ielvgfjqFjM
3.11B: https://youtu.be/l1j-OAwBM5w

3.12. Functional sequences and series, uniform convergence
3.12A: https://youtu.be/vlcY9UpBHGg
3.12B: https://youtu.be/PRXEFme2sV0

3.13. Properties of functional sequences and series
3.13A: https://youtu.be/pJywld91FOs
3.13B: https://youtu.be/cyHCvVqlDGw

3.14. Power series
3.14A: https://youtu.be/lkbV5-3O7Ps
3.14B: https://youtu.be/uOH9-hFgtbM

3.15. Properties of power series
3.15A: https://youtu.be/gvKZiJVjOhE
3.15B: https://youtu.be/JzgBm_z7OqI

3.16. Taylor series
3.16A: https://youtu.be/jM7_Gc7vThE
3.16B: https://youtu.be/8Js_Dl29pX0

3.17. Fourier series in Euclidean space
3.17A: https://youtu.be/yT2KwZh8XVQ
3.17B: https://youtu.be/vnHwF6qCLRU

3.18. Fourier series in the space of integrable functions
3.18A: https://youtu.be/2_hb1tefg7U
3.18B: https://youtu.be/yJqsGKaYgmw

3.19. Properties of Fourier series for various classes of functions
3.19A: https://youtu.be/Y6ftBOrijqk
3.19B: https://youtu.be/5UJfMwBOpx4

You can create a link that immediately plays the video lecture, starting from the specified time. Let us describe this additional feature using the previously mentioned fragment `3.19B/33:49 (06:32)` as an example. This is a fragment of part B of video lecture 3.19B, its short link has the form `5UJfMwBOpx4`. We need to play a lecture starting at 33:49.

To do this, use the Internet link `https://www.youtube.com/watch?` specifying two options after it: a short link to the video lecture (option `v=`) and the start time of playback (option `t=`). The options themselves must be separated by the `&` character.

In our case, the full text of the Internet link will be as follows:

`https://www.youtube.com/watch?v=5UJfMwBOpx4&t=33m49s`

Pay attention to the time format: after the number of minutes, the letter `m` is indicated; after the number of seconds, the letter `s` is indicated. If the number of seconds is 0, then only the number of minutes can be specified.

A set of hyperlinks to video lectures, which also contains the names of the corresponding chapters, sections, and subsections of this book, is presented on the website `mmcs.sfedu.ru` of the Institute of Mathematics, Mechanics, and Computer Science of the Southern Federal University (Moodle environment, link `http://edu.mmcs.sfedu.ru/course/view.php?id=271` for the lecture set 2 and link `http://edu.mmcs.sfedu.ru/course/view.php?id=379` for the lecture set 3). At the top of each specified page, a set of hyperlinks is displayed with titles in Russian and then in English.

1. Antiderivative and indefinite integral

Definition of an antiderivative and indefinite integral

2.1A/00:00 (16:47)

Definition.

Let the function f be defined on the interval (a,b), where a and b are finite points or points at infinity. Let the function F be a differentiable function on this interval, with $F'(x) = f(x)$ for $x \in (a,b)$. Then the function F is called the *antiderivative* (or *primitive function*) of the function f on a given interval.

The process of finding an antiderivative is called *indefinite integration* (or *antidifferentiation*). If a function has an antiderivative on (a,b), then it is called *integrable* on (a,b).

Hereinafter we, as a rule, will not specify interval on which the function is integrable.

The question arises: how many different antiderivatives exist? Let F_1 be the antiderivative of the function f, that is, $F_1'(x) = f(x)$. Let $F_2(x) = F_1(x) + C$, where C is a constant. Then the function F_2 is also the antiderivative of the function f, since

$$F_2'(x) = \big(F_1(x) + C\big)' = F_1'(x) = f(x).$$

Therefore, if we add a constant to some antiderivative, then we will also get a primitive function. So, there exists an infinite number of antiderivatives, differing from each other by a constant term.

There are no other antiderivatives: *all possible antiderivatives can be obtained by adding a constant to some selected antiderivative.* Let us formalize this fact as a theorem.

Theorem (on antiderivatives of a given function).

Let F_1 and F_2 be antiderivatives of f on (a,b). Then there exists a constant $C \in \mathbb{R}$ such that $F_2(x) = F_1(x) + C$.

Proof.

We introduce the auxiliary function $h(x) = F_2(x) - F_1(x)$. The function $h(x)$ is differentiable on (a,b) as the difference of differentiable functions. Let us find its derivative:

$$h'(x) = \big(F_2(x) - F_1(x)\big)' = F_2'(x) - F_1'(x) = f(x) - f(x) = 0.$$

Thus, $h'(x)$ is equal to 0 at any point $x \in (a, b)$. Then, by corollary 1 of Lagrange's theorem [1, Ch. 21], the function $h(x)$ is a constant on the interval (a, b):

$$h(x) = C, \quad x \in (a, b).$$

Therefore, $F_2(x) - F_1(x) = C$, $F_2(x) = F_1(x) + C$. □

So, knowing one antiderivative, we can obtain all the other antiderivatives, since they all differ from the chosen antiderivative by a constant term.

Definition.

The *indefinite integral* $\int f(x)\,dx$ of the function f is the set of all its antiderivatives: if F_1 is some antiderivative of the function f (that is, $F_1'(x) = f(x)$), then

$$\int f(x)\,dx \stackrel{\text{def}}{=} \{F_1(x) + C, \ \ C \in \mathbb{R}\}.$$

The symbol $\int$ is called the *integral sign*, the function $f(x)$ is called the *integrand*, and the expression $f(x)\,dx$ under the integral sign is called the *element of integration*.

As a rule, curly braces are not used and, moreover, it is not indicated that C is an arbitrary real constant:

$$\int f(x)\,dx = F_1(x) + C.$$

Table of indefinite integrals

2.1A/16:47 (12:44)

$$\int 0\,dx = C.$$

$$\int A\,dx = Ax + C, \quad A \in \mathbb{R}.$$

$$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad x > 0, \quad \alpha \in \mathbb{R} \setminus \{-1\}.$$

$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad x \neq 0.$$

To prove the last formula, it suffices to differentiate the superposition $\ln|x| = \ln y \circ |x|$ for $x \neq 0$:

$$(\ln|x|)' = (\ln y)'\big|_{y=|x|} \cdot (|x|)' = \frac{1}{y}\bigg|_{y=|x|} \cdot \operatorname{sign} x = \frac{\operatorname{sign} x}{|x|} = \frac{1}{x}.$$

$$\int e^x\,dx = e^x + C.$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C, \quad a > 0, \quad a \neq 1.$$

$$\int \sin x\,dx = -\cos x + C.$$

$$\int \cos x\,dx = \sin x + C.$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + C.$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

$$\int \operatorname{sh} x\,dx = \operatorname{ch} x + C.$$

$$\int \operatorname{ch} x\,dx = \operatorname{sh} x + C.$$

The simplest properties of an indefinite integral

2.1A/29:31 (09:54), 2.1B/00:00 (02:28)

1. If the function f is integrable, then

$$\Big(\int f(x)\,dx\Big)' = f(x).$$

Proof.

Let $F(x)$ be the antiderivative of the function $f(x)$, then

$$\Big(\int f(x)\,dx\Big)' = (F(x) + C)' = F'(x) = f(x). \ \square$$

2. If the function f is differentiable, then

$$\int f'(x)\,dx = f(x) + C.$$

Proof.

In this case, $f(x)$ is one of the antiderivatives of the function $f'(x)$, whence the formula to be proved follows. $\square$

3. Additivity of the indefinite integral.

Let f and g be integrable, then the function $f+g$ is also integrable and the formula holds:

$$\int (f+g)\,dx = \int f\,dx + \int g\,dx. \tag{1}$$

Proof.

Equality (1) must be interpreted as the equality of two sets. Therefore, we should prove that the set from the left-hand side of equality (1) is equal to the set from the right-hand side of (1). Let F be some antiderivative of the function f, G be some antiderivative of the function g. Then $F+G$ is the antiderivative of the function $f+g$, since $(F+G)' = F' + G' = f + g$. Therefore, equality (1) can be rewritten in the form:

$$F + G + C = (F + C_1) + (G + C_2), \quad C, C_1, C_2 \in \mathbb{R}.$$

Obviously, if we choose the constants C_1 and C_2, that is, if we select some element of the right-hand set, then this element will also belong to the left-hand set (we can just put $C = C_1 + C_2$).

If we select some element $F + G + C$ of the left-hand set, then, by representing the constant C as the sum of two constants C_1 and C_2, we obtain that this element also belongs to the right-hand set.

Thus, we have proved the equality of these sets. □

4. Homogeneity of the indefinite integral.

Let f be integrable, $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then the function αf is integrable and the formula holds:

$$\int \alpha f\,dx = \alpha \int f\,dx. \tag{2}$$

Formula (2) means that the constant factor can be taken out of the integral sign.

The proof is similar to the proof of property 3. □

Remark.

In the case of $\alpha = 0$, formula (2) turns out to be incorrect, as we noted earlier that $\int 0\,dx = C$.

If we combine the properties of additivity and homogeneity, then we get the property of linearity.

5. Linearity of the indefinite integral.

Let f and g be integrable, $\alpha, \beta \in \mathbb{R}$, with α and β not turning into 0 at the same time: $|\alpha| + |\beta| \neq 0$. Then the function $\alpha f + \beta g$ is also integrable and the formula holds:

$$\int (\alpha f + \beta g)\,dx = \alpha \int f\,dx + \beta \int g\,dx.$$

Example.

Using the simplest properties of the indefinite integral and the table of indefinite integrals, one can find the integrals of linear combinations of functions, for example:

$$\int (5e^x + 6\cos x)\,dx = 5\int e^x\,dx + 6\int \cos x\,dx = 5e^x + 6\sin x + C.$$

To verify the resulting relation, it suffices to differentiate the expression on the right-hand side.

Change of variables in an indefinite integral

2.1B/02:28 (13:37)

THEOREM (ON THE CHANGE OF VARIABLES).

Let $f(x)$ be an integrable function on (a, b) and one of its antiderivatives is the function $F(x)$. Let $\varphi(t)$ be a differentiable function on the interval (α, β) and $\varphi(t) \in (a, b)$ as $t \in (\alpha, \beta)$. Then

$$\int f\big(\varphi(t)\big)\varphi'(t)\,dt = F\big(\varphi(t)\big) + C. \tag{3}$$

PROOF.

It is enough for us to verify that the right-hand side of equality (3) is the antiderivative of the integrand of the left-hand side of (3). We use the superposition differentiation theorem and the condition that $F'(x) = f(x)$:

$$\big(F(\varphi(t))\big)' = F'(x)\big|_{x=\varphi(t)}\,\varphi'(t) = f(x)\big|_{x=\varphi(t)}\,\varphi'(t) = f\big(\varphi(t)\big)\varphi'(t).\ \square$$

REMARK.

Considering that the expression $\varphi'(t)\,dt$ is the differential of the function φ, the left-hand side of equality (3) can be written as $\int f(\varphi)d\varphi$.

If we assume that φ is an independent variable, then equality (3) turns into the definition of an indefinite integral:

$$\int f(\varphi)d\varphi = F(\varphi) + C. \tag{4}$$

However, the proved theorem means that equality (4) also holds for the case when φ is a *dependent variable*, that is, it represents a differentiable function of some independent variable (for example, t). In this case, the expression $d\varphi$ must be understood as the *differential* of the function.

The noted circumstance is an additional justification for including the expression dx in the notation of the indefinite integral. It should be noted that this notation is also convenient for calculating integrals by changing variables.

AN EXAMPLE OF APPLYING THE VARIABLE CHANGING THEOREM.

Find the integral $\int \tan x\,dx$:

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x}.$$

We introduce a new variable: $y = \cos x$. The variable y is the function φ from the variable changing theorem, that is, we can assume that y depends on x. Then dy is the differential of the function $\cos x$, therefore $dy = -\sin x\,dx$. Thus, by virtue of the remark on the variable changing theorem, the expression in the numerator of the initial integral can be replaced with $-dy$, and the expression in the denominator can be replaced with y. As a result of changing the variable $y = \cos x$, the initial integral is significantly simplified and can now be found using the table of indefinite integrals:

$$\int \frac{\sin x\,dx}{\cos x} = \int \frac{-dy}{y} = -\int \frac{dy}{y} = -\ln|y| + C.$$

It remains for us to return to the initial variable x. Finally we obtain

$$\int \tan x\,dx = -\ln|\cos x| + C.$$

Remarks.

1. When finding the last integral, we actually applied the formula (3), representing the initial integral as follows:

$$\int \tan x\,dx = \int f(\cos x)\cdot(\cos x)'\,dx, \quad f(y) = -\frac{1}{y}.$$

However, when performing a variable change in an indefinite integral, formula (3) is not used. Instead, in the integral, both the initial variable x and its differential dx are replaced, as was done in the above example.

2. Using a similar change of variable, one can also find the integral $\int \frac{dx}{\sin x}$. To do this, take into account that $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and additionally transform the resulting expression to obtain the result of differentiation of the function $\tan\frac{x}{2}$ in it.

3. The resulting formula for the integral $\int \tan x\,dx$ makes sense on any interval that does not contain points $\frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$, that is, points at which the tangent function is not defined.

Formula of integration by parts

Derivation of the formula of integration by parts

2.1B/16:05 (07:28)

The integral of the product of functions is not equal to the product of the integrals. This is due to the more complicated form of the formula for differentiating the product, compared with the formula for differentiating the sum:

$$\big(u(x)v(x)\big)' = u'(x)v(x) + u(x)v'(x). \tag{5}$$

Nevertheless, using formula (5) for differentiating the product, we can obtain the *formula of integration by parts*, which in some cases allows us to simplify the calculation of the integral of the product.

Let us express the product $u(x)v'(x)$ from equality (5):

$$u(x)v'(x) = \big(u(x)v(x)\big)' - u'(x)v(x).$$

Integrating the last equality and using the linearity of the indefinite integral (the simplest property 5), we obtain

$$\int u(x)v'(x)\,dx = \int \Big(\big(u(x)v(x)\big)' - u'(x)v(x)\Big)\,dx =$$
$$= \int \big(u(x)v(x)\big)'\,dx - \int u'(x)v(x)\,dx. \tag{6}$$

Given the simplest property 2 of the indefinite integral, we have

$$\int \big(u(x)v(x)\big)'\,dx = u(x)v(x) + C.$$

Since the remaining term $\int u'(x)v(x)\,dx$ on the right-hand side of equality (6) also contains an arbitrary constant, we can add the constant C to this arbitrary constant and not specify it explicitly. Finally, we obtain the following formula:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

This formula is called the *formula of integration by parts*. It holds if the functions u and v are differentiable and there exists at least one of the integrals included in it (in this case, there necessarily exists another integral).

So, the formula of integration by parts allows us to express the integral of the product of the functions u and v' in terms of the integral of the product of u' and v. It is used in situations where the integral on its right-hand side is easier to find than the integral on the left-hand side.

The formula of integration by parts can also be written in the following short form:

$$\int u\,dv = uv - \int v\,du.$$

Examples of applying the formula of integration by parts

`2.1B/23:33 (12:42)`

1. Let us find the integral $\int \ln x\,dx$. We put $u(x) = \ln x$, $dv = dx$, whence $v(x) = x$. Then

$$\int \ln x\,dx = x\ln x - \int x(\ln x)'\,dx =$$
$$= x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$$

2. Let us find the integral $\int e^x \sin x\,dx$.

We put $u(x) = \sin x$, $dv = e^x\,dx$, whence $v(x) = e^x$. Then

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx. \tag{7}$$

Transform the integral on the right-hand side of (7) by the formula of integration by parts with $u(x) = \cos x$, $v(x) = e^x$:

$$\int e^x \cos x\,dx = e^x \cos x - \int e^x(-\sin x)\,dx = e^x \cos x + \int e^x \sin x\,dx.$$

Substituting the found integral in (7), we obtain

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx.$$

So, after completing two integrations by parts, we get the initial integral $\int e^x \sin x\,dx$. If we denote one of the antiderivatives of the initial integrand by $I(x)$ and transfer arbitrary constants to the right-hand side of the last equality, then this equality can be written as follows:

$$I(x) = e^x(\sin x - \cos x) - I(x) + C. \tag{8}$$

Solving equation (8) with respect to $I(x)$, we obtain

$$I(x) = \frac{e^x}{2}(\sin x - \cos x) + \frac{C}{2}.$$

The final formula takes the form

$$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

A similar technique can be applied when integrating functions of a more general form $e^{bx}\sin ax$ and $e^{bx}\cos ax$.

Remark.

It is also advisable to apply the formula of integration by parts in the case of an integrand of the form $P(x)f(x)$, where $P(x)$ is a polynomial and $f(x)$ is a function for which there exists a simple antiderivative (such as $\sin x$, $\cos x$, a^x). In this case, we put $u(x) = P(x)$; as a result of differentiation of the function $u(x)$, a polynomial of a lesser degree will be obtained. The integration by parts is again applied to the obtained integral, and the process is repeated until the polynomial at the next differentiation turns into a constant.

If there is a function $\ln x$ in the integrand, we can put $u(x) = \ln x$, since we obtain a simpler function $\frac{1}{x}$ after differentiation.

Sometimes it is convenient to apply the formula of integration by parts, not dividing the integrand into two factors, but setting $dv = dx$, $v(x) = x$ (as in example 1).

2. Integration of rational functions

Partial fraction decomposition of a rational function

2.1B/36:15 (11:49)

Definition.

The *rational function* $R(x)$ is the ratio of two polynomials:

$$R(x) = \frac{P_m(x)}{Q_n(x)}.$$

In studying the question of integrating rational functions, the following facts from the course of algebra are used.

Theorem 1 (on the factorization of a real polynomial).

A polynomial $Q_n(x)$ of degree n with real coefficients can be decomposed into the following irreducible factors:

$$Q_n(x) = a_0(x-c_1)^{\alpha_1}\dots(x-c_k)^{\alpha_k}(x^2+p_1x+q_1)^{\beta_1}\dots(x^2+p_lx+q_l)^{\beta_l}. \tag{1}$$

Here a_0 is the coefficient of the highest degree of the polynomial $Q_n(x)$, $c_1,\dots,c_k$ are the real roots of the polynomial $Q_n(x)$ of multiplicity $\alpha_1,\dots,\alpha_k$, quadratic factors of the form $x^2+p_ix+q_i$ with real coefficients p_i, q_i have a negative discriminant: $p_i^2-4q_i<0$; each factor $(x^2+p_ix+q_i)^{\beta_i}$ corresponds to a pair of complex conjugate roots of the polynomial $Q_n(x)$ of multiplicity β_i, $i=1,\dots,l$. In addition, the following relation holds:

$$\alpha_1+\dots+\alpha_k+2(\beta_1+\dots+\beta_l)=n.$$

Theorem 2 (on the partial fraction decomposition of a real rational function).

Let $R(x)$ be a rational function of the form $\frac{P_m(x)}{Q_n(x)}$ and decomposition (1) takes place for the polynomial $Q_n(x)$. Then $R(x)$ can be represented as follows:

$$R(x) = \tilde{P}(x) + \sum_{i=1}^{k}\sum_{j=1}^{\alpha_i}\frac{A_{ij}}{(x-c_i)^j} + \sum_{i=1}^{l}\sum_{j=1}^{\beta_i}\frac{B_{ij}x+D_{ij}}{(x^2+p_ix+q_i)^j}. \tag{2}$$

The term $\tilde{P}(x)$ appears if the degree m of the polynomial $P_m(x)$ is greater than or equal to the degree n of the polynomial $Q_n(x)$. This term $\tilde{P}(x)$ is a polynomial of degree $m-n$ obtained by dividing the polynomial $P_m(x)$ by the polynomial $Q_n(x)$.

For all remaining terms in formula (2) (called *partial fractions*), the degree of the numerator is less than the degree of the denominator.

Methods for finding the decomposition of a rational function

2.2A/00:00 (09:34)

Consider the following rational function as an example:

$$R(x) = \frac{5x^3+3x+2}{(x-1)^2(x^2+2x+2)}.$$

The degree of its numerator is less than the degree of the denominator, therefore, the term $\tilde{P}(x)$ will not be present in the decomposition. The decomposition will consist of three partial fractions:

$$R(x) = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+2}.$$

It remains for us to find the coefficients of these fractions. To do this, we can use the so-called *method of equating coefficients*. Let us reduce the expression on the right-hand side to a common denominator and equate the resulting numerators:

$$5x^3+3x+2 = A(x-1)(x^2+2x+2)+B(x^2+2x+2)+(Cx+D)(x-1)^2. \tag{3}$$

Now we remove parentheses on the right-hand side and group the terms with the same powers of x:

$$5x^3+3x+2 = (A+C)x^3 + (A+B-2C+D)x^2 + \\ + (2B+C-2D)x + (-2A+2B+D).$$

Let us equate the coefficients at the same powers of x:

$$5 = A+C,$$
$$0 = A+B-2C+D,$$
$$3 = 2B+C-2D,$$
$$2 = -2A+2B+D.$$

As a result, we obtained a system of four linear equations in four unknowns. According to the theorem on the partial fraction decomposition of a rational function, this system has a solution. Having solved this system, we will find the required coefficients: $A=2$, $B=2$, $C=3$, $D=2$.

There is another way: we can consider the specific values of x. If we put $x = 1$ in equality (3), then two terms will disappear in its right-hand side and the only term will remain. Using this term, we can immediately find the coefficient B:

$$5 \cdot 1^3 + 3 + 2 = A(1-1)(1^2+2+2) + B(1^2+2+2) + (C+D)(1-1)^2,$$

$$10 = 5B,$$

$$B = 2.$$

This method is convenient if there are no quadratic factors in the factorization of a denominator. However, even in our case, this method allows us to simplify the resulting system by reducing the number of unknowns:

$$5 = A + C,$$

$$-2 = A - 2C + D,$$

$$-1 = C - 2D,$$

$$-2 = -2A + D.$$

Integration of terms in the partial fraction decomposition of a rational function

Simple cases based on the direct use of the table of integrals

2.2A/09:34 (06:46)

After finding the partial fraction decomposition of the rational function, it remains to integrate separately all the obtained terms.

1. The integral of the polynomial $\tilde{P}(x)$.

This integral is a polynomial whose free term is an arbitrary constant C.

2. The integral of a partial fraction of the form $\frac{A}{(x-c)^k}$ corresponding to the real root c of multiplicity k.

For $k \neq 1$, we have

$$\int \frac{A}{(x-c)^k}\,dx = \frac{A}{(1-k)(x-c)^{k-1}} + C.$$

For $k = 1$, we have

$$\int \frac{A}{x-c}\,dx = A\ln|x-c| + C.$$

Using change of variable

2.2A/16:20 (09:56)

3. The integral of a partial fraction of the form $\frac{Bx+D}{(x^2+px+q)^k}$, provided that the discriminant of the quadratic trinomial is less than zero: $p^2-4q<0$ (this fraction corresponds to complex conjugate roots of multiplicity k).

Let us transform the polynomial in the denominator by complete the square:

$$x^2+px+q=x^2+2x\cdot\frac{p}{2}+\left(\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2+q=\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}.$$

Note that the expression $q-\frac{p^2}{4}$ is greater than zero, since, by assumption, the discriminant p^2-4q is less than zero. Denote $q-\frac{p^2}{4}=\Delta^2$.

As a result of the transformation, we decrease the number of variables x in the integral:

$$\int\frac{(Bx+D)\,dx}{(x^2+px+q)^k}=\int\frac{(Bx+D)\,dx}{\left(\left(x+\frac{p}{2}\right)^2+\Delta^2\right)^k}.$$

Let us change of variable: $t=x+\frac{p}{2}$. Differentials will not change: $dt=dx$. This variable changing will further simplify the denominator:

$$\int\frac{(Bx+D)\,dx}{\left(\left(x+\frac{p}{2}\right)^2+\Delta^2\right)^k}=\int\frac{\left(B\left(t-\frac{p}{2}\right)+D\right)dt}{(t^2+\Delta^2)^k}.$$

Now transform the numerator by grouping the free terms and denoting the difference $D-\frac{Bp}{2}$ by D':

$$\int\frac{\left(B\left(t-\frac{p}{2}\right)+D\right)dt}{(t^2+\Delta^2)^k}=\int\frac{(Bt+D')\,dt}{(t^2+\Delta^2)^k}.$$

Let us split the resulting integral into two:

$$\int\frac{(Bt+D')\,dt}{(t^2+\Delta^2)^k}=\int\frac{Bt\,dt}{(t^2+\Delta^2)^k}+\int\frac{D'\,dt}{(t^2+\Delta^2)^k}.$$

Thus, it remains for us to analyze the integrals of two types: $\int\frac{t\,dt}{(t^2+\Delta^2)^k}$ and $\int\frac{dt}{(t^2+\Delta^2)^k}$.

3a. Find the integral $\int\frac{t\,dt}{(t^2+\Delta^2)^k}$. We make the following variable change in it: $y=t^2+\Delta^2$. Then $dy=2t\,dt$ and as a result we get

$$\int\frac{t\,dt}{(t^2+\Delta^2)^k}=\frac{1}{2}\int\frac{dy}{y^k}.$$

The integral on the right-hand side can be found using the same formulas as the integrals considered in subsection 2.

Using recurrence relation

2.2A/26:16 (12:00)

3b. Now let us turn to the last integral: $\int \frac{dt}{(t^2+\Delta^2)^k}$.

In this case, we perform integration by parts, setting $u = \frac{1}{(t^2+\Delta^2)^k}$, $dv = dt$, $v = t$:

$$\int \frac{dt}{(t^2+\Delta^2)^k} = \frac{t}{(t^2+\Delta^2)^k} - \int \frac{2(-k)t^2\,dt}{(t^2+\Delta^2)^{k+1}} =$$
$$= \frac{t}{(t^2+\Delta^2)^k} + 2k\int \frac{t^2\,dt}{(t^2+\Delta^2)^{k+1}}.$$

In the numerator of the last integral, we add and subtract Δ^2:

$$\frac{t}{(t^2+\Delta^2)^k} + 2k\int \frac{(t^2+\Delta^2-\Delta^2)\,dt}{(t^2+\Delta^2)^{k+1}} =$$
$$= \frac{t}{(t^2+\Delta^2)^k} + 2k\int \frac{dt}{(t^2+\Delta^2)^k} - 2k\Delta^2\int \frac{dt}{(t^2+\Delta^2)^{k+1}}.$$

If we denote $I_k = \int \frac{dt}{(t^2+\Delta^2)^k}$, then we can write the resulting relation as follows:

$$I_k = \frac{t}{(t^2+\Delta^2)^k} + 2k(I_k - \Delta^2 I_{k+1}).$$

Express I_{k+1} in terms of I_k:

$$I_{k+1} = \frac{1}{2k\Delta^2}\left(\frac{t}{(t^2+\Delta^2)^k} + (2k-1)I_k\right).$$

We have obtained a recurrence relation that allows us to reduce the finding of the integral I_{k+1} to I_k. Applying it the required number of times, we can reduce the integral I_k to the integral I_1, which can be found explicitly:

$$I_1 = \int \frac{dt}{t^2+\Delta^2} = \frac{1}{\Delta^2}\int \frac{dt}{\left(\frac{t}{\Delta}\right)^2+1} = \frac{1}{\Delta}\int \frac{d\left(\frac{t}{\Delta}\right)}{\left(\frac{t}{\Delta}\right)^2+1} =$$
$$= \frac{1}{\Delta}\arctan\frac{t}{\Delta} + C.$$

Theorem on the integration of a rational function

2.2B/00:00 (06:08)

Thus, we have shown that all the integrals arising during the integration of a rational function are expressed in terms of elementary functions and the following theorem holds.

Theorem (on the integration of a rational function).

Any rational function can be integrated in elementary functions.

This is an important fact, since there are elementary functions whose integrals are not expressed in terms of elementary functions. Examples of such functions are $\frac{e^x}{x}$, $\frac{\sin x}{x}$, $\frac{\cos x}{x}$.

Having proved the theorem on the integration of a rational function, we can use it to study the integrability of other types of functions. If we can reduce (for example, by changing a variable) a certain integrand to a rational function, then we can state that the original function is also integrated in elementary functions.

Remark.

When integrating rational functions, we “go beyond” the set of rational functions, because as a result of integrating rational functions, logarithms and arctangents can arise.

3. Integration of trigonometric functions

Rational expressions for trigonometric functions

2.2B/06:08 (03:45)

DEFINITION.

A *polynomial in two variables* $P(u, v)$ is a function of the form

$$P(u, v) = \sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij} u^i v^j.$$

A *rational function of two variables* $R(u, v)$ is the ratio of polynomials $P(u, v)$ and $Q(u, v)$ in two variables:

$$R(u, v) = \frac{P(u, v)}{Q(u, v)}.$$

We want to study the integrability of a rational function $R(u, v)$, in which the trigonometric functions $\sin x$ and $\cos x$ are indicated as arguments: $R(\sin x, \cos x)$. This is a wide class of functions containing any arithmetic combinations of integer (including negative) degrees of trigonometric functions $\sin x$ and $\cos x$.

It turns out that such functions, like rational functions, can be integrated in elementary functions.

Universal trigonometric substitution

2.2B/09:53 (04:18)

To find the integral $\int R(\sin x, \cos x)\, dx$, we can use the following variable change, called the *universal trigonometric substitution* (or *tangent half-angle substitution*):

$$t = \tan \frac{x}{2}.$$

This substitution is based on the fact that both sine and cosine are expressed in terms of the tangent of a half angle:

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1 + t^2},$$

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2}.$$

In addition, since $\arctan t = \frac{x}{2}$, we get $dx = \frac{2\,dt}{1+t^2}$.

Thus, when using this substitution, both the functions $\sin x$, $\cos x$ and the differential dx are represented as rational functions of the new variable t.

Performing such a variable change, we obtain an integral of the following form:

$$\int R(\sin x, \cos x)\,dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2\,dt}{1+t^2}.$$

The integrand in the right-hand integral, being a superposition of rational functions, is itself a rational function of the argument t. Since any rational function is integrated in elementary functions, we get that the initial integral can also be integrated in elementary functions.

Features of the use of universal trigonometric substitution

2.2B/14:11 (15:03)

When performing a universal trigonometric substitution, a narrowing of the domain of definition of the integrand can occur, since the function $\tan\frac{x}{2}$ does not exist for some values of x.

In order to illustrate such a feature, we consider the following integral, which is defined for all real numbers x: $\int(\sin^2 x + \cos^2 x)\,dx$. By virtue of the Pythagorean trigonometric identity, this integral is $x + C$.

We find this integral using the universal trigonometric substitution:

$$\int(\sin^2 x + \cos^2 x)\,dx = \int\left(\frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2}\right)\frac{2\,dt}{1+t^2} =$$
$$= 2\int\frac{(4t^2 + 1 - 2t^2 + t^4)\,dt}{(1+t^2)^3} = 2\int\frac{(1+2t^2+t^4)\,dt}{(1+t^2)^3} =$$
$$= 2\int\frac{(1+t^2)^2\,dt}{(1+t^2)^3} = 2\int\frac{dt}{1+t^2} = 2\arctan t + C.$$

Returning to the initial variable x, we get

$$\int(\sin^2 x + \cos^x)\;dx = 2\arctan\left(\tan\frac{x}{2}\right) + C.$$

The resulting expression is not equal to $x+C$ on the entire real axis: it is equal to $x+C$ only on the interval $(-\pi,\pi)$, since the function $\arctan x$ takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. If, for example, we consider the interval $(\pi, 3\pi)$, then our function $2\arctan\bigl(\tan\frac{x}{2}\bigr)$ will also take values from $-\pi$ to π; thus, its graph will be a graph of the linear function $x - 2\pi$. We can say that the graph of the function $2\arctan\bigl(\tan\frac{x}{2}\bigr)$ on the interval $(\pi, 3\pi)$ is obtained by shifting to the right by 2π the graph of the same function on the interval $(-\pi, \pi)$. Using similar shifts, we can obtain graphs of this function on any interval of the form $(-\pi + 2k\pi, \pi + 2k\pi)$, $k \in \mathbb{Z}$. We also note that at the points $\pi + 2k\pi$, $k \in \mathbb{Z}$, the function $2\arctan\bigl(\tan\frac{x}{2}\bigr)$ is not defined.

Thus, the function $2\arctan\bigl(\tan\frac{x}{2}\bigr)$ is a piecewise linear function, it takes values from $-\pi$ to π and has discontinuities of the first kind at points $\pi+2k\pi$, $k \in \mathbb{Z}$ (Fig. 1).

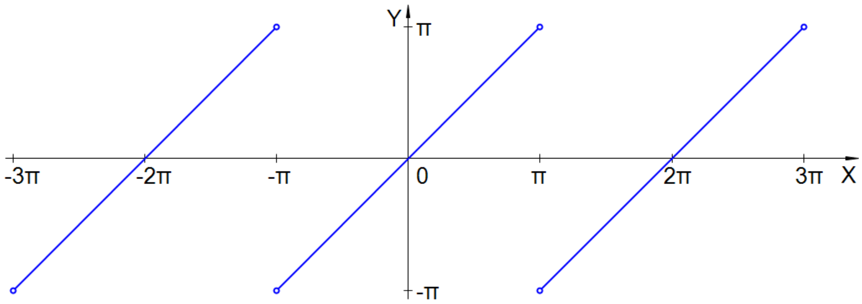

Fig. 1. Graph of the function $2\arctan\bigl(\tan\frac{x}{2}\bigr)$

Since the antiderivative must be a differentiable function and therefore a continuous function, we obtain that the found representation $2\arctan\bigl(\tan\frac{x}{2}\bigr) + C$ of the initial integral makes sense only on intervals $(-\pi + 2k\pi, \pi + 2k\pi)$, $k \in \mathbb{Z}$.

However, in the obtained formula, there exists a constant C that can be chosen differently on the intervals $(-\pi + 2k\pi, \pi + 2k\pi)$, $k \in \mathbb{Z}$, on which antiderivative makes sense. Due to this choice, it is possible to “combine” parts of a piecewise linear function in such a way as to turn discontinuity points of the first kind into points of removable discontinuity. This can be achieved, for example, by setting $C = 0$ for the antiderivative on the interval $(-\pi, \pi)$, $C = 2\pi$ for the antiderivative on the interval $(\pi, 3\pi)$, $C = -2\pi$ for the antiderivative on the interval $(-3\pi, -\pi)$, etc. (Fig. 2).

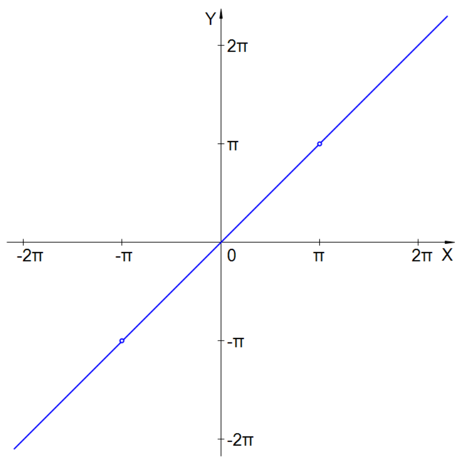

Fig. 2. Combining antiderivative segments

Having defined a combined antiderivative by continuity at the points $\pi + 2k\pi$, $k \in \mathbb{Z}$, we obtain a linear function x, i. e., one of the initial integral antiderivatives defined on the entire real axis.

Thus, the universal trigonometric substitution allows us to obtain a representation of the initial integral on intervals that do not contain singular points of the tangent function, but additional transformations are required (related to the selection of arbitrary terms C and to the additional definition the obtained function by continuity at its singular points) to get the antiderivatives defined on the entire real axis (if such antiderivatives exist).

Other types of variable change for trigonometric expressions

Variable change using the cosine and sine functions

2.2B/29:14 (08:12), 2.3A/00:00 (15:04)

In some cases, when integrating trigonometric expressions, more simple change of variable can be used, which does not narrow the domain of definition of the original expression.

Such types of variable change are possible if the rational function of two variables $R(\sin x, \cos x)$ has additional properties.

Case 1.

Let $R(-u, v) = -R(u, v)$ for any u, v from the domain of definition of this function. This property is called *oddness with respect to the first argument.*

In this case, for an integral of the form $\int R(\sin x, \cos x)\, dx$, we can use a simpler variable change, which reduces it to an integral of a rational function. To justify such a variable change, we need to analyze the properties of the function that is an odd function with respect to the first argument.

First, consider the auxiliary function $\tilde{R}(u, v)$, which is an even function with respect to the first argument:

$$\tilde{R}(-u, v) = \tilde{R}(u, v).$$

Obviously, in such a situation, all powers of the argument u in the function $\tilde{R}$ have an even degree. This means that the function $\tilde{R}(u, v)$ can be represented as follows:

$$\tilde{R}(u, v) = R^*(u^2, v).$$

Here R^* is a new rational function.

For example, the function $\tilde{R}(u,v) = \frac{u^2+u^4v+v^3}{v^4u^2}$ is an even function with respect to the first argument and can be represented in the form $R^*(u^2,v)$, where $R^*(y,v) = \frac{y+y^2v+v^3}{v^4y}$.

Let us return to the initial function R, which is an odd function with respect to the first argument, and construct the following function:

$$\tilde{R}(u,v) = \frac{R(u,v)}{u}.$$

The function being constructed is a rational function and it is an even function with respect to the first argument because, given the property of the function R, we have

$$\tilde{R}(-u,v) = \frac{R(-u,v)}{-u} = \frac{-R(u,v)}{-u} = \frac{R(u,v)}{u} = \tilde{R}(u,v).$$

Since the function $\tilde{R}(u,v)$ is an even function with respect to the first argument, it can be represented as follows:

$$\tilde{R}(u,v) = R^*(u^2,v).$$

Given the definition of the function $\tilde{R}(u,v)$, we obtain that the expression $\frac{R(u,v)}{u}$ can also be represented as $R^*(u^2,v)$:

$$\frac{R(u,v)}{u} = R^*(u^2,v).$$

Therefore, for a rational function $R(u,v)$ which is an odd function with respect to the first argument, there exists the following representation using some rational function R^*:

$$R(u,v) = R^*(u^2,v)u.$$

Let us return to our integral and rewrite it as follows:

$$\int R(\sin x, \cos x)\,dx = \int R^*(\sin^2 x, \cos x)\sin x\,dx.$$

We make the variable change $t = \cos x$, then $dt = -\sin x\,dx$. In addition, $\sin^2 x = 1 - \cos^2 x = 1 - t^2$. Thus, the initial integral takes the form

$$\int R(\sin x, \cos x)\,dx = -\int R^*(1-t^2,t)t\,dx.$$

The function R^* and its arguments are rational functions, therefore the integrand is a rational function and it can be integrated in elementary functions.

Thus, if the function $R(u,v)$ is an odd function with respect to the first argument, then, for the integral $\int R(\sin x, \cos x)\,dx$, we can use the variable change $t = \cos x$, and as a result we obtain the integral of the rational function.

CASE 2.

Let the function $R(u, v)$ be an odd function with respect to the second argument: $R(u, -v) = -R(u, v)$.

Then, using similar reasoning, we can prove that the function $\tilde{R}(u, v) = \frac{R(u,v)}{v}$ is an even function with respect to the second argument and therefore it can be represented in the form $R^*(u, v^2)$. Finally, we obtain the following representation for R:

$$R(u, v) = R^*(u, v^2)v.$$

In this case, the integral $\int R(\sin x, \cos x)\,dx$ takes the form

$$\int R(\sin x, \cos x)\,dx = \int R^*(\sin x, \cos^2 x)\cos x\,dx.$$

For its transformation to the integral of a rational function, we can use the variable change $t = \sin x$ (with $dt = \cos x\,dx$):

$$\int R(\sin x, \cos x)\,dx = \int R^*(t, 1 - t^2)t\,dx.$$

REMARK.

Both considered variable changes ($t = \cos x$ and $t = \sin x$), unlike the universal trigonometric substitution, do not narrow the domain of definition of the initial function.

Variable change using the tangent function

2.3A/15:04 (09:28)

CASE 3.

Let the function $R(u, v)$ be an even function with respect to both arguments, that is, for any u and v from its domain of definition, the following relation holds:

$$R(-u, -v) = R(u, v).$$

We transform the function R as follows:

$$R(u, v) = R\Big(\frac{u}{v} \cdot v, v\Big).$$

The expression on the right-hand side means that the initial function can be represented as a rational function of the arguments $\frac{u}{v}$ and v:

$$R(u, v) = \tilde{R}\Big(\frac{u}{v}, v\Big).$$

Since the initial function R is an even function with respect to both arguments, the function $\tilde{R}$ is an even function with respect to the second argument:

$$\tilde{R}\Big(\frac{u}{v}, -v\Big) = \tilde{R}\Big(\frac{-u}{-v}, -v\Big) = R(-u, -v) = R(u, v) = \tilde{R}\Big(\frac{u}{v}, v\Big).$$

Therefore, the function $\tilde{R}\big(\frac{u}{v}, v\big)$ has the following representation using some rational function R^*:

$$\tilde{R}\Big(\frac{u}{v}, v\Big) = R^*\Big(\frac{u}{v}, v^2\Big).$$

So, we have proved that a function $R(u, v)$, which is an even function with respect to both arguments, can be represented using some rational function R^* as follows:

$$R(u, v) = R^*\Big(\frac{u}{v}, v^2\Big).$$

Thus, for the integral $\int R(\sin x, \cos x)\, dx$ we can get the following representation:

$$\int R(\sin x, \cos x)\, dx = \int R^*(\tan x, \cos^2 x)\, dx.$$

In this case, it is convenient to make the variable change $t = \tan x$, then $x = \arctan t$, $dx = \frac{dt}{1+t^2}$. In addition, $\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2}$. Given the obtained relations, the initial integral takes the form

$$\int R(\sin x, \cos x)\, dx = \int R^*\Big(t, \frac{1}{1+t^2}\Big)\frac{dx}{1+t^2}.$$

So, we can represent the initial integral as the integral of rational function, although when replacing $t = \tan x$, the domain of definition of the original function can be narrowed (as in the case of universal trigonometric substitution $t = \tan\frac{x}{2}$). However, the resulting superposition of rational functions is more simple than the superposition obtained as a result of universal trigonometric substitution.

Using multiple variable changes

2.3A/24:32 (05:08)

Any rational function $R(u, v)$ can be represented as the sum of three rational functions, the first of which is an odd function with respect to the first argument, the second one is an odd function with respect to the second argument, and the third one is an even function with respect to both arguments. Therefore, to integrate the initial function $R(\sin x, \cos x)$, we can represent it in the form of the indicated sum of three functions and apply the corresponding change of variable described for cases 1, 2, 3 to each of these functions.

The required representation of the function $R(u, v)$ can be obtained using the following equality:

$$R(u,v) = \frac{R(u,v) - R(-u,v)}{2} + \frac{R(-u,v) - R(-u,-v)}{2} + \\ + \frac{R(-u,-v) + R(u,v)}{2}.$$

It is easy to verify that on the right-hand side of the equality obtained, the first term $\frac{R(u,v)-R(-u,v)}{2}$ is a rational function which is an odd function with respect to the first argument, the second term $\frac{R(-u,v)-R(-u,-v)}{2}$ is a rational function which is an odd function with respect to the second argument, and the third term $\frac{R(-u,-v)+R(u,v)}{2}$ is a rational function which is an even function with respect to both arguments.

For example, if we denote the first term by $R_1(u,v)$ and replace u with $(-u)$ in it, then we get

$$R_1(-u,v) = \frac{R(-u,v) - R(u,v)}{2} = -\frac{R(u,v) - R(-u,v)}{2} = \\ = -R_1(u,v).$$

This relation means that the function $R_1(u,v)$ is an odd function with respect to the first argument.

4. Integration of irrational functions

Integration of a rational function with an irrational argument

2.3A/29:40 (08:42)

Let $R(u, v)$ be a rational function, $a, b, c, d \in \mathbb{R}$, $p \in \mathbb{Q}$. Consider the following integral:

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^p\right) dx.$$

This integral contains an irrational function if the exponent p is not an integer: $p = \frac{q}{r}$, where $q \in \mathbb{Z}$, $r \in \mathbb{N}$, $r > 1$.

We also require that the determinant consisting of coefficients a, b, c, d be nonzero:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0. \tag{1}$$

If condition (1) is violated, then the coefficients a and b will be proportional to the coefficients c and d. In this case, the fraction $\frac{ax+b}{cx+d}$ will be a constant and the irrationality in the integrand will disappear. Therefore, this case is not interesting.

This integral can be transformed, by an appropriate change of variable, to the integral of a rational function. Let us use the following variable change:

$$t = \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{r}}. \tag{2}$$

Then the second argument of the function R takes the form t^q, i. e., it does not contain irrationality. It remains for us to express the argument x and the differential dx in terms of the new variable t. We transform equality (2) as follows:

$$t^r = \frac{ax+b}{cx+d},$$
$$(cx+d)t^r = ax+b,$$
$$x = \frac{b - dt^r}{ct^r - a}.$$

Condition (1) ensures that the terms t^r on the right-hand side of the equality obtained do not disappear. Thus, we have obtained that x is expressed as a rational function of the argument t: $R_1(t) = \frac{b-dt^r}{ct^r-a}$.

Since $x = R_1(t)$, we get that $dx = R_1'(t)\,dt$, where $R_1'(t)$ is also a rational function (as a derivative of a rational function).

As a result of this variable change, the initial integral takes the form

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^p\right) dx = \int R\big(R_1(t), t^q\big) R_1'(t)\,dt.$$

The integrand in the right-hand integral is a rational function as a superposition and a product of rational functions. Therefore, applying the theorem on integrating a rational function, we obtain that the initial integral can also be expressed in elementary functions.

Generalization to the case of several irrational arguments

2.3A/38:22 (05:28), 2.3B/00:00 (03:45)

DEFINITION.

A *polynomial in n variables* is a function $P(x_1, x_2, \dots, x_n)$ of the following form:

$$P(x_1, x_2, \dots, x_n) = \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{i_n=0}^{m_n} c_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}.$$

A *rational function of n variables* is the ratio of two polynomials in n variables.

We generalize the previous result on the integration of a rational function with an irrational argument to the case of a rational function of $n+1$ variables containing n irrational arguments *with the same bases*:

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{p_1}, \left(\frac{ax+b}{cx+d}\right)^{p_2}, \dots, \left(\frac{ax+b}{cx+d}\right)^{p_n}\right) dx.$$

Here $p_1, p_2, \dots, p_n \in \mathbb{Q}$, $p_i = \frac{q_i}{r_i}$, where $q_i \in \mathbb{Z}$, $r_i \in \mathbb{N}$, $r_i > 1$, $i = 1, \dots, n$. We require that condition (1) is satisfied for the coefficients a, b, c, d.

Let us introduce an auxiliary value s which is the least common multiple of the numbers $r_1, r_2, \dots, r_n$:

$$s = \mathrm{LCM}\,\{r_1, r_2, \dots, r_n\}. \tag{3}$$

Then p_i, $i = 1, \dots, n$, can be represented as follows:

$$p_i = \frac{q_i}{r_i} = \frac{q_i}{s} \cdot \frac{s}{r_i} = \frac{q_i l_i}{s}.$$

Here $l_i = \frac{s}{r_i}$ is an integer by (3).

Thus, we expressed all the numbers p_i as fractions with the same denominator s. Perform the following change of variable in the integral:

$$t = \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}}. \tag{4}$$

Then the arguments $\left(\frac{ax+b}{cx+d}\right)^{p_i}$ of the function R will take the form $t^{q_i l_i}$, that is, they will not contain irrationality.

Acting in the same way as in the previously considered case, we can express x from relation (4) as a rational function of the argument t:

$$x = \frac{b - dt^s}{ct^s - a} = R_1(t).$$

As a result of this variable change, the initial integral takes the form

$$\int R\left(R_1(t), t^{q_1 l_1}, t^{q_2 l_2}, \ldots, t^{q_n l_n}\right) R_1'(t)\, dt.$$

The integrand in the resulting integral is a rational function, so this integral and therefore the initial integral can also be expressed in elementary functions.

Integration of the binomial differential 2.3B/03:45 (15:59)

Consider the following integral (called the integral of the *binomial differential*):

$$\int x^m (a + bx^n)^p\, dx.$$

Here m, n, p are some rational numbers.

Due to the power function x^n, this integral differs from the previously considered one (for which the base of the power function with a rational exponent was the ratio of linear functions).

Therefore, at the first stage of the transformation, we get rid of the power function x^n by performing the variable change $t = x^n$. Then $x = t^{\frac{1}{n}}$, $dx = \frac{1}{n} t^{\frac{1}{n}-1}\, dt$, and the integral takes the form

$$\int x^m (a+bx^n)^p\, dx = \frac{1}{n}\int t^{\frac{m}{n}}(a+bt)^p t^{\frac{1-n}{n}}\, dt = \frac{1}{n}\int t^{\frac{m+1}{n}-1}(a+bt)^p\, dt.$$

For brevity, we denote the resulting rational exponent $\frac{m+1}{n} - 1$ by q. Using this notation we get

$$\int x^m (a + bx^n)^p\, dx = \frac{1}{n}\int t^q (a + bt)^p\, dt.$$

Thus, it remains for us to consider the following integral containing rational exponents p and q:

$$\int t^q (a + bt)^p\, dt.$$

There are three simple cases in which a given integral can be transformed to an integral of a rational function.

CASE 1: $q \in \mathbb{Z}$, $p = \frac{r}{s}$, $r \in \mathbb{Z}$, $s \in \mathbb{N}$.

This case has already been analyzed previously. To transform the integral to the integral of a rational function, it suffices to use the variable change $u = (a + bt)^{\frac{1}{s}}$.

CASE 2: $p \in \mathbb{Z}$, $q = \frac{r}{s}$, $r \in \mathbb{Z}$, $s \in \mathbb{N}$.

In this case, we use the variable change $u = t^{\frac{1}{s}}$. Then $t = u^s$, $dt = su^{s-1}\,du$, and we get the integral of a polynomial:

$$\int t^q(a+bt)^p\,dt = \int u^r(a+bu^s)^p su^{s-1}\,du.$$

CASE 3: $p + q \in \mathbb{Z}$, $p = \frac{r}{s}$, $r \in \mathbb{Z}$, $s \in \mathbb{N}$.

We transform the integral as follows:

$$\int t^q(a+bt)^p\,dt = \int t^{p+q}\Big(\frac{a+bt}{t}\Big)^p\,dt\ .$$

We again got the integral considered earlier, since the irrational factor has a base which is a ratio of linear functions. To transform the integral to the integral of a rational function, it suffices to use the variable change $u = \left(\frac{a+bt}{t}\right)^{\frac{1}{s}}$.

REMARK.

In all other cases, the integral of the binomial differential cannot be represented as the integral of a rational function. This fact was established by the Russian mathematician P. L. Chebyshev.

Euler's substitutions

Three types of Euler's substitutions 2.3B/19:44 (12:18)

Let $R(u,v)$ be a rational function of two variables, $a, b, c \in \mathbb{R}$, $a \neq 0$. Consider the following integral:

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)\,dx. \tag{5}$$

We exclude from consideration those values of x for which $ax^2+bx+c < 0$.

L. Euler showed that the integral (5) can always be transformed to the integral of a rational function using one of three variable changes called *Euler's substitutions.*

CASE 1: $a > 0$.

The substitution $\sqrt{ax^2+bx+c} = \sqrt{a}x + t$ is used.

Square both sides of the equality:

$$ax^2 + bx + c = ax^2 + 2\sqrt{a}xt + t^2,$$
$$bx + c = 2\sqrt{a}xt + t^2.$$

The last equality contains only the first power of the variable x, so we can easily express x through the new variable t, resulting in a rational function $R_1(t)$:

$$x = \frac{t^2 - c}{b - 2\sqrt{a}t} = R_1(t).$$

Thus, the initial integral takes the form

$$\int R\left(x, \sqrt{ax^2 + bx + c}\right) dx = \int R\left(R_1(t), \sqrt{a}R_1(t) + t\right) R_1'(t)\, dt.$$

We got the integral of the rational function.

CASE 2: $c > 0$.

The substitution $\sqrt{ax^2 + bx + c} = xt + \sqrt{c}$ is used.

When squaring both sides of this equality, we obtain

$$ax^2 + bx + c = x^2t^2 + 2\sqrt{c}xt + c,$$
$$ax^2 + bx = x^2t^2 + 2\sqrt{c}xt,$$
$$ax + b = xt^2 + 2\sqrt{c}t.$$

The last equality contains only the first power of the variable x, which allows us to express x through the new variable t, resulting in a rational function $R_2(t)$:

$$x = \frac{2\sqrt{c}t - b}{a - t^2} = R_2(t).$$

The initial integral will take the form

$$\int R\left(x, \sqrt{ax^2 + bx + c}\right) dx = \int R\left(R_2(t), tR_2(t) + \sqrt{c}\right) R_2'(t)\, dt.$$

We again got the integral of the rational function.

CASE 3: the quadratic trinomial $ax^2 + bx + c$ has real roots, i. e., it can be represented as follows:

$$ax^2 + bx + c = a(x - \alpha)(x - \beta), \quad \alpha, \beta \in \mathbb{R}. \tag{6}$$

Then the substitution $\sqrt{ax^2 + bx + c} = (x - \alpha)t$ is used.

We square both sides of this equality and then apply relation (6) on the left-hand side:

$$a(x - \alpha)(x - \beta) = (x - \alpha)^2t^2,$$

$$a(x-\beta) = (x-\alpha)t^2.$$

The last equality contains only the first power of the variable x, which allows us to express the variable x through the new variable t, resulting in a rational function $R_3(t)$:

$$x = \frac{a\beta - \alpha t^2}{a - t^2} = R_3(t).$$

The initial integral takes the form

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)\,dx = \int R\left(R_3(t), (R_3(t)-\alpha)t\right) R_3'(t)\,dt.$$

We again got the integral of the rational function.

Remark.

In each case, we can use one more formula for the corresponding substitution. For case 1, this is $\sqrt{ax^2+bx+c} = \sqrt{a}x - t$, for case 2, this is $\sqrt{ax^2+bx+c} = xt - \sqrt{c}$, for case 3, this is $\sqrt{ax^2+bx+c} = (x-\beta)t$. For some types of quadratic trinomials, we can use substitution corresponding to different cases (for example, if the inequalities $a > 0$ and $c > 0$ are satisfied simultaneously). The choice of a substitution formula is determined by which formula leads to a simpler form of the resulting rational function.

On the possibility of applying Euler's substitutions to any quadratic trinomial

2.3B/32:02 (08:38)

It turns out that, for any quadratic trinomial ax^2+bx+c, at least one of the three cases listed above necessarily takes place, therefore any integral (5) can be transformed to the integral of a rational function. Moreover, for any quadratic trinomial, either case 1 or case 3 takes place. Let us prove this statement.

Recall that we consider the integral (5) on the set of x such that $ax^2+bx+c \geq 0$. In particular, if the entire graph of the parabola $y = ax^2+bx+c$ is located below the OX axis, then it does not make sense to analyze the integral (5).

Therefore, only the following types for the location of the parabola $y = ax^2+bx+c$ relative to the OX axis are of interest:

1) the entire parabola (or the entire parabola, except for its vertex) is located above the OX axis (see the left-hand part of Fig. 3);

2) the parabola intersects the OX axis at two points (see the right-hand part of Fig. 3).

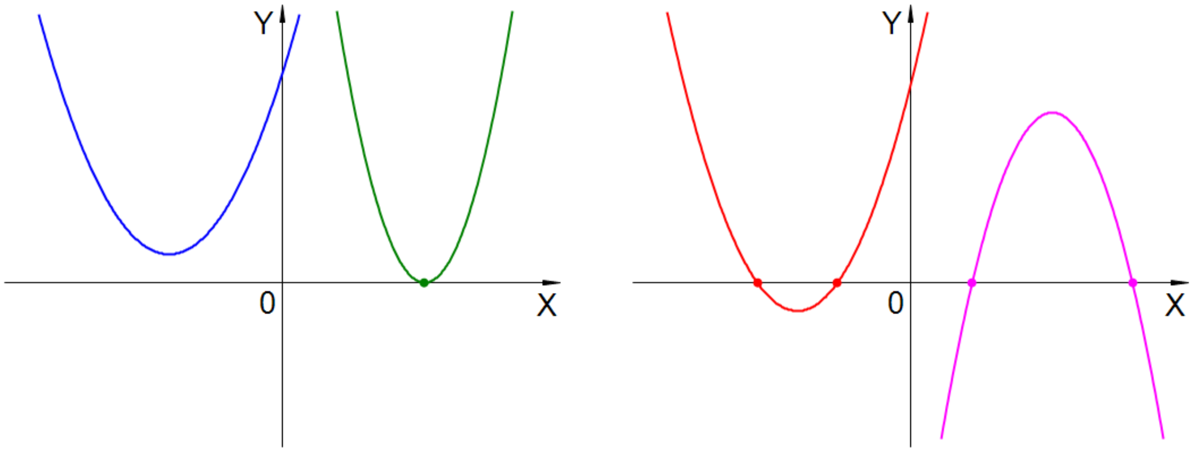

Fig. 3. Types of parabola location

For type 2 of parabola location, case 3 of Euler's substitutions takes place: the quadratic trinomial has two real roots.

It remains for us to show that, for type 1 of parabola location, case 1 of Euler's substitutions necessarily takes place, i. e., $a > 0$. So, suppose that the following relation holds for all $x \in \mathbb{R}$:

$$ax^2 + bx + c \geq 0. \tag{7}$$

We transform the left-hand side of inequality (7) by complete the square:

$$a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right) =$$
$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right).$$

Now let us transform the term $\frac{c}{a} - \frac{b^2}{4a^2}$:

$$\frac{c}{a} - \frac{b^2}{4a^2} = \frac{4ac - b^2}{4a^2}.$$

The denominator of the fraction obtained is greater than zero. The numerator is non-negative, because it is $(-D)$, where $D = b^2 - 4ac$ is the discriminant that cannot be positive, since the parabola crosses the OX axis at no more than one point. Therefore, the term $\frac{c}{a} - \frac{b^2}{4a^2}$ is also non-negative and can be denoted by Δ^2.

Thus, inequality (7) can be rewritten in the form

$$a\left(\left(x + \frac{b}{2a}\right)^2 + \Delta^2\right) \geq 0.$$

Since the expression in parentheses is non-negative, the coefficient a must also be non-negative. Since $a \neq 0$ by condition, we finally obtain that $a > 0$ and case 1 holds.

Remark.

The required result can be obtained more easily without transforming the original quadratic trinomial. It is enough to note that the term ax^2 grows faster than the term bx as $x \to \infty$; this means that the sign of the expression $ax^2 + bx + c$ for sufficiently large x is determined by the sign of the term ax^2. Therefore, if the inequality $a < 0$ holds, then estimate (7) could not be valid for all $x \in \mathbb{R}$.

5. Definite integral and Darboux sums

Definite integral

The problem of finding the area of a curvilinear trapezoid

2.4A/00:00 (07:00)

The basic concepts related to a definite integral can be considered by the example of the geometric problem of finding the area of a curvilinear trapezoid.

Let a function $f(x)$ be defined on the segment (closed interval) $[a, b]$ and taking non-negative values on this segment: $f(x) \geq 0$, $x \in [a, b]$. It is required to find the area of the figure G bounded by the OX axis, the vertical lines $x = a$ and $x = b$, and the graph of the function $y = f(x)$. Such a figure is called a *curvilinear trapezoid* with the base $[a, b]$ (Fig. 4).

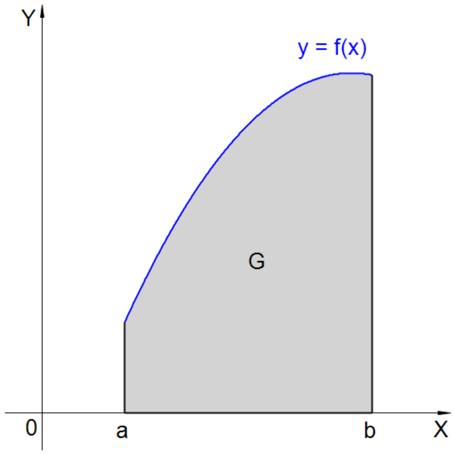

Fig. 4. Curvilinear trapezoid

How to find the approximate area of a curvilinear trapezoid?

Let us divide the segment $[a, b]$ into smaller segments (not necessarily of equal length) with endpoints $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. For brevity, we denote the obtained segments as follows: $\Delta_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$. We also introduce the notation for the length of the segment Δ_i: $\Delta x_i = x_i - x_{i-1}$, $i = 1, \ldots, n$.

Choose a point ξ_i on each of the segments Δ_i: $\xi_i \in \Delta_i$, $i = 1, \ldots, n$.

Provided that the function f has sufficiently "good" properties, we can assume that the area of the curvilinear trapezoid with the base Δ_i will be close to the area of the rectangle with the same base Δ_i and a height equal to the value of the function f at the point ξ_i. The area of this rectangle is $f(\xi_i)\Delta x_i$.

Summing up the areas of all such rectangles, we get the approximate value of the area of the initial curvilinear trapezoid: $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ (see the left-hand part of Fig. 5).

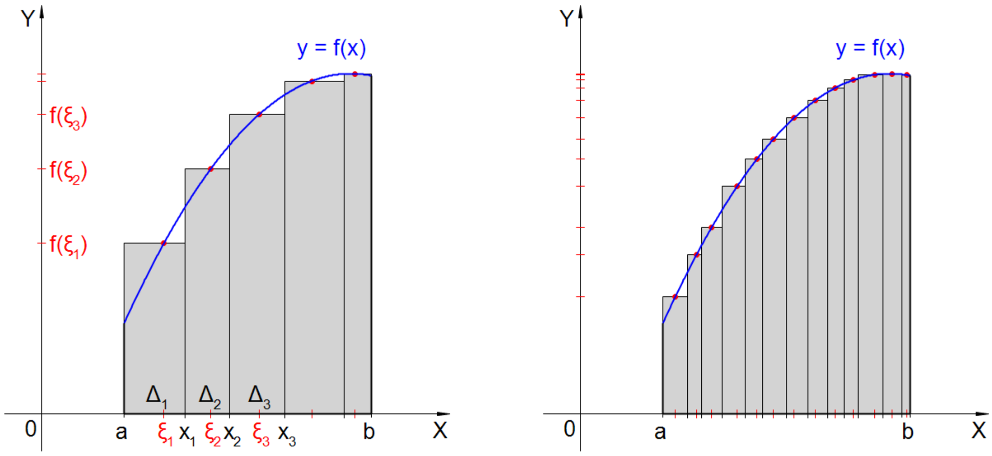

Fig. 5. Curvilinear trapezoid approximation by a set of rectangles

As the length of segments Δ_i decreases, the resulting union of the rectangles will be even closer to the initial curvilinear trapezoid (see the right-hand part of Fig. 5).

If the expression $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ has a limit as the length of segments Δ_i unlimitedly decreases (and, accordingly, as the number of points x_i unlimitedly increases), then it is natural to consider this limit as the area of the initial curvilinear trapezoid.

It is this limit that is called the *definite integral* of the function f over the segment $[a, b]$.

Definition of a definite integral

2.4A/07:00 (14:01)

DEFINITION.

Let the function f be defined on the segment $[a, b]$. The *partition* T of the segment $[a, b]$ is the ordered set of points x_i, $i = 0, \ldots, n$, which has the following property:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

For the segments $[x_{i-1}, x_i]$ with endpoints at adjacent points of the partition T, as well as for their lengths $x_i - x_{i-1}$, we will use the notation introduced above:

$$\Delta_i \stackrel{\text{def}}{=} [x_{i-1}, x_i], \quad \Delta x_i \stackrel{\text{def}}{=} x_i - x_{i-1}, \quad i = 1, \ldots, n.$$

Obviously, $\Delta x_i > 0$.

The *mesh* of the partition T (notation $l(T)$) is the maximum of the lengths of the segments Δ_i:

$$l(T) \overset{\text{def}}{=} \max_{i=1,\ldots,n} \Delta x_i.$$

A *sample* ξ constructed on the basis of a partition T is an arbitrary set of points $\xi_i \in \Delta_i$, $i = 1, \ldots, n$.

The *integral sum* $\sigma_T(f, \xi)$ of the function f by the partition T and the sample ξ is the following expression:

$$\sigma_T(f, \xi) \overset{\text{def}}{=} \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

A function f is called *Riemann integrable* on the segment $[a, b]$ if there exists a number I such that

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, T, l(T) < \delta, \quad \forall\, \xi \quad |\sigma_T(f, \xi) - I| < \varepsilon. \tag{1}$$

Briefly, condition (1) can be written using the limit notation:

$$I = \lim_{l(T)\to 0, \forall\, \xi} \sigma_T(f, \xi).$$

The number I is called the *Riemann integral*, or the *definite integral*, of the function f over the segment $[a, b]$, and it is denoted as follows: $\int_a^b f(x)\, dx$.

So, the Riemann integral of the function f is the limit of the integral sums $\sigma_T(f, \xi)$ as $l(T) \to 0$, $\forall\, \xi$, if this limit exists:

$$\int_a^b f(x)dx \overset{\text{def}}{=} \lim_{l(T)\to 0, \forall\, \xi} \sigma_T(f, \xi).$$

In what follows, Riemann integrability and the Riemann integral will be called simply integrability and integral, respectively.

Remarks.

1. Although the limit of integral sums used in the definition of the integral I differs from the usual limit of a function at a point, it is easy to prove, using condition (1), that this limit satisfies both the theorem on arithmetic properties of the limit and the theorem on passing to the limit in the inequalities. We will use these theorems in the next chapter to prove the properties of a definite integral.

2. One can extend the class of integrable functions by giving other definitions of integrability. Among such types of integrability, one of the most important is *Lebesgue integrability*. Lebesgue integrability is not considered in this book.

A necessary condition for integrability

2.4A/21:01 (11:50)

Theorem (a necessary condition for integrability).

If the function is integrable on a segment, then it is bounded on this segment.

Proof.

Let the function f be integrable on the segment $[a, b]$. This means that there exists a number I for which condition (1) is satisfied. In this condition, we choose the value of ε, setting it equal to 1. Then there exists a value $\delta > 0$ such that the following estimate holds for any partition T of the segment $[a, b]$ satisfying the additional condition $l(T) < \delta$ and any sample ξ constructed on the basis the partition T:

$$|\sigma_T(f, \xi) - I| < 1. \tag{2}$$

We select some partition T satisfying the condition $l(T) < \delta$.

Let us prove the statement of the theorem by contradiction: suppose that the function f is not bounded on $[a, b]$. This means that it is unbounded on at least one segment Δ_i associated with the previously selected partition T. For definiteness, we assume that such a segment is the segment Δ_1.

From the integral sum $\sigma_T(f, \xi)$, we extract the term associated with this segment:

$$\sigma_T(f, \xi) = f(\xi_1)\Delta x_1 + \sum_{i=2}^{n} f(\xi_i)\Delta x_i. \tag{3}$$

Let us fix all the elements of the sample ξ except for the first one, i. e., let us fix the values $\xi_2, \xi_3, \ldots, \xi_n$. In this case, the second term on the right-hand side of equality (3) will be uniquely determined. Denote the value of this term by A:

$$A = \sum_{i=2}^{n} f(\xi_i)\Delta x_i.$$

Then inequality (2) can be transformed as follows:

$$I - 1 < \sigma_T(f, \xi) < I + 1,$$

$$I - 1 < f(\xi_1)\Delta x_1 + A < I + 1.$$

In the resulting relation, we move A from the middle part to the left-hand and right-hand part, after which we divide all parts of the double inequality by Δx_1 (this can be done since $\Delta x_1 > 0$):

$$\frac{I - 1 - A}{\Delta x_1} < f(\xi_1) < \frac{I + 1 - A}{\Delta x_1}. \tag{4}$$

In this double inequality, all values are fixed except for the point ξ_1, which can vary within the segment Δ_1. Thus, we have obtained that the double inequality (4) holds for all $\xi_1 \in \Delta_1$, which implies that the function f is bounded on the segment Δ_1. But this contradicts our assumption that the function f is unbounded on this segment. The obtained contradiction proves the theorem. $\square$

Remarks.

1. Taking into account this theorem, we will consider only bounded functions hereinafter, not always noting this condition.

2. The converse of the proved theorem is false: if the function is bounded, then this does not follow that it is integrable. We give a corresponding example at the end of this chapter.

Darboux sums and Darboux integrals

Definition of Darboux sums

2.4A/32:51 (05:52)

Definition.

Let the function f be defined and bounded on the segment $[a, b]$.

We choose some partition T of this segment and introduce the following notation:

$$M_i = \sup_{x \in \Delta_i} f(x), \quad m_i = \inf_{x \in \Delta_i} f(x), \quad i = 1, \ldots, n.$$

Since the function f is bounded on $[a, b]$, the values M_i and m_i exist for all $i = 1, \ldots, n$.

The *upper Darboux sum* $S_T^+(f)$ and the *lower Darboux sum* $S_T^-(f)$ are defined as follows:

$$S_T^+(f) \stackrel{\text{def}}{=} \sum_{i=1}^{n} M_i \Delta x_i, \quad S_T^-(f) \stackrel{\text{def}}{=} \sum_{i=1}^{n} m_i \Delta x_i.$$

If it is clear which function f is associated with Darboux sums, then the short notation S_T^+ and S_T^- can be used for them.

Remark.

The main difference between Darboux sums and integral sums is that the notion of sample ξ is not used in the definition of Darboux sums: Darboux sums depend only on the function f itself and the partition T of the initial segment.

The simplest properties of Darboux sums

2.4A/38:43 (01:33), 2.4B/00:00 (12:05)

In the formulations of all properties, it is assumed that the function f is defined and bounded on the segment $[a, b]$.

1. Let T be some partition of the segment $[a, b]$, ξ be an arbitrary sample associated with this partition. Then the following double inequality holds:

$$S_T^-(f) \le \sigma_T(f, \xi) \le S_T^+(f). \tag{5}$$

PROOF.

From the definition of the supremum M_i and the infimum m_i, it follows

$$\forall\, \xi_i \quad m_i \le f(\xi_i) \le M_i. \tag{6}$$

Multiply all parts of the double inequality (6) by $\Delta x_i > 0$ and take a sum of all inequalities for $i = 1, \dots, n$:

$$\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i.$$

Given the definitions of the integral sum and Darboux sums, we obtain (5). □

2. For a fixed partition T of the segment $[a, b]$, the following relations hold:

$$S_T^+(f) = \sup_{\xi} \sigma_T(f, \xi), \qquad S_T^-(f) = \inf_{\xi} \sigma_T(f, \xi).$$

PROOF.

Let us prove this property for the upper Darboux sum. Given the definition of supremum, it is necessary to prove two statements:

1) $\forall\, \xi \quad \sigma_T(f, \xi) \le S_T^+(f)$,

2) $\forall\, \varepsilon > 0 \quad \exists\, \xi' \quad \sigma_T(f, \xi') > S_T^+(f) - \varepsilon$.

Statement 1 has already been proved (see property 1). Let us prove statement 2. From the definition of the supremum M_i, it follows

$$\forall\, \varepsilon > 0 \quad \exists\, \xi_i' \in \Delta_i \quad f(\xi_i') > M_i - \frac{\varepsilon}{b - a}. \tag{7}$$

Multiply both sides of inequality (7) by $\Delta x_i > 0$ and take a sum of all inequalities for $i = 1, \dots, n$:

$$\sum_{i=1}^{n} f(\xi_i')\Delta x_i > \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} \frac{\varepsilon}{b - a}\Delta x_i.$$

Given the definitions of the integral sum and the upper Darboux sum, as well as the fact that $\sum_{i=1}^{n} \frac{\varepsilon}{b-a}\Delta x_i = \frac{\varepsilon}{b-a}\sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$, we obtain statement 2.

The property for the lower Darboux sum is proved similarly, using the definition of the infimum. □

Darboux sum property related to refinement of a partition

2.4B/12:05 (15:11)

Before stating the next property, we introduce the concept of refinement of a partition.

Definition.

The partition T_2 is called the *refinement* of the partition T_1 if any element of the partition T_1 belongs to the partition T_2, i. e., $T_1 \subset T_2$. In other words, the refinement T_2 of the partition T_1 contains all points of the partition T_1 and possibly some other points of the original segment.

3. If the partition T_2 is a refinement of the partition T_1, then the following chain of inequalities holds:

$$S^-_{T_1} \le S^-_{T_2} \le S^+_{T_2} \le S^+_{T_1}. \tag{8}$$

Proof.

The middle inequality in (8) immediately follows from property 1. Let us prove the right-hand inequality: $S^+_{T_2} \le S^+_{T_1}$.

It is enough for us to consider the case when the refinement T_2 of the partition T_1 differs from T_1 by only one additional point. The case when there are several additional points can be reduced to the case with one point if we add these points to the partition sequentiallyand apply the proved estimate to the resulting refinements.

So, we assume that the refinement T_2 contains one additional point x': $T_1 = \{x_i, i = 0, \ldots, n\}, T_2 = T_1 \cup \{x'\}$. For definiteness, we also assume that $x' \in \Delta_1$, i. e., $x_0 < x' < x_1$. We also introduce the following notation:

$$\Delta_1' = [x_0, x'], \quad \Delta x_1' = x' - x_0, \quad M_1' = \sup_{x \in \Delta_1'} f(x),$$

$$\Delta_1'' = [x', x_1], \quad \Delta x_1'' = x_1 - x', \quad M_1'' = \sup_{x \in \Delta_1''} f(x).$$

We need to prove that

$$S^+_{T_1} - S^+_{T_2} \ge 0. \tag{9}$$

The indicated Darboux sums contain the coinciding terms $M_i\Delta x_i$ for $i = 2, \dots, n$. After reducing these coinciding terms, the difference $S^+_{T_1} - S^+_{T_2}$ takes the following form:

$$S^+_{T_1} - S^+_{T_2} = M_1\Delta x_1 - (M'_1\Delta x'_1 + M''_1\Delta x''_1). \tag{10}$$

Since $\sup A \le \sup B$ for $A \subset B$ and in our case $\Delta'_1 \subset \Delta_1$ and $\Delta''_1 \subset \Delta_1$, we get

$$M'_1 \le M_1, \quad M''_1 \le M_1.$$

Therefore, the right-hand side of equality (10) can be estimated as follows:

$$M_1\Delta x_1 - (M'_1\Delta x'_1 + M''_1\Delta x''_1) \ge M_1\Delta x_1 - (M_1\Delta x'_1 + M_1\Delta x''_1).$$

The right-hand side of the last inequality is 0, since $\Delta x_1 = \Delta x'_1 + \Delta x''_1$. We proved the validity of inequality (9) and thereby the validity of the right-hand inequality in (8).

The left-hand inequality in (8) is proved similarly, by taking into account the following property of the infimum: $\inf A \ge \inf B$ for $A \subset B$. □

Darboux sums associated with different partitions

2.4B/27:16 (05:14)

4. If T' and T'' are some partitions of the segment $[a, b]$, then the estimate holds:

$$S^-_{T'} \le S^+_{T''}. \tag{11}$$

Thus, any lower Darboux sum of the function f is less than or equal to any of its upper Darboux sums.

Proof.

Consider the union of two given partitions: $T = T' \cup T''$. The resulting partition T is a refinement of both the partition T' and the partition T''. Therefore, applying property 3, we obtain the following chain of inequalities:

$$S^-_{T'} \le S^-_{T} \le S^+_{T} \le S^+_{T''}. \tag{12}$$

In this case, we applied the left-hand inequality from (8) for T' and its refinement T, the middle inequality from (8) for T, and the right-hand inequality from (8) for T'' and its refinement T.

Consequently, the boundary terms of the obtained chain of inequalities (12) satisfy inequality (11). □

Darboux integrals

2.4B/32:30 (07:05)

5. There exist values $I^-(f) = \sup_T S^-_T(f)$, $I^+(f) = \inf_T S^+_T(f)$ and the following estimate holds for them:

$$I^-(f) \le I^+(f). \tag{13}$$

Proof.

Consider the previously proved inequality (11), fix the partition T'' in it, and consider the arbitrary partition T of the segment $[a, b]$ as the partition T':

$$S^-_T(f) \le S^+_{T''}(f).$$

This inequality means that the set of all lower Darboux sums over arbitrary partitions T is bounded from above by $S^+_{T''}(f)$. Therefore, the set of all lower Darboux sums for the function f is bounded from above, which means that it has the least upper bound $I^-(f)$.

Since the value $S^+_{T''}(f)$ is the upper bound for the set of all lower Darboux sums and the value $I^-(f)$ is the least upper bound for these sums, we obtain the following inequality:

$$I^-(f) \le S^+_{T''}(f).$$

In the last inequality, we can assume that T'' is an arbitrary partition of the segment $[a, b]$. Therefore, the set of all upper Darboux sums of the function f over an arbitrary partition T'' is bounded from below by the value $I^-(f)$. So, this set has the greatest lower bound $I^+(f)$.

The estimate (13) follows from the fact that the quantity $I^-(f)$ is the lower bound for the set of all upper Darboux sums and the value $I^+(f)$ is the greatest lower bound for these sums.

Definition.

The values $I^-(f)$ and $I^+(f)$ are called the *lower and upper Darboux integrals* for the function f on the segment $[a, b]$, respectively. Thus, by virtue of property 5, any bounded function has the lower and upper Darboux integrals and inequality (13) holds for them.

Integrability criterion in terms of Darboux sums

Formulation of the integrability criterion

`2.5A/00:00 (09:13)`

Theorem (integrability criterion in terms of Darboux sums).

The function f is integrable on the segment $[a, b]$ if and only if two conditions are satisfied:

1) f is bounded on $[a, b]$,

2) $\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, T, l(T) < \delta, \quad S^+_T(f) - S^-_T(f) < \varepsilon.$

Remark.

Condition 2 of the theorem can be written as follows:

$$\lim_{l(T)\to 0}\bigl(S_T^+(f)-S_T^-(f)\bigr)=0.$$

Proof of necessity

2.5A/09:13 (09:43)

Given: the function f is integrable on $[a,b]$. Prove: conditions 1 and 2 are satisfied.

The validity of condition 1 follows from the necessary condition for integrability. It remains for us to prove the validity of condition 2.

Since the function f is integrable, the following limit exists:

$$\lim_{l(T)\to 0,\forall\,\xi}\sigma_T(f,\xi)=I.$$

We choose some value of $\varepsilon>0$. Due to the integrability of the function f, we obtain

$$\exists\,\delta>0\quad\forall\,T,l(T)<\delta,\quad\forall\,\xi\quad|\sigma_T(f,\xi)-I|<\frac{\varepsilon}{3}.\tag{14}$$

Let us show that the choice of the same value δ ensures the fulfillment of condition 2 of the theorem.

Transform estimate (14) as follows:

$$I-\frac{\varepsilon}{3}<\sigma_T(f,\xi)<I+\frac{\varepsilon}{3}.\tag{15}$$

This double estimate is valid for any sample ξ. Thus, we have lower and upper bounds for the set of integral sums $\sigma_T(f,\xi)$ for a fixed partition T and any sample ξ.

By property 2 of Darboux sums, we have

$$S_T^+(f)=\sup_{\xi}\sigma_T(f,\xi),\quad S_T^-(f)=\inf_{\xi}\sigma_T(f,\xi).$$

Since the expressions $I-\frac{\varepsilon}{3}$ and $I+\frac{\varepsilon}{3}$ are, by virtue of (15), the lower and upper bounds of the integral sums, respectively, and $S_T^-(f)$ and $S_T^+(f)$ are, by virtue of property 2 of Darboux sums, the greatest lower bound and the least upper bound of the integral sums, we obtain the following chain of inequalities:

$$I-\frac{\varepsilon}{3}\le S_T^-(f)\le S_T^+(f)\le I+\frac{\varepsilon}{3}.\tag{16}$$

Since the distance between the internal terms of the triple inequality (16) cannot exceed the distance between its external terms, the following estimate follows from this triple inequality:

$$S_T^+(f) - S_T^-(f) \le \left(I + \frac{\varepsilon}{3}\right) - \left(I - \frac{\varepsilon}{3}\right). \tag{17}$$

Simplify the right-hand side:

$$\left(I + \frac{\varepsilon}{3}\right) - \left(I - \frac{\varepsilon}{3}\right) = \frac{2\varepsilon}{3} < \varepsilon.$$

Thus, estimate (17) can be rewritten in the form

$$S_T^+(f) - S_T^-(f) < \varepsilon.$$

We obtained an estimate from condition 2 of the theorem. The necessity is proven.

Proof of sufficiency

2.5A/18:56 (14:25)

Given: conditions 1 and 2 are satisfied. Prove: the function f is integrable on $[a, b]$.

Condition 1 (i. e., the boundedness of the function f) is required only to guarantee the existence of lower and upper Darboux sums for the function f.

By condition 2, for any $\varepsilon > 0$, there exists a value $\delta > 0$ such that, for all partitions T with mesh $l(T) < \delta$, the estimate holds:

$$S_T^+(f) - S_T^-(f) < \varepsilon. \tag{18}$$

On the other hand, for any partition T, by virtue of property 5 of the Darboux sums, the following triple estimate holds:

$$S_T^-(f) \le I^-(f) \le I^+(f) \le S_T^+(f).$$

This estimate implies the inequality

$$I^+(f) - I^-(f) \le S_T^+(f) - S_T^-(f).$$

Given (18), we obtain

$$I^+(f) - I^-(f) < \varepsilon.$$

The left-hand side of the resulting inequality does not depend on ε, therefore, this inequality can be true for arbitrary $\varepsilon > 0$ only if $I^+(f) - I^-(f) = 0$, i. e., $I^+(f) = I^-(f)$.

Thus, we have proved that, under condition 2 of the theorem, the lower and upper Darboux integrals coincide. We denote their value by I and show that the value of I is equal to the integral of the function f on the segment $[a, b]$, i. e., that the following condition is true:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, T, l(T) < \delta, \quad \forall\, \xi \quad |\sigma_T(f, \xi) - I| < \varepsilon. \tag{19}$$

We choose the value $\varepsilon > 0$ and select the value $\delta > 0$ from it using condition 2 of the theorem. Then, for any partition T satisfying the condition $l(T) < \delta$, estimate (18) holds.

Using property 1 of the Darboux sums, we obtain

$$\forall \xi \quad S_T^-(f) \le \sigma_T(f, \xi) \le S_T^+(f).$$

In addition, by virtue of property 5 of the Darboux sums, we have

$$S_T^-(f) \le I \le S_T^+(f).$$

Thus, the values of $\sigma_T(f, \xi)$ and I are between the values of $S_T^-(f)$ and $S_T^+(f)$. Therefore, the following estimate is true:

$$\forall \xi \quad |\sigma_T(f, \xi) - I| \le S_T^+(f) - S_T^-(f).$$

Given that $S_T^+(f) - S_T^-(f) < \varepsilon$, we finally get

$$\forall \xi \quad |\sigma_T(f, \xi) - I| < \varepsilon.$$

We proved that condition (19) is satisfied for the function f; therefore, the function f is integrable. □

Corollary of the criterion and an example of a non-integrable function 2.5A/33:21 (08:48)

Corollary.

If the function f is integrable on the segment $[a, b]$, then its upper and lower Darboux integrals coincide and, moreover, they are equal to the integral of the function f over the segment $[a, b]$.

Proof.

If the function is integrable, then condition 2 of the theorem is fulfilled for it, which implies both the coincidence of the upper and lower Darboux integrals and their equality to the integral of this function (see the proof of sufficiency). □

Remark.

It follows from the corollary that if the upper and lower Darboux integrals are different, then the function is not integrable.

An example of a bounded function that is not integrable.

We define the following function, called the *Dirichlet function*:

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Thus, the Dirichlet function is equal to 1 at rational points and is equal to 0 at irrational points.

This function is bounded. However, it is not integrable on any segment $[a, b]$ of nonzero length. We show this for the segment $[0, 1]$.

It is easy to prove that any segment of nonzero length contains both rational and irrational numbers. This means that, for any partition T of the segment $[0, 1]$, the following relations hold:

$$m_i = \inf_{x \in \Delta_i} D(x) = 0, \quad M_i = \sup_{x \in \Delta_i} D(x) = 1.$$

Then, for Darboux sums of the Dirichlet function over any partition T of the segment $[0, 1]$, we have

$$S_T^+(D) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1,$$

$$S_T^-(f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0.$$

Similar relations hold for Darboux integrals:

$$I^+(D) = \inf_T S_T^+(D) = \inf_T 1 = 1,$$

$$I^-(D) = \sup_T S_T^-(D) = \sup_T 0 = 0.$$

We proved that $I^-(D) \neq I^+(D)$, therefore, the Dirichlet function is not integrable on the interval $[0, 1]$.

6. Classes of integrable functions. Properties of a definite integral

Classes of integrable functions

The simplest example of an integrable function: a constant function

2.5B/00:00 (02:05)

Consider the constant function $f(x) = c$ and show that it is integrable on any segment $[a, b]$.

To do this, we calculate the integral sum for the function f on this segment:

$$\sigma_T(\xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = c\sum_{i=1}^{n} \Delta x_i = c(b-a).$$

Thus, for any partition T and any sample ξ, the integral sum takes the same value, therefore, when passing to the limit as $l(T) \to 0$, $\forall\,\xi$, this value will not change.

We have proved that

$$\int_a^b c\,dx = c(b-a).$$

Oscillation of a function and its use in integrability criterion

2.5B/02:05 (04:12)

We noted earlier that the condition for integrability criterion in terms of Darboux sums can be written as follows:

$$\lim_{l(T)\to 0}(S_T^+ - S_T^-) = 0.$$

Using the definition of Darboux sums, we can transform an expression under the limit sign:

$$S_T^+ - S_T^- = \sum_{i=1}^{n} M_i\Delta x_i - \sum_{i=1}^{n} m_i\Delta x_i = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i.$$

Under the sum sign, the expression $M_i - m_i$ arises, which determines the maximum difference of the values of the function f on the segment Δ_i. This characteristic is called the *oscillation* of the function f on the segment Δ_i and is denoted by $\omega_i(f)$:

$$\omega_i(f) \stackrel{\text{def}}{=} M_i - m_i.$$

Thus, the condition from the criterion of integrability of the function can be represented as follows:

$$\lim_{l(T)\to 0} \sum_{i=1}^{n} \omega_i(f)\Delta x_i = 0.$$

REMARK.

It can be proved that the following formula holds for the oscillation of a function:

$$\omega_i(f) = \sup_{x', x'' \in \Delta_i} |f(x') - f(x'')|. \qquad (1)$$

We will use this formula to prove the integrability of the product of functions.

Integrability of continuous functions

2.5B/06:17 (13:33)

THEOREM (INTEGRABILITY THEOREM FOR CONTINUOUS FUNCTIONS).

If the function is continuous on a segment, then it is integrable on this segment.

REMARK.

Continuity is not a necessary condition for integrability. An integrable function may have points of discontinuity.

PROOF.

Let the function f be continuous on $[a, b]$. Let us prove that the conditions of the integrability criterion are satisfied for it.

Condition 1 of the criterion (boundedness of a function on $[a, b]$) follows from the first Weierstrass theorem, which states that any function continuous on a segment is bounded on this segment.

To prove condition 2 of the criterion, we use Cantor's theorem, which states that a function continuous on an segment is uniformly continuous on this segment.

Let us write the definition of uniform continuity for the function f on the segment $[a, b]$ in the following form:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x', x'' \in [a, b], |x' - x''| < \delta,$$
$$|f(x') - f(x'')| < \frac{\varepsilon}{b - a}. \qquad (2)$$

We choose the value $\varepsilon > 0$, select the value $\delta > 0$ from (2) and show that condition 2 of the integrability criterion will be satisfied for the given value δ, i. e., that, for all partitions T such that $l(T) < \delta$, the estimate $S_T^+ - S_T^- < \varepsilon$ holds.

So, let us choose some partition T satisfying the condition $l(T) < \delta$.

Let $x', x'' \in \Delta_i$, where Δ_i is some segment defined by the partition T, $i = 1, \ldots, n$. Obviously, $|x' - x''| \leq \Delta x_i$. Considering that the mesh of the partition $l(T)$ is the maximum length of the segments Δ_i and, by the condition, $l(T) < \delta$, we obtain the following chain of inequalities:

$$|x' - x''| \leq \Delta x_i \leq l(T) < \delta.$$

Therefore, if $x', x'' \in \Delta_i$, then the inequality $|x' - x''| < \delta$ holds for these points. Then, by the condition of uniform continuity (2), $|f(x') - f(x'')| < \frac{\varepsilon}{b-a}$.

Since the points x' and x'' can be arbitrarily selected on the segment Δ_i, we choose them so that the maximum value of the function f on the segment Δ_i is reached at the point x', and the minimum value of the function f on this segment is reached at the point x''. Such points exist by virtue of the second Weierstrass theorem, which states that a function continuous on the segment takes its maximum and minimum value:

$$f(x') = \max_{x \in \Delta_i} f(x) = M_i, \quad f(x'') = \min_{x \in \Delta_i} f(x) = m_i.$$

Since the estimate $|x' - x''| < \delta$ is also valid for these points, which means that the estimate $|f(x') - f(x'')| < \frac{\varepsilon}{b-a}$ holds, we obtain

$$|M_i - m_i| < \frac{\varepsilon}{b-a}.$$

In this estimate it is not necessary to use the absolute value sign, since the difference $M_i - m_i$ is always non-negative.

So, we have proved that if for a given $\varepsilon > 0$, we choose the value $\delta > 0$ from condition (2), then, for any partition T for which $l(T) < \delta$, the following relation holds:

$$M_i - m_i < \frac{\varepsilon}{b-a}, \quad i = 1, \ldots, n.$$

Then, for the difference $S_T^+ - S_T^-$, we get

$$S_T^+ - S_T^- = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i <$$
$$< \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \Delta x_i = \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b-a}(b-a).$$

We got the estimate $S_T^+ - S_T^- < \varepsilon$. Thus, condition 2 of the integrability criterion is also satisfied, and, by virtue of this criterion, the function f is integrable on the segment $[a, b]$. $\square$

Integrability of monotone functions

2.5B/19:50 (10:18)

Theorem (integrability theorem for monotone functions).

If the function is monotone on the segment, then it is integrable on this segment.

Remark.

This fact does not follow from the previous theorem, since a monotone function can have a finite or even infinite number of discontinuity points (of the first kind).

Proof.

Let the function f be monotone on the segment $[a, b]$. For definiteness, we assume that f is non-decreasing on $[a, b]$. Let us prove its integrability using the integrability criterion in terms of Darboux sums.

First, we prove the validity of condition 1 of the criterion, i. e., let us prove the boundedness of the function f.

Since the function f is non-decreasing, we have

$$\forall\, x \in [a, b] \quad f(a) \le f(x) \le f(b).$$

The resulting double inequality means that the function f is bounded on $[a, b]$.

Now we prove the validity of condition 2 of the criterion. This condition can be represented as

$$\lim_{l(T)\to 0} (S_T^+ - S_T^-) = 0.$$

Choose some partition T. Since the function f is non-decreasing, we have for any segment Δ_i, $i = 1, \dots, n$,

$$m_i = \min_{x\in\Delta_i} f(x) = f(x_{i-1}), \quad M_i = \max_{x\in\Delta_i} f(x) = f(x_i).$$

Then the difference $S_T^+ - S_T^-$ can be transformed as follows:

$$S_T^+ - S_T^- = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} \bigl(f(x_i) - f(x_{i-1})\bigr)\Delta x_i.$$

By the definition of the mesh of the partition, we get $\Delta x_i \le l(T)$. Since all factors are non-negative, the following estimate holds:

$$\sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big)\Delta x_i \le \sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big)l(T) =$$
$$= l(T)\sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big).$$

We write out the terms of the last sum in the reverse order and reduce similar terms:

$$\sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big) = \big(f(x_n)-f(x_{n-1})\big) + \big(f(x_{n-1})-f(x_{n-2})\big) +$$
$$+ \big(f(x_{n-2})-f(x_{n-3})\big) + \dots + \big(f(x_2)-f(x_1)\big) + \big(f(x_1)-f(x_0)\big) =$$
$$= f(x_n)-f(x_0).$$

Thus, we obtained the following double inequality (in which we took into account that $x_0 = a$, $x_n = b$):

$$0 \le S_T^+ - S_T^- \le \big(f(b)-f(a)\big)l(T).$$

If we pass to the limit in the resulting double inequality as $l(T) \to 0$, then the left-hand and right-hand sides of the inequality will be 0; therefore, by virtue of the theorem on passing to the limit in inequalities, the difference $S_T^+ - S_T^-$ will also be 0.

So, we have proved that condition 2 of the integrability criterion also holds. By virtue of this criterion, the function f is integrable on the segment $[a, b]$. □

Integral properties associated with integrands

Linearity of a definite integral

2.5B/30:08 (09:46)

Theorem 1 (on linearity of a definite integral with respect to the integrand).

Let the functions f and g be integrable on the segment $[a, b]$, $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha f + \beta g$ is also integrable on $[a, b]$ and the following equality holds:

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx. \tag{3}$$

Proof.

Let us prove this fact using the definition of a definite integral. We write down the integral sum for the function $\alpha f + \beta g$ and transform it:

$$\sigma_T(\alpha f + \beta g, \xi) = \sum_{i=1}^{n} \bigl(\alpha f(\xi_i) + \beta g(\xi_i)\bigr)\Delta x_i =$$

$$= \alpha \sum_{i=1}^{n} f(\xi_i)\Delta x_i + \beta \sum_{i=1}^{n} g(\xi_i)\Delta x_i = \alpha\sigma_T(f, \xi) + \beta\sigma_T(g, \xi).$$

We have obtained the following relation, which is valid for any partition T and any sample ξ:

$$\sigma_T(\alpha f + \beta g, \xi) = \alpha\sigma_T(f, \xi) + \beta\sigma_T(g, \xi). \tag{4}$$

Since, by condition, the functions f and g are integrable on $[a, b]$, the limits $\lim_{l(T)\to 0, \forall\,\xi} \sigma_T(f, \xi)$ and $\lim_{l(T)\to 0, \forall\,\xi} \sigma_T(g, \xi)$ exist and are equal to $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$, respectively.

Then the limit on the right-hand side of equality (4), as $l(T) \to 0$, $\forall\,\xi$, exists and equals $\alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$. Therefore, for the left-hand side of equality (4), there also exists a limit with the same value. Thus, we simultaneously proved the integrability of the function $\alpha f + \beta g$ and the validity of formula (3). □

Integrability of the product 2.6A/00:00 (16:44)

Theorem 2 (on integrability of the product of integrable functions).

Let the functions f and g be integrable on the segment $[a, b]$. Then the function fg is also integrable on $[a, b]$.

Remark.

In this case, we can only establish the fact of integrability, since there is no formula expressing the integral of the product of functions in terms of the integrals of the factors.

Proof.

Let us use the integrability criterion in terms of the oscillation of a function, which can be formulated as follows: the function f is integrable if and only if it is bounded and $\sum_{i=1}^{n} \omega_i(f)\Delta x_i \to 0$ as $l(T) \to 0$. To find the oscillation of the function, we apply the formula (1).

First, we note that if the functions f and g are integrable, then they are bounded on $[a, b]$ due to the necessary integrability condition:

$$\exists\, C > 0 \quad \forall\, x \in [a, b] \quad |f(x)| \le C, \quad |g(x)| \le C. \tag{5}$$

Therefore, the product fg is also bounded.

Taking into account (5), we transform the absolute value of the difference $f(x')g(x') - f(x'')g(x'')$ in such a way that it allows us to estimate the oscillation of the product fg through the oscillations of the factors f and g:

$$\begin{aligned}&|f(x')g(x') - f(x'')g(x'')| = \\ &\quad = |f(x')g(x') - f(x'')g(x') + f(x'')g(x') - f(x'')g(x'')| \le \\ &\quad \le |g(x')||f(x') - f(x'')| + |f(x'')||g(x') - g(x'')| \le \\ &\quad \le C\big(|f(x') - f(x'')| + |g(x') - g(x'')|\big). \qquad (6)\end{aligned}$$

We assume that $x', x'' \in \Delta_i$, $i = 1, \ldots, n$. Then, by virtue of (1), we obtain

$$|f(x') - f(x'')| \le \sup_{x',x''\in\Delta_i} |f(x') - f(x'')| = \omega_i(f).$$

Similarly,

$$|g(x') - g(x'')| \le \omega_i(g).$$

Given the estimates obtained, relation (6) can be written in the form

$$\forall\, x', x'' \in \Delta_i \quad |f(x')g(x') - f(x'')g(x'')| \le C\big(\omega_i(f) + \omega_i(g)\big).$$

We have obtained an upper bound for the set of differences of the form $|f(x')g(x') - f(x'')g(x'')|$ when $x', x'' \in \Delta_i$. Therefore, this set is bounded from above and we have the following estimate for its least upper bound:

$$\sup_{x',x''\in\Delta_i} |f(x')g(x') - f(x'')g(x'')| \le C\big(\omega_i(f) + \omega_i(g)\big).$$

The expression on the left-hand side of the last inequality is, by virtue of (1), an oscillation of the function fg. Thus, the resulting inequality takes the form

$$\omega_i(fg) \le C\big(\omega_i(f) + \omega_i(g)\big).$$

So, we have estimated the oscillation of the product fg through the oscillations of the factors. It remains to multiply both sides by Δx_i and summarize these inequalities by $i = 1, \ldots, n$:

$$\sum_{i=1}^{n} \omega_i(fg)\Delta x_i \le C\Big(\sum_{i=1}^{n} \omega_i(f)\Delta x_i + \sum_{i=1}^{n} \omega_i(g)\Delta x_i\Big).$$

Since, by condition, the functions f and g are integrable on $[a, b]$, we obtain, by the necessary condition of the integrability criterion in terms of the oscillation of the function, that each term on the right-hand side of the inequality approaches 0 as $l(T) \to 0$.

Consequently, the quantity indicated on the left side of the inequality also approaches 0 by virtue of the theorem on passing to the limit in inequalities. Therefore, by virtue of the sufficient condition for the integrability criterion, the function fg is integrable on $[a, b]$. □

Properties associated with integration segments

Integrability on a nested segment

2.6A/16:44 (07:03)

THEOREM 3 (ON INTEGRABILITY ON A NESTED SEGMENT).

If the function f is integrable on the segment $[a, b]$, then it is integrable on any segment $[c, d] \subset [a, b]$.

PROOF.

It is enough for us to prove, by virtue of the integrability criterion in terms of the oscillation of the function, that

$$\sum_{T} \omega_i(f)\Delta x_i \to 0, \quad l(T) \to 0. \tag{7}$$

Here, T denotes the partition of the segment $[c, d]$. To make the notation more clear, we used the partition T, according to which the segments Δ_i are constructed, as the summation parameter.

For any partition T, we can add to it new points in such a way as to obtain a partition of the original segment $[a, b]$ as a result. We will denote the resulting partition of the segment $[a, b]$ by T' and we will use the index k to indicate the segments obtained for this partition: Δ_k (such a notation allows us to distinguish these segments from the segments connected with the partition T and marked with the index i). We require that the mesh of the constructed partition T' coincides with $l(T)$: $l(T') = l(T)$. This can be satisfied by choosing new points so that neighboring points are located at a distance not exceeding $l(T)$.

If we consider all possible partitions T' constructed on the basis of partitions T and pass to the limit as $l(T')$ approaches 0, then the mesh of partitions T will also approach 0.

Since, by condition, the function f is integrable on $[a, b]$, we obtain, by virtue of the necessary part of the integrability criterion in terms of the oscillation of the function, that

$$\sum_{T'} \omega_k(f)\Delta x_k \to 0, \quad l(T') \to 0. \tag{8}$$

Note that the integrability criterion assumes that the indicated limit relation is valid for all possible partitions of the interval $[a, b]$. But if this relation is valid for all partitions, then it remains valid for a part of these partitions, namely, a part that is constructed on the basis of partitions T of segment $[c, d]$ as described above.

Since the sum $\sum_{T'} \omega_k(f)\Delta x_k$ contains all terms from the sum $\sum_T \omega_i(f)\Delta x_i$, as well as some additional non-negative terms, corresponding to the segments Δ_k not lying on $[c, d]$, the estimate holds:

$$\sum_T \omega_i(f)\Delta x_i \le \sum_{T'} \omega_k(f)\Delta x_k. \tag{9}$$

It follows from (8) and (9) that $\sum_T \omega_i(f)\Delta x_i \to 0$ as $l(T') \to 0$. Since, by construction, $l(T') = l(T)$, we obtain that relation (7) also holds. □

The first theorem on the additivity of a definite integral with respect to the integration segment 2.6A/23:47 (06:27)

Theorem 4 (the first theorem on the additivity of a definite integral with respect to the integration segment).

Let the function f be integrable on $[a, b]$, $c \in (a, b)$ (note that, by virtue of Theorem 3, this function is integrable on the segments $[a, c]$ and $[c, b]$). Then the following equality holds:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{10}$$

Proof.

Let T' be some partition of the segment $[a, c]$, T'' be some partition of the segment $[c, b]$. Then $T = T' \cup T''$ is a partition of the segment $[a, b]$. The partition T necessarily contains the point c and, in addition, we have $l(T) \to 0$ as $l(T') \to 0$ and $l(T'') \to 0$.

Let ξ' and ξ'' be the samples corresponding to the partitions T' and T''. By ξ we denote the sample, which is the union of ξ' and ξ''; this sample corresponds to the partition T.

Then, for the integral sums corresponding to the constructed partitions and samples, the following equality holds:

$$\sigma_T(f, \xi) = \sigma_{T'}(f, \xi') + \sigma_{T''}(f, \xi'').$$

We pass to the limit as $l(T') \to 0$, $\forall\, \xi'$, and $l(T'') \to 0$, $\forall\, \xi''$. By virtue of the already proved integrability of the function f on $[a, c]$ and $[c, b]$, we obtain that the right-hand side of the equality approaches $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

On the other hand, since the function f is integrable on $[a, b]$, we get that the limit of integral sums exists (and is equal to $\int_a^b f(x)\,dx$) for any partitions whose mesh approaches 0 and for any samples related to these partitions. But then the same will be true for the part of the possible partitions T that are constructed on the basis of the partitions T', T'' such that $l(T') \to 0$, $\forall\, \xi'$, and $l(T'') \to 0$, $\forall\, \xi''$.

Passing to the limit in both sides of the previous equality, we obtain the proved relation (10). □

Remark.

The converse statement, which we accept without proof, is also true: if the function is integrable on the segments $[a, c]$ and $[c, b]$, then it is integrable on the segment $[a, b]$ and equality (10) holds. This fact implies that any function that has a finite number of discontinuities of the first kind on the segment $[a, b]$ is integrable on this segment.

The second theorem on the additivity of a definite integral with respect to the integration segment

2.6A/30:14 (11:39)

Definition.

We assume that the integral of any function defined at a over a segment of zero length $[a, a]$ is 0:

$$\int_a^a f(x)\,dx \stackrel{\text{def}}{=} 0.$$

In addition, we define the integral from b to a for $a < b$ as follows:

$$\int_b^a f(x)\,dx \stackrel{\text{def}}{=} -\int_a^b f(x)\,dx.$$

This is a quite natural definition, which follows from the initial definition of a definite integral if we allow the situation $x_{i-1} > x_i$ (for which $\Delta x_i < 0$).

So, we can say that if we swap the limits of integration, then the sign of the integral changes to the opposite.

Theorem 5 (the second theorem on the additivity of a definite integral with respect to the integration segment).

Let the function f be integrable on $[a, b]$, $c_1, c_2, c_3 \in [a, b]$. Then the equality holds:

$$\int_{c_1}^{c_3} f(x)\,dx = \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx. \tag{11}$$

Proof.

Let us prove equality (11) for one of the cases of the location of the points c_1, c_2, c_3 that is different from the case $c_1 < c_2 < c_3$, which is already considered in Theorem 4.

Let, for example, $c_2 < c_1 < c_3$. By virtue of Theorem 4, we have

$$\int_{c_2}^{c_3} f(x)\,dx = \int_{c_2}^{c_1} f(x)\,dx + \int_{c_1}^{c_3} f(x)\,dx.$$

In the obtained relation, we transform the integrals so that their limits correspond to the limits indicated in (11). In this case, we only need to transform the integral from c_2 to c_1, changing its sign:

$$\int_{c_2}^{c_3} f(x)\,dx = -\int_{c_1}^{c_2} f(x)\,dx + \int_{c_1}^{c_3} f(x)\,dx.$$

If we transfer the integral preceded by a minus sign to another part of the equality and swap the left-hand and right-hand sides of this equality, then we obtain (11).

Any other arrangement of points c_1, c_2, c_3 can be analyzed in a similar way. For example, for the case $c_3 < c_2 < c_1$, we have

$$\int_{c_3}^{c_1} f(x)\,dx = \int_{c_3}^{c_2} f(x)\,dx + \int_{c_2}^{c_1} f(x)\,dx,$$

$$-\int_{c_1}^{c_3} f(x)\,dx = -\int_{c_2}^{c_3} f(x)\,dx - \int_{c_1}^{c_2} f(x)\,dx,$$

Multiplying the resulting equality by -1, we obtain (11). It is even easier to analyze situations in which some points coincide. □

Estimates for integrals

Simple estimates of integrals

2.6A/41:53 (01:17), 2.6B/00:00 (06:47)

Theorem 6 (on the non-negativity of the integral of a non-negative function).

If the function f is integrable on $[a, b]$ and $\forall\, x \in [a, b]$ $f(x) \ge 0$, then

$$\int_a^b f(x)\,dx \ge 0. \tag{12}$$

Proof.

Consider the integral sum for some partition T and a sample ξ:

$$\sigma_T(f,\xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

Since $\Delta x_i > 0$ and, by condition, $f(\xi_i) \ge 0$, all terms of this sum are non-negative, therefore the integral sum itself is non-negative too:

$$\sigma_T(f,\xi) \ge 0.$$

When passing to the limit as $l(T) \to 0$, $\forall\,\xi$, the sign of the non-strict inequality is preserved, therefore estimate (12) holds. □

Theorem 7 (on the comparison of integrals).

If the functions f and g are integrable on $[a,b]$ and $\forall\,x \in [a,b]$ $f(x) \le g(x)$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx. \tag{13}$$

Proof.

We use the previously proved Theorems 1 and 6. Let us introduce the auxiliary function $h(x) = g(x) - f(x)$. Obviously, this function is non-negative. In addition, by virtue of Theorem 1, this function is integrable; moreover,

$$\int_a^b h(x)\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx.$$

According to Theorem 6, the left-hand side of the resulting equality is non-negative:

$$\int_a^b h(x)\,dx \ge 0.$$

Therefore, the right-hand side is also non-negative, therefore estimate (13) holds. □

Corollary.

If the function f is integrable on $[a,b]$ and $\forall\,x \in [a,b]$ $m \le f(x) \le M$ for some $m, M \in \mathbb{R}$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \tag{14}$$

Proof.

Earlier, we established that the constant function $f(x) = c$ is integrable on any interval and

$$\int_a^b c\,dx = c(b-a).$$

We apply Theorem 7 to the double inequality $m \le f(x) \le M$:

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx.$$

Given the formula for the integral of the constant, we obtain relation (14). □

Integral of a positive continuous function 2.6B/06:47 (11:25)

Theorem 8 (on the integral of a positive continuous function).

Let the function f be integrable and non-negative on $[a,b]$. Also suppose that the function f is continuous at the point $c \in [a,b]$ and, moreover, $f(c) > 0$. Then

$$\int_a^b f(x)\,dx > 0.$$

Proof.

We use the simplest property of a continuous function: if the function is continuous at the point c and takes a positive value at it, then there exists a neighborhood of this point at which the function remains positive.

If we denote $f(c) = D > 0$, then it can be argued that there exists a neighborhood U_c^δ such that the estimate $f(x) \ge \frac{D}{2}$ holds for any point $x \in U_c^\delta$.

We assume that the neighborhood U_c^δ lies inside the segment $[a,b]$, and also that the estimate $f(x) \ge \frac{D}{2}$ is satisfied at the boundary of the neighborhood U_c^δ (otherwise, it's enough to simply reduce the neighborhood). Then the integral from a to b can be represented as the sum of three integrals:

$$\int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^b f(x)\,dx.$$

The first and third integrals on the right-hand side are non-negative by virtue of Theorem 6. Let us turn to the second integral. Since $\forall\, x \in [c-\delta, c+\delta]\ f(x) \ge \frac{D}{2}$, applying the corollary of Theorem 7, we obtain

$$\int_{c-\delta}^{c+\delta} f(x)\,dx \ge \frac{D}{2} \cdot \big(c+\delta-(c-\delta)\big) = D\delta > 0.$$

Thus, the second integral is positive. Therefore, the sum of the three integrals is also positive. □

COROLLARY.

If the function f is continuous on $[a,b]$ and $\forall\, x \in [a,b]$ $f(x) < M$, then

$$\int_a^b f(x)\,dx < M(b-a).$$

PROOF.

Consider the function $h(x) = M - f(x)$. This function is continuous and positive on $[a,b]$. Therefore, by the previous theorem, we obtain $\int_a^b h(x)\,dx > 0$. To get the required estimate, it remains to use the linearity of the integral and the formula for the integral of a constant function. □

Properties of the integral of the absolute value of a function 2.6B/18:12 (16:20)

THEOREM 9 (ON THE INTEGRAL OF THE ABSOLUTE VALUE OF A FUNCTION).

If the function f is integrable on $[a,b]$, then its absolute value $|f|$ is also integrable on $[a,b]$ and the estimate holds:

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx. \tag{15}$$

PROOF.

First, we prove the integrability of the function $|f|$. Let us use the lower bound for the difference $|t'-t''|$:

$$|t'-t''| \ge \bigl||t'|-|t''|\bigr|. \tag{16}$$

We choose the partition T of the segment $[a,b]$, choose some segment Δ_i defined by this partition, and write the estimate (16) for $f(x')$ and $f(x'')$ when $x', x'' \in \Delta_i$, swapping the left-hand and right-hand sides of the estimate:

$$\bigl||f(x')|-|f(x'')|\bigr| \le |f(x')-f(x'')|.$$

We will argue in the same way as in the proof of the integrability of the product (see Theorem 2). First, it is obvious that the right-hand side of the resulting inequality is bounded from above by the value $\sup_{x',x''\in\Delta_i}|f(x')-f(x'')|$, which is equal to the oscillation of the function f on the segment Δ_i. Therefore,

$$\bigl||f(x')|-|f(x'')|\bigr| \le \omega_i(f).$$

Further, since this estimate is valid for all $x', x'' \in \Delta_i$, we find that a similar estimate holds for the least upper boundary of the left-hand side:

$$\sup_{x',x''\in\Delta_i} \big||f(x')| - |f(x'')|\big| \le \omega_i(f).$$

The left-hand side of the last estimate is the oscillation of the function $|f|$:

$$\omega_i(|f|) \le \omega_i(f).$$

So, we have proved that the oscillation of the absolute value of a function does not exceed the oscillation of the function itself. It remains for us to multiply both sides of the resulting estimate by Δx_i and summarize the resulting inequalities for i from 1 to n:

$$\sum_{i=1}^{n} \omega_i(|f|)\Delta x_i \le \sum_{i=1}^{n} \omega_i(f)\Delta x_i.$$

This estimate is valid for an arbitrary partition T. Passing to the limit as $l(T) \to 0$ and taking into account that, by condition, the function f is integrable on $[a, b]$, we obtain, by virtue of the integrability criterion, that the right-hand side of the inequality approaches 0. Then, by virtue of the theorem on passing to the limit in inequalities, the left-hand side also approaches 0; therefore, due to the same integrability criterion, the function $|f|$ is also integrable on $[a, b]$. The first part of the theorem is proved.

Now let us turn to the proof of estimate (15). We choose an arbitrary partition T of the segment $[a, b]$ and a sample ξ, consider the absolute value of the integral sum for the function f, and transform it using a generalization of the triangle inequality $|t' + t''| \le |t'| + |t''|$ for the case of n terms:

$$|\sigma_T(f, \xi)| = \Big|\sum_{i=1}^{n} f(\xi_i)\Delta x_i\Big| \le \sum_{i=1}^{n} |f(\xi_i)|\Delta x_i.$$

On the right-hand side, we get the integral sum for the function $|f|$ over the same partition T and the sample ξ. Therefore,

$$|\sigma_T(f, \xi)| \le \sigma_T(|f|, \xi).$$

Since we have already proved that the function $|f|$ is integrable, the limits of the integral sums as $l(T) \to 0$, $\forall\, \xi$, exist both on the left-hand side and on the right-hand side. These limits are equal to the integrals of the corresponding functions and the same estimate holds for them. □

Remark.

The integrability of the absolute value of a function does not imply the integrability of the function itself. To prove this statement, it suffices to give an example. Consider the following function (which can be obtained from the Dirichlet function by stretching and shifting along the OY axis):

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ -1, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function, like the Dirichlet function, is not integrable on any segment of positive length, because for any segment $[a, b]$, its upper Darboux integral is $(b - a)$, and it differs from the lower Darboux integral equal to $-(b - a)$. At the same time, the absolute value of this function is a constant: $|f(x)| = 1$, and the constant is integrable on any interval.

Mean value theorems for definite integrals

The first mean value theorem

2.6B/34:32 (10:39)

Theorem 10 (the first mean value theorem).

Suppose that the functions f and g are integrable on $[a, b]$ and the following conditions are satisfied for them:

1) for the function f, a double estimate holds: $m \le f(x) \le M$, $x \in [a, b]$;

2) the function g preserves the sign on $[a, b]$, i. e., either $g(x) \ge 0$ for $x \in [a, b]$ or $g(x) \le 0$ for $x \in [a, b]$.

Then there exists a value $\mu \in [m, M]$ such that the following equality holds:

$$\int_a^b f(x)g(x)\,dx = \mu \int_a^b g(x)\,dx. \tag{17}$$

Proof.

First, we consider the case when $g(x) \ge 0$ for $x \in [a, b]$.

We multiply all the terms of the estimate from condition 1 by $g(x)$. The signs of inequality will not change, since, by our assumption, the function g is non-negative:

$$mg(x) \le f(x)g(x) \le Mg(x).$$

By virtue of Theorem 2, each of the obtained products is an integrable function. We integrate all the terms of the double inequality from a to b. By virtue of Theorem 7, the signs of inequality will not change. In addition, the constants m and M can be taken out of the signs of the integrals:

$$m \int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M \int_a^b g(x)\,dx.$$

Thus, we obtain the integral $\int_a^b g(x)\,dx$ on the left-hand and right-hand sides of the resulting double inequality.

If $\int_a^b g(x)\,dx = 0$, then the last double inequality takes the form $0 \le \int_a^b f(x)g(x)\,dx \le 0$, which implies that $\int_a^b f(x)g(x)\,dx = 0$. In this case, equality (17) is satisfied and any value from the interval $[m, M]$ can be taken as μ.

If $\int_a^b g(x)\,dx \ne 0$, then we can divide all parts of the double inequality by this nonzero value. As a result, we get

$$m \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le M.$$

Denote the obtained quotient of integrals by μ:

$$\mu = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}. \tag{18}$$

Thus, the double inequality $m \le \mu \le M$ holds for μ and, in addition, relation (18) can be transformed to (17) by multiplying both sides of the equality by $\int_a^b g(x)\,dx$.

So, we have proved the theorem for the case $g(x) \ge 0$.

Now suppose that $g(x) \le 0$ for $x \in [a, b]$. Consider the auxiliary function $\tilde{g}(x) = -g(x)$. The function $\tilde{g}(x)$ is non-negative: $\tilde{g}(x) \ge 0$ for $x \in [a, b]$ and the theorem has already been proved for the case of non-negative functions. Therefore, there exists a value $\mu \in [m, M]$ such that

$$\int_a^b f(x)\tilde{g}(x)\,dx = \mu \int_a^b \tilde{g}(x)\,dx.$$

Let's get back to the function $g(x)$:

$$\int_a^b f(x)\big(-g(x)\big)\,dx = \mu \int_a^b \big(-g(x)\big)\,dx.$$

To obtain equality (17), it suffices to put the signs "minus" behind the signs of the integrals and multiply both sides of the resulting equality by (-1). Thus, equality (17) is valid for the function $g(x)$ also in the case $g(x) \le 0$. $\square$

The second and the third mean value theorems

`2.7A/00:00 (12:56)`

Theorem 11 (the second mean value theorem).

Suppose that the functions f and g are defined on $[a, b]$ and the following conditions are satisfied for them:

1) the function f is continuous on $[a, b]$ (this condition immediately implies the integrability of the function f on $[a, b]$);

2) the function g is integrable on $[a, b]$ and preserves the sign on this segment, i. e., either $g(x) \geq 0$ for $x \in [a, b]$ or $g(x) \leq 0$ for $x \in [a, b]$.

Then there exists a point $c \in [a, b]$ such that the following equality holds:

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx. \tag{19}$$

Proof.

We use the already proved Theorem 10, for which all conditions are satisfied. In particular, since the function f is continuous on a segment, for it, by virtue of the first Weierstrass theorem, there exist numbers $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for $x \in [a, b]$ (note that the boundedness of the function f follows not only from the first Weierstrass theorem, but also from the necessary integrability condition).

As m and M, we can take the values $\inf_{x\in[a,b]} f(x)$ and $\sup_{x\in[a,b]} f(x)$, respectively:

$$m = \inf_{x\in[a,b]} f(x), \quad M = \sup_{x\in[a,b]} f(x).$$

By virtue of Theorem 10, there exists a value $\mu \in [m, M]$ for which equality (17) holds.

Since the function f is continuous on the segment $[a, b]$, we obtain, by virtue of the second Weierstrass theorem, that the values of m and M are reached at some points, i. e., there exist points $c_1, c_2 \in [a, b]$ for which the equalities $f(c_1) = m$, $f(c_2) = M$ hold.

By virtue of the corollary of the intermediate value theorem, for the function f, there exists a point c lying on a segment with endpoints c_1 and c_2, in which the function f takes the value μ: $f(c) = \mu$. Since $c_1, c_2 \in [a, b]$, we obtain that the point c also belongs to the segment $[a, b]$.

Substituting the value $f(c)$ into (17) instead of μ, we get equality (19). □

Theorem 12 (the third mean value theorem).

Let the function f be continuous on $[a, b]$. Then there exists a point $c \in [a, b]$ such that the following equality holds:

$$\int_a^b f(x)\,dx = f(c)(b - a). \tag{20}$$

Remark (geometric sense of the third mean value theorem).

Assume that $f(x) > 0$ for $x \in [a, b]$. We noted earlier that the value of a definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area of a curvilinear trapezoid bounded by the graph $y = f(x)$, the segment of the axis OX, and the lines $x = a$ and $x = b$ (this fact will be proved later when we give a rigorous definition of area). Formula (20) means that there exists a point $c \in [a, b]$ for which a rectangle with the base $[a, b]$ and the height $f(c)$ has an area equal to the area of this curvilinear trapezoid (Fig. 6).

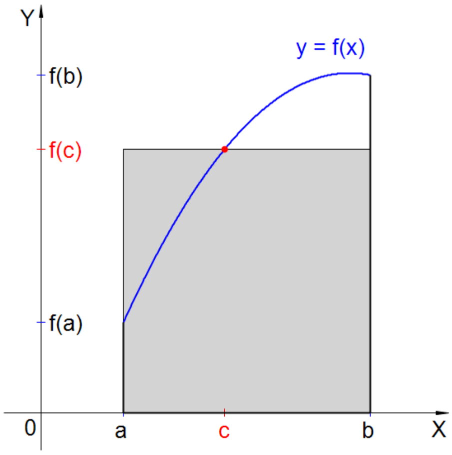

Fig. 6. Geometric sense of the third mean value theorem

Proof.

It is enough to use the second mean value theorem (Theorem 11) by putting $g(x) \equiv 1$ in it. Obviously, in this case the function $g(x)$ preserves the sign. Then

$$\int_a^b g(x)\,dx = \int_a^b dx = b - a.$$

Substituting the function $g(x) \equiv 1$ and the found value of the integral of this function into formula (19), we obtain (20). $\square$

7. Integral with a variable upper limit. Newton–Leibniz formula

Integral with a variable upper limit

Definition of an integral with a variable upper limit

2.7A/12:56 (04:01)

Definition.

Let the function f be integrable on the segment $[a, b]$. Then, by the integrability theorem on the embedded segment, it is integrable on the segment $[a, x]$ for any $x \in [a, b]$. Therefore, for any $x \in [a, b]$, there exists an integral $\int_a^x f(t)\,dt$. Denote this integral by $F(x)$:

$$F(x) \overset{\text{def}}{=} \int_a^x f(t)\,dt.$$

The function $F(x)$ is called an *integral with a variable upper limit*. Obviously, $F(a) = 0$ as an integral over a segment of zero length.

Theorem on the continuity of an integral with a variable upper limit

2.7A/16:57 (16:48)

Theorem 1 (on the continuity of an integral with a variable upper limit).

For any function f integrable on the segment $[a, b]$, its integral with a variable upper limit F is a continuous function on this segment.

Proof.

We choose an arbitrary point $x_0 \in [a, b]$ and prove that the function $F(x)$ is continuous at this point. For definiteness, we assume that $x_0 \in (a, b)$.

We want to prove that the limit of the function $F(x)$ as $x \to x_0$ is equal to the value of the function at the point x_0:

$$\lim_{\Delta x \to 0} \bigl(F(x_0 + \Delta x) - F(x_0)\bigr) = 0.$$

We assume that $x_0 + \Delta x \in [a, b]$; the increment Δx can be both positive and negative.

Consider the difference $|F(x_0+\Delta x)-F(x_0)|$ and transform it using the definition of an integral with a variable upper limit and the additivity theorem for the integral with respect to the integration segment:

$$|F(x_0+\Delta x)-F(x_0)| = \Big|\int_a^{x_0+\Delta x} f(t)\,dt - \int_a^{x_0} f(t)\,dt\Big| =$$

$$= \Big|\int_a^{x_0} f(t)\,dt + \int_{x_0}^{x_0+\Delta x} f(t)\,dt - \int_a^{x_0} f(t)\,dt\Big| = \Big|\int_{x_0}^{x_0+\Delta x} f(t)\,dt\Big|.$$

If $\Delta x > 0$, then the right-hand side of the resulting equality can be estimated using the property of the integral of the absolute value of a function:

$$\Big|\int_{x_0}^{x_0+\Delta x} f(t)\,dt\Big| \le \int_{x_0}^{x_0+\Delta x} |f(t)|\,dt.$$

A similar estimate can be obtained for the case $\Delta x < 0$; in this case, we must use the integral $\int_{x_0+\Delta x}^{x_0} |f(t)|\,dt$ on the right-hand side of the estimate.

If we do not impose additional conditions on Δx, then we can write the following version of the estimate, which is valid for both positive and negative values of Δx:

$$\Big|\int_{x_0}^{x_0+\Delta x} f(t)\,dt\Big| \le \Big|\int_{x_0}^{x_0+\Delta x} |f(t)|\,dt\Big|.$$

Since the function f is integrable, it is bounded:

$$\exists\, C > 0 \quad \forall\, x \in [a,b] \quad |f(x)| \le C.$$

If we assume that $\Delta x > 0$, then from the estimate $|f(x)| \le C$, using the theorem on the comparison of integrals, we obtain the following estimate:

$$\int_{x_0}^{x_0+\Delta x} |f(t)|\,dt \le \int_{x_0}^{x_0+\Delta x} C\,dt = C\Delta x\ .$$

If we do not impose additional conditions on Δx, then we have a similar estimate containing the absolute value of the integral and the absolute value of Δx:

$$\Big|\int_{x_0}^{x_0+\Delta x} |f(t)|\,dt\Big| \le C|\Delta x|.$$

Indeed, in the case $\Delta x < 0$ we get

$$\Big|\int_{x_0}^{x_0+\Delta x} |f(t)|\,dt\Big| = \int_{x_0+\Delta x}^{x_0} |f(t)|\,dt \le C(-\Delta x) = C|\Delta x|.$$

So, we started with the expression $|F(x_0+\Delta x)-F(x_0)|$ and, as a result, evaluated it from above with the expression $C|\Delta x|$:

$$|F(x_0 + \Delta x) - F(x_0)| \le C|\Delta x|.$$

If Δx approaches 0, then the right-hand side of the resulting estimate also approaches 0; therefore, by virtue of the theorem on passing to the limit in inequalities, the left-hand side also approaches 0. We have proved that the function F is continuous at an arbitrary point $x_0 \in (a, b)$.

The case when x_0 coincides with one of the endpoints of the initial segment is considered similarly, taking into account the fact that in this case the limit at the endpoints of the segment should be understood as one-sided limit (and it suffices to consider the positive increment Δx for the point a and negative increment for the point b). $\square$

Theorem on the differentiability of an integral with a variable upper limit and a continuous integrand

2.7A/33:45 (13:39), 2.7B/00:00 (04:04)

Theorem 2 (on the differentiability of an integral with a variable upper limit and a continuous integrand).

If the function f is integrable on the segment $[a, b]$ and continuous at the point $x_0 \in (a, b)$, then its integral with a variable upper limit F is a differentiable function at the point x_0 and the formula holds:

$$F'(x_0) = f(x_0).$$

Remarks.

1. It can be proved that the integral with a variable upper limit and an integrand continuous on $[a, b]$ is a differentiable function also at the endpoints of the segment $[a, b]$ if, in this case, we consider the one-sided derivative, that is, one-sided limit of the ratio of the increment of the function to the increment of the argument. However, we will not need this fact.

2. Theorems 1 and 2 indicate that the integration operation “improves” the properties of functions: if the original function is integrable, then its integral with a variable upper limit is a continuous function and if the original function is continuous, then its integral with a variable upper limit is a differentiable function.

Proof.

We need to prove that there exists a limit $\lim_{\Delta x \to 0} \frac{F(x_0+\Delta x)-F(x_0)}{\Delta x}$ and the limit value is $f(x_0)$. In other words, we need to prove that the following equality holds:

$$\lim_{\Delta x \to 0} \Big(\frac{F(x_0 + \Delta x) - F(x_0)}{\Delta x} - f(x_0) \Big) = 0\ .$$

Let us write down what the last equality means in the language ε–δ:

$$\forall\,\varepsilon > 0 \quad \exists\,\delta > 0 \quad \forall\,\Delta x, |\Delta x| < \delta,$$
$$\Big|\frac{F(x_0+\Delta x) - F(x_0)}{\Delta x} - f(x_0)\Big| < \varepsilon. \tag{1}$$

We select some value of $\varepsilon > 0$. By condition, the function f is continuous at the point x_0. This means that the following condition is true for the selected ε:

$$\exists\,\delta > 0 \quad \forall\,\Delta x, |\Delta x| < \delta, \quad |f(x_0+\Delta x) - f(x_0)| < \frac{\varepsilon}{2}. \tag{2}$$

Let us show that the value δ from condition (2) also ensures that condition (1) is satisfied, i. e., that the estimate $|f(x_0+\Delta x) - f(x_0)| < \frac{\varepsilon}{2}$ implies the validity of the estimate $\big|\frac{F(x_0+\Delta x)-F(x_0)}{\Delta x} - f(x_0)\big| < \varepsilon$.

Transform the difference $\big|\frac{F(x_0+\Delta x)-F(x_0)}{\Delta x} - f(x_0)\big|$ by taking out the factor $\frac{1}{\Delta x}$ and then use the definition of an integral with a variable upper limit:

$$\Big|\frac{1}{\Delta x}\Big(\int_a^{x_0+\Delta x} f(t)\,dt - \int_a^{x_0} f(t)\,dt - f(x_0)\Delta x\Big)\Big|. \tag{3}$$

In the proof of Theorem 1, we have already established that the difference $\int_a^{x_0+\Delta x} f(t)\,dt - \int_a^{x_0} f(t)\,dt$ is an integral from x_0 to $x_0+\Delta x$. Further, the factor Δx in the last term $f(x_0)\Delta x$ of expression (3) can be represented as the integral $\int_{x_0}^{x_0+\Delta x} dt$. Thus, expression (3) takes the form

$$\frac{1}{|\Delta x|}\Big|\int_{x_0}^{x_0+\Delta x} f(t)\,dt - f(x_0)\int_{x_0}^{x_0+\Delta x} dt\Big|.$$

Since the obtained integrals have the same integration limits, we can write the last expression as a single integral of the difference of functions:

$$\frac{1}{|\Delta x|}\Big|\int_{x_0}^{x_0+\Delta x} \big(f(t) - f(x_0)\big)\,dt\Big|.$$

This expression can be estimated from above by an expression containing the integral of the absolute value of the difference of functions:

$$\frac{1}{|\Delta x|}\Big|\int_{x_0}^{x_0+\Delta x} \big(f(t) - f(x_0)\big)\,dt\Big| \le \frac{1}{|\Delta x|}\Big|\int_{x_0}^{x_0+\Delta x} |f(t) - f(x_0)|\,dt\Big|. \tag{4}$$

We did not remove the absolute value sign for the integral, since the value of Δx can be either positive or negative.

Now let us turn to the estimate $|f(x_0+\Delta x) - f(x_0)| < \frac{\varepsilon}{2}$ from (2). The points x_0 and $x_0+\Delta x$ appearing in this estimate are the limits of the integral on the right-hand side of (4). Any point t located between x_0 and $x_0+\Delta x$ can be represented as $x_0+\delta'$, where $|\delta'| < |\Delta x|$. Since it is assumed in condition (2) that $|\Delta x| < \delta$, we see that the same estimate holds for $|\delta'|$: $|\delta'| < \delta$. This means that, for the point $t = x_0 + \delta'$, the estimate $|f(t) - f(x_0)| < \frac{\varepsilon}{2}$ is also valid.

Thus, the integrand on the right-hand side of (4) is estimated by $\frac{\varepsilon}{2}$ for all points t:

$$|f(t) - f(x_0)| < \frac{\varepsilon}{2}.$$

In this estimate, the "$<$" sign can be replaced with the "$\le$" sign. Using the theorem on the comparison of integrals, we obtain

$$\frac{1}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x} |f(t) - f(x_0)|\,dt\right| \le \frac{1}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x} \frac{\varepsilon}{2}\,dt\right| =$$
$$= \frac{1}{|\Delta x|}\cdot\frac{\varepsilon}{2}\left|\int_{x_0}^{x_0+\Delta x} dt\right| = \frac{1}{|\Delta x|}\cdot\frac{\varepsilon}{2}|\Delta x| = \frac{\varepsilon}{2} < \varepsilon.$$

So, we have proved that, for any values of Δx satisfying the condition $|\Delta x| < \delta$, the estimate holds:

$$\left|\frac{F(x_0+\Delta x) - F(x_0)}{\Delta x} - f(x_0)\right| < \varepsilon.$$

This means that condition (1) is satisfied. Therefore, the function $F(x)$ has a derivative at the point x_0 and this derivative is equal to $f(x_0)$. $\square$

Newton–Leibniz formula

Theorems on antiderivatives for continuous functions

2.7B/04:04 (06:23)

THEOREM 3 (ON THE EXISTENCE OF AN ANTIDERIVATIVE FOR A CONTINUOUS FUNCTION).

Any function f continuous on $[a,b]$ has an antiderivative on (a,b), which is an integral with a variable upper limit: $F(x) = \int_a^x f(t)\,dt$.

PROOF.

Since f is continuous on $[a,b]$, it follows from Theorem 2 that its integral with a variable upper limit F is a differentiable function on (a,b) and, for any point $x \in (a,b)$, the equality $F'(x) = f(x)$ is true. We have obtained that $F(x)$ satisfies the definition of the antiderivative of the function $f(x)$ for $x \in (a,b)$. $\square$

Corollary.

If f is a continuous function on $[a,b]$ and $\Phi(x)$ is its antiderivative on (a,b), then this antiderivative can be represented in the following form, where C is some constant:

$$\Phi(x) = \int_a^x f(t)\,dt + C. \tag{5}$$

Proof.

By Theorem 3, we obtain that the integral with a variable upper limit $\int_a^x f(t)\,dt$ is an antiderivative of the function f. The theorem on antiderivatives of a given function states that any two antiderivatives of the function f are different by some constant term C. □

Newton–Leibniz formula

2.7B/10:27 (05:55)

Theorem 4 (the fundamental theorem of calculus).

If the function f is continuous on $[a,b]$, $\Phi(x)$ is a continuous function on $[a,b]$, and Φ is the antiderivative of the function f on (a,b) (a function Φ with the indicated properties exists by virtue of Theorem 3), then

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a). \tag{6}$$

Formula (6) is called the *Newton–Leibniz formula.*

Remarks.

1. The antiderivative (and the indefinite integral) is defined by means of the differentiation operation, but the definite integral is defined by means of the limit of integral sums and therefore its definition is not related with the differentiation operation. Nevertheless, there is a relation between the operations of differentiation (that is, finding the derivative) and integration (that is, finding the definite integral), which is established by the Newton–Leibniz formula. That is why Theorem 4 is called the *fundamental theorem of calculus.*

2. The Newton–Leibniz formula (6) allows us to reduce the problem of finding a definite integral to the problem of finding the antiderivative of an integrand over a given interval.

3. Formula (6) remains valid for the case $a \ge b$.

4. Formula (6) is often written in the following form:

$$\int_a^b f(x)\,dx = \Phi(x)\Big|_a^b.$$

Proof.

By the corollary of Theorem 3, there exists a constant $C \in \mathbb{R}$ such that the antiderivative $\Phi(x)$ of the function $f(x)$ is representable in the form (5). Given this form, we find the values of the antiderivative $\Phi(x)$ at the endpoints of the segment $[a, b]$:

$$\Phi(a) = \int_a^a f(t)\,dt + C = C, \quad \Phi(b) = \int_a^b f(t)\,dt + C.$$

The difference $\Phi(b) - \Phi(a)$ is $\int_a^b f(t)\,dt + C - C = \int_a^b f(t)\,dt$. Thus, the Newton–Leibniz formula is proved, since the value of the integral does not depend on the choice of a letter for the integration parameter (in this case, x or t). $\square$

Additional techniques for calculating definite integrals

Change of variables in a definite integral 2.7B/16:22 (11:37)

Theorem 5 (on the change of variables in a definite integral).

Let the function $f(x)$ be continuous on $[a_0, b_0]$, the function $\varphi(t)$ act from (α_0, β_0) to (a_0, b_0) and be continuously differentiable on (α_0, β_0) (this means that the derivative $\varphi'(t)$ is defined and continuous on (α_0, β_0)). Let, in addition, $\alpha, \beta \in (\alpha_0, \beta_0)$ and $\varphi(\alpha) = a$, $\varphi(\beta) = b$ (moreover, $a, b \in (a_0, b_0)$ due to the properties of the function $\varphi(t)$).

Then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\big(\varphi(t)\big)\varphi'(t)\,dt. \tag{7}$$

Remark.

When using Theorem 5 to transform the integral $\int_a^b f(x)\,dx$, the function $\varphi(t)$ arises when we change the previous integration parameter x by the new parameter t: $x = \varphi(t)$. In this case, the differentials will be related as follows: $dx = \varphi'(t)\,dt$. This is similar to the relation used to change of variables in an indefinite integral. The only difference from the case of changing variables in an indefinite integral is that in the case of a definite integral, it is also necessary to change the integration limits using the relations $a = \varphi(\alpha)$, $b = \varphi(\beta)$.

Proof.

First, we note that the integrals on the left-hand and right-hand side of (7) exist, since their integrands are continuous over the entire integration segment.

Since the function $f(x)$ is continuous on $[a_0, b_0]$, it has an antiderivative on (a, b) by virtue of Theorem 3. Denote this antiderivative by $\Phi(x)$.

Let us differentiate the superposition $\Phi\bigl(\varphi(t)\bigr)$, which is defined for $t \in (\alpha_0, \beta_0)$:

$$\Bigl(\Phi\bigl(\varphi(t)\bigr)\Bigr)' = \Phi'(x)\big|_{x=\varphi(t)} \cdot \varphi'(t) = f\bigl(\varphi(t)\bigr)\varphi'(t).$$

Thus, the superposition $\Phi\bigl(\varphi(t)\bigr)$ is the antiderivative for the integrand of the right-hand side of equality (7) on the interval (α_0, β_0).

Now we apply the Newton–Leibniz formula for the integrals indicated on the left-hand side and the right-hand side of (7):

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a),$$

$$\int_\alpha^\beta f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt = \Phi\bigl(\varphi(\beta)\bigr) - \Phi\bigl(\varphi(\alpha)\bigr) = \Phi(b) - \Phi(a).$$

Since the right-hand sides of the obtained equalities coincide, we conclude that the left-hand sides coincide too, i. e., that equality (7) holds. □

Corollaries of the theorem on the change of variables in a definite integral

2.7B/27:59 (09:28)

1. Let the function f be an odd function defined and continuous on the segment $[-a, a]$. Then $\int_{-a}^{a} f(t)\,dt = 0$.

Proof.

We represent this integral as the sum of the integrals:

$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt. \tag{8}$$

In the first integral from the right-hand side of equality (8), we make the variable change $t = -x$. Then $dt = -dx$, the integration limits $-a$ and 0 will change by a and 0, respectively, and this integral will take the form

$$\int_{-a}^{0} f(t)\,dt = \int_{a}^{0} f(-x)\,(-dx).$$

Since the function f is odd, the equality $f(-x) = -f(x)$ holds. Thus,

$$\int_a^0 f(-x)\,(-dx) = \int_a^0 \big(-f(x)\big)\,(-dx) = \int_a^0 f(x)\,dx.$$

Now change the integration limits:

$$\int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx.$$

Substituting this representation for the first integral in the right-hand side of (8), we obtain

$$-\int_0^a f(x)\,dx + \int_0^a f(t)\,dt = 0. \ \square$$

2. Let the function f be an even function defined and continuous on the segment $[-a, a]$. Then $\int_{-a}^a f(t)\,dt = 2\int_0^a f(t)\,dt$.

PROOF.

As in the proof of corollary 1, we represent this integral as the sum of integrals (8) and make the same variable change $t = -x$ in the first integral from the right-hand side of (8):

$$\int_{-a}^0 f(t)\,dt = \int_a^0 f(-x)\,(-dx).$$

In this case, the function is even, i. e., $f(-x) = f(x)$, so further transformations of the integral will be as follows:

$$\int_a^0 f(-x)\,(-dx) = -\int_a^0 f(x)\,dx = \int_0^a f(x)\,dx.$$

Substituting this representation for the first integral in the right-hand side of (8), we obtain the required expression:

$$\int_0^a f(x)\,dx + \int_0^a f(t)\,dt = 2\int_0^a f(t)\,dt. \ \square$$

3. Let the function f be a continuous periodic function with period T. Then

$$\forall\, a \in \mathbb{R} \quad \int_a^{a+T} f(t)\,dt = \int_0^T f(t)\,dt. \tag{9}$$

Thus, the integral of a periodic function over any segment whose length is equal to its period T is equal to the integral over the segment $[0, T]$.

PROOF.

Using the second theorem on the additivity of a definite integral with respect to the integration segment, we transform the integral $\int_a^{a+T} f(t)\,dt$ as follows:

$$\int_a^{a+T} f(t)\,dt = \int_a^0 f(t)\,dt + \int_0^T f(t)\,dt + \int_T^{a+T} f(t)\,dt. \tag{10}$$

In the last integral of the right-hand side of (10), we make the variable change $x = t - T$. Then $dx = dt$, the integration limits T, $a + T$ change by 0, a, and this integral takes the form

$$\int_T^{a+T} f(t)\,dt = \int_0^a f(x+T)\,dx = -\int_a^0 f(x+T)\,dx.$$

Since the function f is periodic with the period T, the equality $f(x + T) = f(x)$ holds. We got that the third integral on the right-hand side of (10) is $-\int_a^0 f(x)\,dx$ and, in combination with the first integral, gives the value 0. Thus, equality (10) turns into equality (9). □

Version of the theorem on the change of variables in a definite integral

2.8A/00:00 (10:45)

The theorem on the change of variables in a definite integral considers the intervals (a_0, b_0) and (α_0, β_0) containing segments with endpoints a, b and α, β, over which the integration is carried out in (7). The purpose of this formulation is to guarantee the existence of the derivative $\varphi'(t)$ at all points of the integration segment.

If we consider the derivatives defined on the segment, assuming that the derivatives are calculated as one-sided limits at the endpoints of the segment, then the condition of the theorem can be simplified by requiring that the function $f(x)$ is continuous on $[a, b]$, the function $\varphi(t)$ acts from $[\alpha, \beta]$ to $[a, b]$ and is continuously differentiable on $[\alpha, \beta]$, and the equalities $\varphi(\alpha) = a$, $\varphi(\beta) = b$ hold.

Integration formula by parts for a definite integral

2.8A/10:45 (04:49)

Theorem (on integration by parts of a definite integral)..

Let the functions u, v be continuously differentiable on the interval (a_0, b_0) and the segment $[a, b]$ be contained in the interval (a_0, b_0). Then the following formula holds:

$$\int_a^b uv'\,dx = uv\big|_a^b - \int_a^b u'v\,dx. \tag{11}$$

Formula (11) is called the *integration formula by parts* for a definite integral. Recall that the expression $uv|_a^b$ means the difference $u(b)v(b)-u(a)v(a)$.

REMARK.

As in the case of the theorem on changing a variable in a definite integral, if we consider the derivatives defined on the segment, assuming that the derivatives at the endpoints of the segment are calculated as one-sided limits, then the condition of the theorem can be simplified by requiring only that the functions u, v were continuously differentiable on the segment $[a, b]$.

PROOF.

By the formula of the derivative of the product, we have

$$\big(u(x)v(x)\big)' = u'(x)v(x) + u(x)v'(x).$$

Let us express the product $u(x)v'(x)$ from the last equality:

$$u(x)v'(x) = \big(u(x)v(x)\big)' - u'(x)v(x).$$

The expressions on the left and on the right are continuous functions and therefore they are integrable. Integrating the left-hand side and the right-hand side of the equality from a to b and using the linearity of a definite integral with respect to the integrand, we obtain

$$\int_a^b u(x)v'(x)\,dx = \int_a^b \big(u(x)v(x)\big)'\,dx - \int_a^b u'(x)v(x)\,dx. \tag{12}$$

Obviously, the function $F(x) = u(x)v(x)$ is the antiderivative for the function $\big(u(x)v(x)\big)'$. Then, according to the Newton–Leibniz formula, we have

$$\int_a^b \big(u(x)v(x)\big)'\,dx = F(b) - F(a) = F(x)\big|_a^b = u(x)v(x)\big|_a^b.$$

Substituting the obtained representation of the integral $\int_a^b \big(u(x)v(x)\big)'\,dx$ into (12), we get equality (11). $\square$

8. Calculation of areas and volumes

Quadrable figures on a plane

Plane figures. Cell figures

2.8A/15:34 (09:49)

We began the study of definite integrals by formulating the problem of finding the area of a curvilinear trapezoid. But a strict definition of the area was not given.

To prove that a definite integral is equal to the area of a curvilinear trapezoid, we need, first of all, to define the area for a sufficiently wide class of sets on the plane. For this we need to introduce a number of auxiliary definitions.

A *figure* is any nonempty bounded set of points on the plane. Recall that the boundedness of the set G on the plane means that there exists a circle that contains all points of the set G.

We define the *area of the rectangle* Π with sides parallel to the coordinate axes as follows: if $\Pi = \{(x, y) : a_1 \le x \le b_1,\ a_2 \le y \le b_2\}$, then the area of the rectangle (denoted by $S(\Pi)$) is

$$S(\Pi) \stackrel{\text{def}}{=} (b_1 - a_1)(b_2 - a_2). \tag{1}$$

We will also assume that formula (1) determines the area of the rectangle even if its boundary (or part of it) does not belong to this rectangle. In particular, for the rectangle $\Pi = \{(x, y) : a_1 < x < b_1,\ a_2 < y < b_2\}$, the area is also calculated by formula (1). Rectangles can degenerate into segments or points; the areas of such degenerate rectangles are equal to 0.

A *cell figure* is a figure Q, which can be represented as the union of a finite number of pairwise disjoint rectangles Π_i, $i = 1, \dots, n$, with sides parallel to the coordinate axes:

$$Q = \bigcup_{i=1}^{n} \Pi_i, \qquad \Pi_i \cap \Pi_j = \varnothing, \quad i \ne j.$$

Part of the boundary of the rectangle Π_i may not belong to this rectangle.

By definition, the *area of the cell figure* Q, which is the union of pairwise disjoint rectangles, is the sum of the areas of these rectangles (notation $S(Q)$):

$$S(Q) \stackrel{\text{def}}{=} \sum_{i=1}^{n} S(\Pi_i).$$

This definition of the area of the cell figure is well-posed, since the following statement can be proved: for any method of splitting the cell figure into pairwise disjoint rectangles, the sum of the areas of these rectangles will be equal to the same value.

We also accept the following statement without proof: if the embedding $q \subset Q$ holds for the cell figures q and Q, then the inequality $S(q) \le S(Q)$ holds for their areas.

Squarable figure and its area 2.8A/25:23 (12:33)

Now we give two basic definitions: the squarable figure and its area.

A figure G is called *squarable* if, for any $\varepsilon > 0$, there exists a pair of cell figures q, Q such that $q \subset G \subset Q$ and $S(Q) - S(q) < \varepsilon$.

The *area of the squarable figure* G is the number $S(G)$ satisfying the double inequality $S(q) \le S(G) \le S(Q)$ for any cell figures q, Q such that $q \subset G \subset Q$.

REMARK.

Not each bounded set on a plane is a squarable figure. For example, one can prove that the set of all points with *rational coordinates* located inside the square $\{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$ is not a squarable figure.

THEOREM (ON THE WELL-POSEDNESS OF THE DEFINITION OF THE SQUARABLE FIGURE AREA).

If G is a squarable figure, then the number $S(G)$ exists and is unique.

PROOF.

The existence of this number follows from the axiom of continuity. Let A be the set of areas of all cell figures q embedded in G and let B be the set of areas of all cell figures Q containing G. These sets are nonempty. Since the embedding $q \subset Q$ holds for q and Q, we obtain that the estimate $S(q) \le S(Q)$ holds for all $S(q) \in A$ and $S(Q) \in B$. Therefore, by virtue of the axiom of continuity, there exists a number $S(G)$ satisfying the double estimate $S(q) \le S(G) \le S(Q)$ for all $S(q) \in A$ and $S(Q) \in B$.

Now we prove uniqueness. Let there exist two numbers S' and S'' satisfying the inequalities $S(q) \le S' \le S(Q)$ and $S(q) \le S'' \le S(Q)$ for any cell figures q, Q such that $q \subset G \subset Q$.

Without loss of generality, we can assume that $S' \le S''$. Then the following chain of inequalities holds:

$$S(q) \le S' \le S'' \le S(Q).$$

From this chain of inequalities we obtain

$$S'' - S' \le S(Q) - S(q). \tag{2}$$

By the definition of a squarable figure, for any value $\varepsilon > 0$, there exists a pair of cell figures q, Q such that $q \subset G \subset Q$ and $S(Q) - S(q) < \varepsilon$. Taking into account (2), we obtain that the following estimate holds for any $\varepsilon > 0$:

$$S'' - S' < \varepsilon.$$

Since the last estimate can be fulfilled only for $S' = S''$, we get that there exists a unique value $S(G)$ satisfying the definition of the area of the squarable figure G. □

Remarks.

1. The existence of the number $S(G)$ can be proved for any figure (not necessarily squarable one). However, only for the squarable figure it can be proved that the number $S(G)$ is unique.

2. It can also be proved that the following equalities are valid for the area of the squarable figure G, (in these equalities cell figures are denoted by q and Q):

$$S(G) = \sup_{q \subset G} S(q) = \inf_{G \subset Q} S(Q).$$

Criterion for the squarability of a figure

2.8A/37:56 (03:38)

The following criterion for the squarability of a figure holds (its proof is given, for example, in [18, Ch. 7, Sec. 37.1]).

Theorem (criterion for the squarability of a figure).

A figure G is a squarable one if and only if, for any $\varepsilon > 0$, there exist squarable figures $\tilde{q}$, $\tilde{Q}$ such that $\tilde{q} \subset G \subset \tilde{Q}$ and $S(\tilde{Q}) - S(\tilde{q}) < \varepsilon$.

This criterion differs from the definition of a squarable figure in that the *cell* figures q and Q are used in the definition, and the *squarable* figures $\tilde{q}$ and $\tilde{Q}$ are used in the criterion.

Remark.

Obviously, the area of the squarable figure is non-negative. In addition, it can be proved that the area of the squarable figure has the following properties:

1) *additivity*: the area of the union of any finite number of pairwise disjoint squarable figures is equal to the sum of the areas of these figures;

2) *invariance*: the area of the figure does not change when it is shifted, rotated or reflected.

Now, after introducing all the required definitions and formulating the necessary statements, we will consider figures of a special kind, prove their squarability, and find formulas for their area.

Area of a curvilinear trapezoid and area of a curvilinear sector

Theorem on the area of a curvilinear trapezoid: formulation and proof of squarability

2.8B/00:00 (13:27)

Theorem (on the area of a curvilinear trapezoid).

Let the function f be continuous on $[a, b]$ and non-negative on this segment: $f(x) \geq 0$, $x \in [a, b]$. Let G be a *curvilinear trapezoid* defined as follows (see Fig. 4 in Chapter 5):

$$G = \{(x, y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}.$$

Then the curvilinear trapezoid G is a squarable figure and its area is calculated by the formula

$$S(G) = \int_a^b f(x)\,dx. \tag{3}$$

Proof.

1. To prove the squarability of the figure G, we must show that, for any $\varepsilon > 0$, there exist cell figures q and Q satisfying two conditions: $q \subset G \subset Q$ and $S(Q) - S(q) < \varepsilon$.

We will construct cell figures q_T and Q_T based on some partition T of the segment $[a, b]$. For each segment Δ_i of this partition, $i = 1, \ldots, n$, we define two numbers:

$$m_i = \min_{x \in \Delta_i} f(x), \qquad M_i = \max_{x \in \Delta_i} f(x).$$

Note that in this case we use the notation min and max instead of inf and sup, since, by virtue of the second Weierstrass theorem, the continuous function takes its minimum and maximum value on the segment.

Define the following rectangles:

$$q_i = \{(x, y) : x \in \Delta_i,\ 0 \leq y \leq m_i\},$$
$$Q_i = \{(x, y) : x \in \Delta_i,\ 0 \leq y \leq M_i\}.$$

By definition of the area of the rectangle, we get $S(q_i) = m_i \Delta x_i$, $S(Q_i) = M_i \Delta x_i$, $i = 1, \ldots, n$, and the area of these rectangles does not change if we remove a part of their boundary.

Now we define the sets q_T and Q_T as unions of pairwise disjoint rectangles:

$$q_T = \bigcup_{i=1}^{n} \tilde{q}_i, \quad Q_T = \bigcup_{i=1}^{n} \tilde{Q}_i.$$

The rectangles $\tilde{q}_i$ and $\tilde{Q}_i$ differ from the previously defined rectangles q_i and Q_i since part of the boundary of $\tilde{q}_i$ and $\tilde{Q}_i$ can be removed. Removing a part of the boundary is necessary to satisfy the conditions $\tilde{q}_i \cap \tilde{q}_j = \varnothing$, $\tilde{Q}_i \cap \tilde{Q}_j = \varnothing$ for $i \neq j$. For definiteness, we can assume that, for any index $i = 2, \dots, n$, the common part of the boundary of the figures q_{i-1}, q_i (and Q_{i-1} and Q_i) is removed from the boundary of the figure q_i (and Q_i, respectively).

By construction, q_T and Q_T are cell figures and its areas are as follows:

$$S(q_T) = \sum_{i=1}^{n} S(\tilde{q}_i) = \sum_{i=1}^{n} S(q_i) = \sum_{i=1}^{n} m_i \Delta x_i,$$

$$S(Q_T) = \sum_{i=1}^{n} S(\tilde{Q}_i) = \sum_{i=1}^{n} S(Q_i) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Note that the obtained values of the areas coincide with the values of the lower and upper Darboux sums for the function f and the partition T: $S(q_T) = S_T^-(f)$, $S(Q_T) = S_T^+(f)$. The left-hand part of Fig. 7 shows the figure q_T and the right-hand part of Fig. 7 shows the figure Q_T.

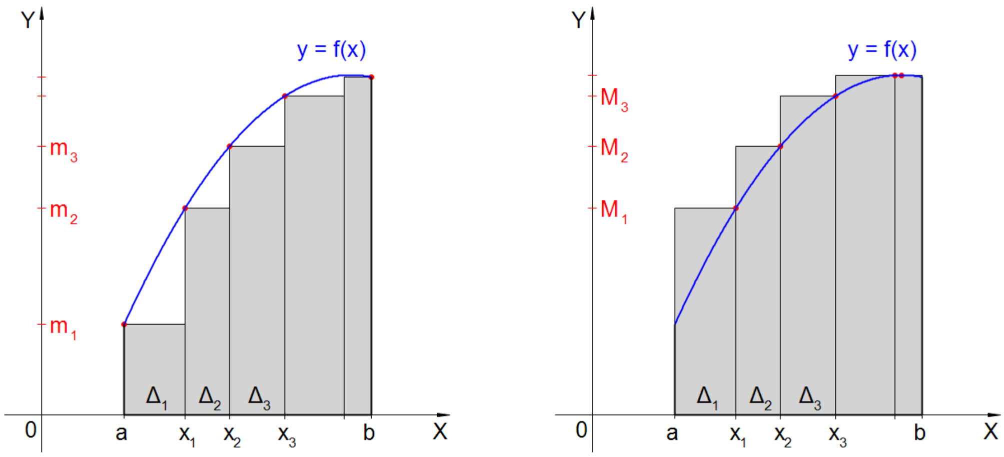

Fig. 7. A curvilinear trapezoid and the associated cell figures q_T and Q_T

In addition, a double embedding $q_T \subset G \subset Q_T$ holds for the figures q_T and Q_T.

So, we have constructed two cell figures q_T and Q_T on the basis of an arbitrary partition T, these figures satisfy the condition $q_T \subset G \subset Q_T$, and the following relations hold: $S(q_T) = S_T^-(f)$, $S(Q_T) = S_T^+(f)$.

By the condition of the theorem, the function f is continuous on $[a, b]$ and therefore is integrable. So, by virtue of the integrability criterion in terms of Darboux sums, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any partition T with $l(T) < \delta$, the estimate $S_T^+(f) - S_T^-(f) < \varepsilon$ holds.

Therefore, for a given $\varepsilon > 0$, we can choose a partition T such that for the cell figures q_T and Q_T constructed on the basis of this partition and satisfying the condition $q_T \subset G \subset Q_T$, the following relation is fulfilled:

$$S(Q_T) - S(q_T) = S_T^+(f) - S_T^-(f) < \varepsilon.$$

So, we have shown that the set G is a squarable figure.

Proof of the formula of the curvilinear trapezoid area

2.8B/13:27 (07:27)

2. It remains for us to prove formula (3). Recall that the area of the squarable figure G is the number $S(G)$ satisfying the double inequality $S(q) \le S(G) \le S(Q)$ for any cell figures q and Q such that $q \subset G \subset Q$.

In the proof of the integrability criterion, we obtained the following relation connecting the integral with Darboux sums:

$$\int_a^b f(x)\,dx = \sup_T S_T^-(f) = \inf_T S_T^+(f).$$

The last relation can be rewritten using the areas of the cell figures q_T and Q_T:

$$\int_a^b f(x)\,dx = \sup_T S(q_T) = \inf_T S(Q_T). \tag{4}$$

Moreover, for all T, the embeddings $q_T \subset G \subset Q_T$ are valid.

Relation (4) means that, for any cell figures q_T, Q_T, the double inequality holds:

$$S(q_T) \le \int_a^b f(x)\,dx \le S(Q_T). \tag{5}$$

No value A other than $\int_a^b f(x)\,dx$ can satisfy the double inequality (5). Indeed, if we take $A > \int_a^b f(x)\,dx$, then, by virtue of (4), the estimate $\inf_T S(Q_T) < A$ will be satisfied for A; therefore, there exists a partition T for which the figure Q_T has an area smaller than A: $S(Q_T) < A$. If we take $A < \int_a^b f(x)\,dx$, then, by (4), the estimate $\sup_T S(q_T) > A$ will be satisfied for A; therefore, there exists a partition T for which the figure q_T has an area greater than A: $S(q_T) > A$.

Since any numbers A except $\int_a^b f(x)\,dx$ do not satisfy the double inequality (5), such numbers A also cannot satisfy the more general inequality $S(q) \le S(G) \le S(Q)$, where q and Q are arbitrary cell figures satisfying the condition $q \subset G \subset Q$.

On the other hand, the value $S(G)$ exists, since we have already proved that the curvilinear trapezoid G is a squarable figure. Therefore, the only possible value for $S(G)$ is the integral $\int_a^b f(x)\,dx$. $\square$

Theorem on the area of a figure with two curvilinear boundary parts 2.8B/20:54 (05:15)

The result proved for a curvilinear trapezoid can be generalized to a figure with two curvilinear parts of the boundary.

THEOREM (ON THE AREA OF A FIGURE WITH TWO CURVILINEAR BOUNDARY PARTS).

Let the functions f and g be continuous on $[a,b]$ and satisfy the inequality $f(x) \le g(x)$, $x \in [a,b]$. We define the figure G as follows (see the left-hand part of Fig. 8):

$$G = \{(x,y) : a \le x \le b,\ f(x) \le y \le g(x)\}.$$

Then G is a squarable figure and its area is calculated by the formula

$$S(G) = \int_a^b \bigl(g(x) - f(x)\bigr)\,dx.$$

PROOF.

Perform a shift of the figure G in the positive direction of the OY axis so that its lower curvilinear boundary is located above the OX axis. Such a shift means that, instead of the functions f and g, we need to consider the functions $f + C$ and $g + C$ with the constant C equal, for example, to the value $\left|\min_{x\in[a,b]} f(x)\right| + 1$.

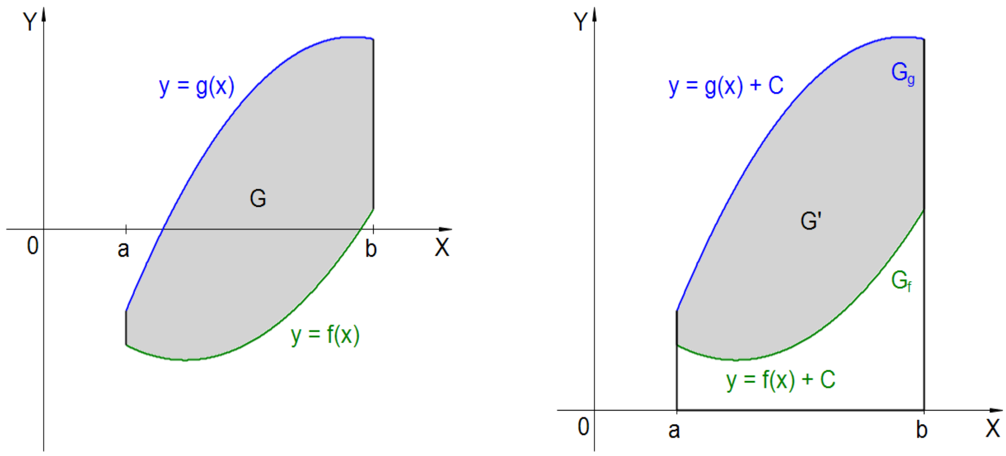

Fig. 8. The figure bounded by the graphs of two continuous functions

Denote by G' the figure obtained as a result of the described shift:

$$G' = \{(x,y) : a \le x \le b,\ f(x) + C \le y \le g(x) + C\}.$$

Then the figure G' can be represented as the difference of two curvilinear trapezoids (see the right-hand part of Fig. 8):

$$G' = G_g \setminus G_f,$$

$$G_f = \{(x, y) : a \le x \le b,\ 0 \le y < f(x) + C\},$$

$$G_g = \{(x, y) : a \le x \le b,\ 0 \le y \le g(x) + C\}.$$

Note that in the definition of the set G_f, we used the strict inequality $y < f(x) + C$. However, this modification of the definition of a curvilinear trapezoid will not affect its squarability and will not change its area.

Using formula (3) from the theorem on the area of a curvilinear trapezoid, we obtain

$$S(G_f) = \int_a^b \bigl(f(x) + C\bigr)\, dx, \quad S(G_g) = \int_a^b \bigl(g(x) + C\bigr)\, dx.$$

Now we use the additivity property of an area and the additivity property of a definite integral with respect to the integrand:

$$S(G') = S(G_g \setminus G_f) = S(G_g) - S(G_f) =$$
$$= \int_a^b \bigl(g(x) + C\bigr)\, dx - \int_a^b \bigl(f(x) + C\bigr)\, dx = \int_a^b \bigl(g(x) - f(x)\bigr)\, dx.$$

It remains for us to notice that, due to the invariance property, the area of the figure does not change with its shift, therefore $S(G) = S(G')$. $\square$

Calculation of the area of an ellipse. Circular sector and curvilinear sector

2.8B/26:09 (07:57)

Find the area of the *ellipse* G_{ab} defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Fig. 9). For symmetry reasons, it is enough for us to find the area of the part G_1 of the ellipse, which is located in the first coordinate quarter, and multiply it by 4: $S(G_{ab}) = 4S(G_1)$.

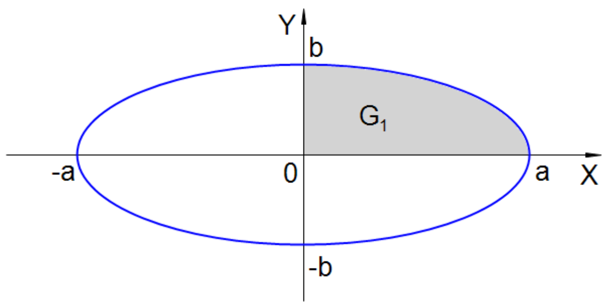

Fig. 9. Ellipse

The set G_1 can be described as follows:

$$G_1 = \left\{(x,y) : 0 \le x \le a,\ 0 \le y \le b\sqrt{1-\frac{x^2}{a^2}}\right\}.$$

Therefore, this set is a curvilinear trapezoid and its area is calculated by the formula

$$S(G_1) = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx.$$

In the resulting integral, we can make the variable change $x = a\sin t$. Then $dx = a\cos t\,dt$, the integration limits will change to 0 and $\frac{\pi}{2}$, and the expression $\sqrt{1-\frac{x^2}{a^2}}$ will take the form $\sqrt{1-\sin^2 t} = \cos t$ (when extracting the square root, we use the "plus" sign, since the cosine takes non-negative values on the segment $\left[0, \frac{\pi}{2}\right]$). So,

$$\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = \int_0^{\frac{\pi}{2}} ab\cos^2 t\,dt = \frac{ab}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2t)\,dt =$$

$$= \frac{ab}{2}\left(t+\frac{\sin 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4}.$$

Finally we get

$$S(G_{ab}) = 4S(G_1) = 4\cdot\frac{\pi ab}{4} = \pi ab.$$

Remark.

In the case $a = b = R$, we obtain the well-known formula for the *area of the circle* G_R: $S(G_R) = \pi R^2$.

Recall also the formula for the area of the *circular sector* G_R^α with an angle α and a radius R:

$$S(G_R^\alpha) = \pi R^2\cdot\frac{\alpha}{2\pi} = \frac{R^2\alpha}{2}. \tag{6}$$

Definition.

Let the function $f(\varphi)$ be defined and continuous on the segment $[\alpha,\beta]$, where $0 \le \alpha < \beta < 2\pi$. The *curvilinear sector* (Fig. 10) is the set G defined in the polar coordinate system (ρ,φ) as follows:

$$G = \{(\rho,\varphi) : \alpha \le \varphi \le \beta,\ 0 \le \rho \le f(\varphi)\}. \tag{7}$$

This notion is a generalization of the notion of a circular sector; if the function f is a constant, $f(\varphi) \equiv R$, then the corresponding curvilinear sector is a circular sector with an angle $\beta-\alpha$ and a radius R.

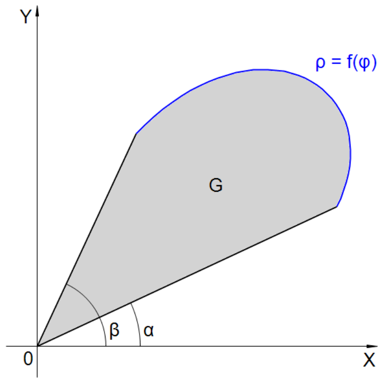

Fig. 10. Curvilinear sector

Theorem on the area of a curvilinear sector: formulation and proof of squarability

2.9A/00:00 (15:06)

Theorem (on the area of a curvilinear sector).

The curvilinear sector G defined by formula (7) for the function f continuous on the segment $[\alpha, \beta]$ is a squarable figure and its area is calculated by the formula

$$S(G) = \frac{1}{2} \int_{\alpha}^{\beta} f^2(x)\, dx. \tag{8}$$

Remark.

If the function f is constant, i. e., $f(\varphi) \equiv R$, then formula (8) turns into formula (6) of the area of the circular sector with an angle $\beta - \alpha$.

Proof.

1. In this case, we cannot construct simple *cell* figures that are "close" to the initial curvilinear sector. However, we can use simple *squarable* figures and then apply the squarability criterion. Such squarable figures can be composed of circular sectors, for which we have already established the squarability.

Let T be a partition of the segment $[\alpha, \beta]$:

$$\alpha = x_0 < x_1 < \cdots < x_{n-1} < x_n = \beta.$$

In this case, the segment Δ_i, $i = 1, \ldots, n$, corresponds to various angles φ in the range from x_{i-1} to x_i. For each segment Δ_i, we define two numbers:

$$m_i = \min_{\varphi \in \Delta_i} f(\varphi), \qquad M_i = \max_{\varphi \in \Delta_i} f(\varphi).$$

We define the following circular sectors (Fig. 11):

$$q_i = \{(\rho, \varphi) : \varphi \in \Delta_i,\ 0 \le \rho \le m_i\},$$

$$Q_i = \{(\rho, \varphi) : \varphi \in \Delta_i,\ 0 \le \rho \le M_i\}.$$

By formula (6), we have $S(q_i) = \frac{1}{2}m_i^2\Delta x_i$, $S(Q_i) = \frac{1}{2}M_i^2\Delta x_i$, $i = 1, \ldots, n$, and the areas of these sectors will not change if we remove part of their boundaries.

Now we define the sets q_T and Q_T as unions of pairwise disjoint circular sectors:

$$q_T = \bigcup_{i=1}^{n} \tilde{q}_i, \quad Q_T = \bigcup_{i=1}^{n} \tilde{Q}_i.$$

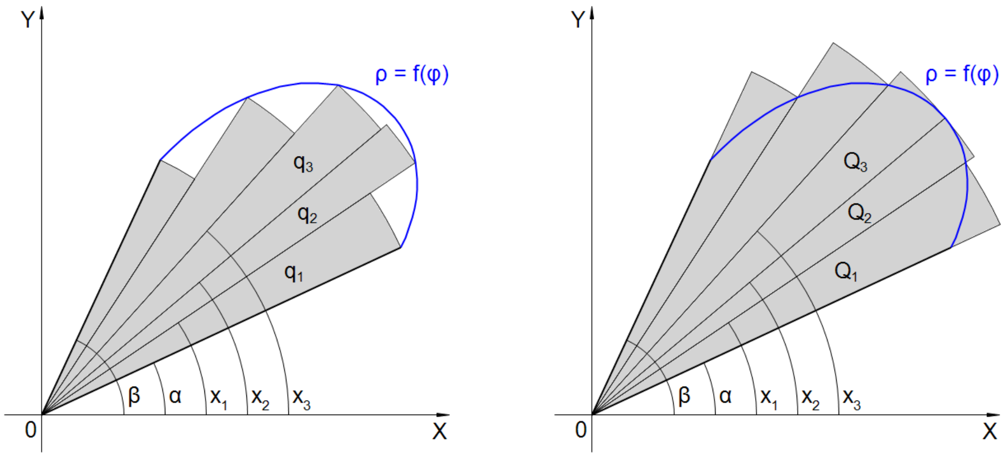

Fig. 11. Curvilinear sector with a set of auxiliary circular sectors

The sectors $\tilde{q}_i$ and $\tilde{Q}_i$ differ from the previously defined sectors q_i and Q_i since part of the boundary of $\tilde{q}_i$ and $\tilde{Q}_i$ can be removed. For definiteness, we can assume that, for any index $i = 2, \ldots, n$, the common part of the boundary of the figures q_{i-1}, q_i (and Q_{i-1} and Q_i) is removed from the boundary of the figure q_i (and Q_i, respectively).

The figures q_T and Q_T are squarable as a union of squarable figures. The circular sectors included into them are pairwise disjoint, therefore, due to the property of area additivity, we obtain

$$S(q_T) = \sum_{i=1}^{n} S(\tilde{q}_i) = \sum_{i=1}^{n} S(q_i) = \frac{1}{2}\sum_{i=1}^{n} m_i^2\Delta x_i,$$

$$S(Q_T) = \sum_{i=1}^{n} S(\tilde{Q}_i) = \sum_{i=1}^{n} S(Q_i) = \frac{1}{2}\sum_{i=1}^{n} M_i^2\Delta x_i.$$

The obtained values of the areas coincide with the values of the lower and upper Darboux sums for the function $\frac{1}{2}f^2$ and the partition T: $S(q_T) = S_T^-\left(\frac{1}{2}f^2\right)$, $S(Q_T) = S_T^+\left(\frac{1}{2}f^2\right)$.

In addition, a double embedding $q_T \subset G \subset Q_T$ holds for the figures q_T and Q_T.

So, we have constructed two squarable figures q_T and Q_T on the basis of an arbitrary partition T, these figures satisfy the condition $q_T \subset G \subset Q_T$, and the following relations hold: $S(q_T) = S_T^-\bigl(\frac{1}{2}f^2\bigr)$, $S(Q_T) = S_T^+\bigl(\frac{1}{2}f^2\bigr)$.

By the condition of the theorem, the function $\frac{1}{2}f^2$ is continuous on $[\alpha, \beta]$ and therefore is integrable. So, by virtue of the integrability criterion in terms of Darboux sums, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any partition T with $l(T) < \delta$, the estimate $S_T^+\bigl(\frac{1}{2}f^2\bigr) - S_T^-\bigl(\frac{1}{2}f^2\bigr) < \varepsilon$ holds.

Therefore, for a given $\varepsilon > 0$, we can choose a partition T such that, for the squarable figures q_T and Q_T constructed on the basis of this partition and satisfying the condition $q_T \subset G \subset Q_T$, the following relation holds:

$$S(Q_T) - S(q_T) = S_T^+\Bigl(\frac{1}{2}f^2\Bigr) - S_T^-\Bigl(\frac{1}{2}f^2\Bigr) < \varepsilon.$$

We have proved that, for any $\varepsilon > 0$, there exist squarable figures q, Q that satisfy the conditions $q \subset G \subset Q$, $S(Q) - S(q) < \varepsilon$. By virtue of the squarability criterion, this means that the set G is a squarable figure.

Proof of the curvilinear sector area formula

2.9A/15:06 (04:48)

2. Formula (8) can be proved by the same reasoning as formula (3) from the theorem on the area of a curvilinear trapezoid, if we replace the segment $[a, b]$ with $[\alpha, \beta]$ and the function f with $\frac{1}{2}f^2$. $\square$

Volume calculation

Cubable solids

2.9A/19:54 (11:52)

As with the study of areas, we begin with auxiliary definitions associated with sets in three-dimensional space.

A *solid* is any nonempty bounded set of points in three-dimensional space.

Consider a *cuboid* (a *rectangular parallelepiped*) P with edges parallel to the coordinate axes:

$$P = \{(x, y, z) : a_1 \le x \le b_1,\ a_2 \le y \le b_2,\ a_3 \le z \le b_3\}.$$

The *volume of the cuboid* P (notation $V(P)$) is defined as follows:

$$V(P) \overset{\text{def}}{=} (b_1 - a_1)(b_2 - a_2)(b_3 - a_3). \tag{9}$$

We will also assume that formula (9) determines the volume of the cuboid even if its boundary (or part of it) does not belong to this cuboid. In particular, for the cuboid $P = \{(x, y, z) : a_1 < x < b_1,\ a_2 < y < b_2,\ a_3 < z < b_3\}$, the volume is also calculated by formula (9).

A *cell solid* is a solid Q, which can be represented as the union of a finite number of pairwise disjoint cuboids P_i, $i = 1, \ldots, n$, with edges parallel to the coordinate axes:

$$Q = \bigcup_{i=1}^{n} P_i, \qquad P_i \cap P_j = \varnothing, \quad i \neq j.$$

Part of the boundary of the cuboid P_i may not belong to this cuboid.

By definition, the *volume of the cell solid* Q, which is the union of pairwise disjoint cuboids, is the sum of the volumes of these cuboids (notation $V(Q)$):

$$V(Q) \stackrel{\text{def}}{=} \sum_{i=1}^{n} V(P_i).$$

This definition of the volume of the cell solid is well-posed, since the following statement can be proved: for any method of dividing the cell solid into pairwise disjoint cuboids, the sum of the volumes of these cuboids will be the same value.

Now we give two basic definitions: a cubable solid and its volume.

A solid Ω is called *cubable* if, for any $\varepsilon > 0$, there exists a pair of cell solids q, Q such that $q \subset \Omega \subset Q$ and $V(Q) - V(q) < \varepsilon$.

The *volume of a cubable solid* Ω is the number $V(\Omega)$ satisfying the double inequality $V(q) \leq V(\Omega) \leq V(Q)$ for any cell solids q, Q such that $q \subset \Omega \subset Q$.

Theorem (on the well-posedness of the definition of the cubable solid volume).

If Ω is a cubable solid, then the number $V(\Omega)$ exists and is unique.

The proof of this theorem is similar to the proof of the theorem on the well-posedness of the definition of the squarable figure area. □

There exists a cubability criterion similar to the squarability criterion given above, which we also accept without proof.

Theorem (criterion for the cubability of a solid).

The solid Ω is cubable if and only if, for any $\varepsilon > 0$, there exist cubable solids q, Q such that $q \subset \Omega \subset Q$, $V(Q) - V(q) < \varepsilon$.

Remark.

It can be proved that the volume of the cubable solid has the same additivity and invariance properties as the area of the squarable figure.

Cylindrical solid and its volume

2.9B/00:00 (13:57)

Definition.

Let G be some figure located on the coordinate plane OXY. Perform its shift by the distance h in the positive direction of the OZ axis (Fig. 12).

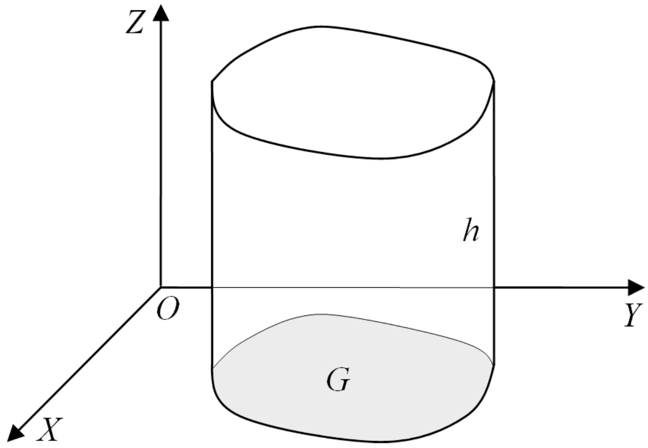

Fig. 12. The cylindrical solid Ω

All points through which the figure G passes as a result of the described shift form a solid Ω called a *cylindrical solid* with the base G and height h:

$$\Omega = \{(x, y, z) : (x, y) \in G,\ 0 \le z \le h\}.$$

THEOREM (ON THE VOLUME OF A CYLINDRICAL SOLID).

If the base G of the cylindrical solid Ω is a squarable figure, then the cylindrical solid Ω is cubable and its volume is calculated by the formula

$$V(\Omega) = S(G)h.$$

PROOF.

By condition, the base G is a squarable figure; therefore, for any $\varepsilon > 0$, there exist cell figures q and Q that satisfy two conditions:

$$q \subset G \subset Q, \quad S(Q) - S(q) < \frac{\varepsilon}{h}. \tag{10}$$

We need to prove that, for any $\varepsilon > 0$, there exist cell solids $\tilde{q}$ and $\tilde{Q}$ that satisfy the following conditions:

$$\tilde{q} \subset \Omega \subset \tilde{Q}, \quad V(\tilde{Q}) - V(\tilde{q}) < \varepsilon. \tag{11}$$

We choose some $\varepsilon > 0$, take cell figures q and Q that satisfy conditions (10), and construct cell solids $\tilde{q}$ and $\tilde{Q}$ performing a shift of figures q and Q to the distance h in the positive direction of the OZ axis:

$$\tilde{q} = \{(x, y, z) : (x, y) \in q,\ 0 \le z \le h\},$$
$$\tilde{Q} = \{(x, y, z) : (x, y) \in Q,\ 0 \le z \le h\}.$$

As a result, we obtain cylindrical solids, which are obviously cell ones, since with such a shift any of the pairwise disjoint rectangles Π, which form the cell figure q or Q, is transformed into a cuboid with the base Π and height h and the union of such (pairwise disjoint) cuboids coincides with the solid $\tilde{q}$ or $\tilde{Q}$, respectively.

Moreover, if the area of the rectangle Π is equal to s, then the volume of the cuboid with the base Π and height h will be equal to sh. Thus, the volume of the cell solid $\tilde{q}$ or $\tilde{Q}$ can be obtained by multiplying the area of its base ($S(q)$ or $S(Q)$, respectively) by h:

$$V(\tilde{q}) = S(q)h, \quad V(\tilde{Q}) = S(Q)h. \tag{12}$$

Obviously, for the constructed cell solids $\tilde{q}$ and $\tilde{Q}$, the embedding chain from conditions (11) holds: $\tilde{q} \subset \Omega \subset \tilde{Q}$. An estimate of the difference $V(\tilde{Q}) - V(\tilde{q})$ from (11) can be obtained by multiplying both parts of the estimate from (10) by h:

$$S(Q)h - S(q)h < \frac{\varepsilon}{h} \cdot h, \qquad V(\tilde{Q}) - V(\tilde{q}) < \varepsilon.$$

So, we have proved that, for arbitrary $\varepsilon > 0$, there exist cell solids $\tilde{q}$ and $\tilde{Q}$ that satisfy conditions (11). Therefore, the cylindrical solid Ω is cubable. It remains to prove the formula for its volume.

By definition of the area of the squarable figure, the value of the area $S(G)$ satisfies the following double inequality, which is valid for any cell figures q and Q such that $q \subset G \subset Q$:

$$S(q) \leq S(G) \leq S(Q). \tag{13}$$

Moreover, there exists a unique number $S(G)$ satisfying condition (13) for any q and Q.

Multiply all parts of the double inequality (13) by h:

$$S(q)h \leq S(G)h \leq S(Q)h.$$

Given the previously obtained relations (12), the last inequality can be rewritten in the form

$$V(\tilde{q}) \leq S(G)h \leq V(\tilde{Q}). \tag{14}$$

We get that the value $S(G)h$ is the only value that satisfies the double inequality (14) for all the cell solids $\tilde{q}$ and $\tilde{Q}$ described above.

Since we have already proved that the solid Ω is cubable, it can be stated that there exists a number $V(\Omega)$ satisfying the condition $V(\tilde{q}) \leq V(\Omega) \leq V(\tilde{Q})$ for any cell solids $\tilde{q}$ and $\tilde{Q}$ such that $\tilde{q} \subset \Omega \subset \tilde{Q}$. Therefore, the only possible value for $V(\Omega)$ is $S(G)h$. $\square$

Remark.

The simplest case of a cylindrical solid is a *circular cylinder* $\Omega_{h,R}$ with height h and a base that is a circle of radius R. The volume of such a circular cylinder is calculated by the formula

$$V(\Omega_{h,R}) = \pi R^2 h. \tag{15}$$

Volume of a solid of revolution

2.9B/13:57 (09:58)

Definition.

Consider a curvilinear trapezoid defined by the continuous function $f(x)$ on the segment $[a,b]$ (we assume that $f(x) \ge 0$) and located on the plane OXY (thus, the curvilinear part of the trapezoid is a graph $y = f(x)$). We will rotate this curvilinear trapezoid around the OX axis. As a result, we get a three-dimensional solid Ω (Fig. 13) called a *solid of revolution*:

$$\Omega = \left\{(x,y,z) : a \le x \le b,\ 0 \le \sqrt{y^2+z^2} \le f(x)\right\}. \tag{16}$$

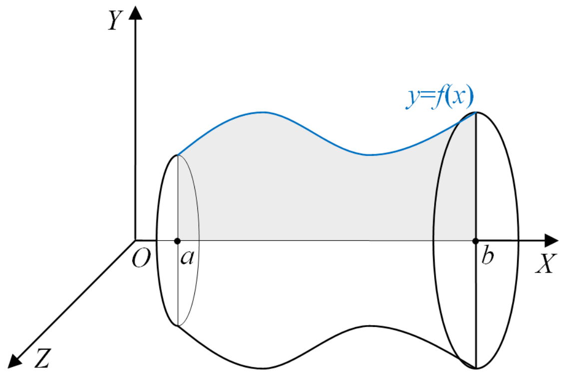

Fig. 13. The solid of revolution Ω

Theorem (on the volume of a solid of revolution).

The solid of revolution Ω defined by formula (16) for a function f continuous on a segment $[a,b]$ is a cubable solid and its volume is calculated by the formula

$$V(\Omega) = \pi \int_a^b f^2(x)\,dx. \tag{17}$$

Remark.

If the function f is constant, i. e., $f(x) \equiv R$, then formula (17) turns into formula (15) of the volume of a circular cylinder with height $b - a$.

Proof.

We use the cubability criterion and show that, for any $\varepsilon > 0$, there exist cubable solids q and Q such that $q \subset \Omega \subset Q$ and $V(Q) - V(q) < \varepsilon$.

The required cubable solids can be composed of circular cylinders, for which we have already established cubability and obtained the volume formula (15).

Let T be a partition of the segment $[a,b]$:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

For each segment Δ_i, $i = 1,\dots,n$, we define two numbers:

$$m_i = \min_{x\in\Delta_i} f(x), \quad M_i = \max_{x\in\Delta_i} f(x).$$

We define the following circular cylinders (Fig. 14):

$$q_i = \Big\{(x,y,z) : x \in \Delta_i,\ 0 \le \sqrt{y^2+z^2} \le m_i\Big\},$$

$$Q_i = \Big\{(x,y,z) : x \in \Delta_i,\ 0 \le \sqrt{y^2+z^2} \le M_i\Big\}.$$

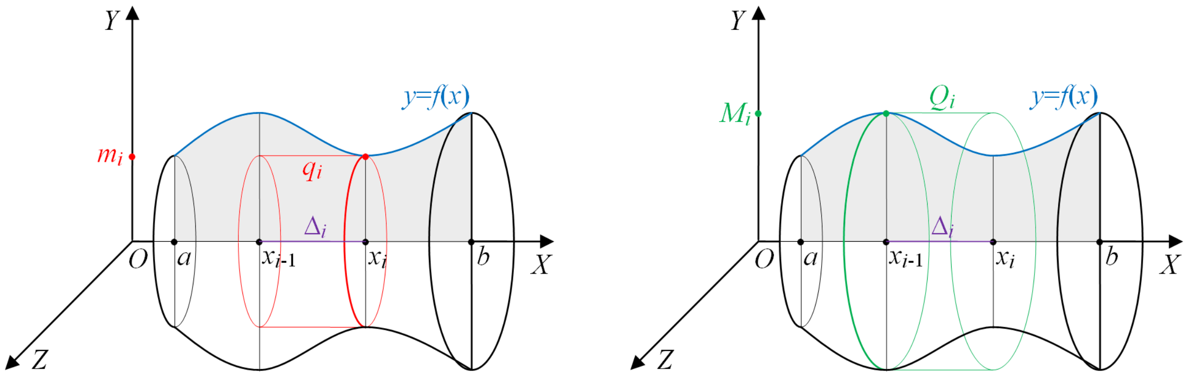

Fig. 14. Circular cylinders q_i and Q_i

By formula (15), we have $V(q_i) = \pi m_i^2 \Delta x_i$, $V(Q_i) = \pi M_i^2 \Delta x_i$, $i = 1,\ldots,n$, moreover, the volume of these cylinders does not change if we remove part of their boundaries.

Now we define the sets q_T and Q_T as unions of pairwise disjoint circular cylinders:

$$q_T = \bigcup_{i=1}^{n} \tilde{q}_i, \quad Q_T = \bigcup_{i=1}^{n} \tilde{Q}_i.$$

The cylinders $\tilde{q}_i$ and $\tilde{Q}_i$ differ from the previously defined cylinders q_i and Q_i since part of the boundary of $\tilde{q}_i$ and $\tilde{Q}_i$ can be removed. For definiteness, we can assume that, for any index $i = 2,\ldots,n$, the common part of the boundary of the solids q_{i-1} and q_i (and Q_{i-1} and Q_i) is removed from the boundary of the solid q_i (and Q_i, respectively).

The solids q_T and Q_T are cubable as the unions of cubable solids. The circular cylinders included into them are pairwise disjoint, therefore, due to the property of volume additivity, we obtain

$$V(q_T) = \sum_{i=1}^{n} V(\tilde{q}_i) = \sum_{i=1}^{n} V(q_i) = \pi \sum_{i=1}^{n} m_i^2 \Delta x_i,$$

$$V(Q_T) = \sum_{i=1}^{n} V(\tilde{Q}_i) = \sum_{i=1}^{n} V(Q_i) = \pi \sum_{i=1}^{n} M_i^2 \Delta x_i.$$

The obtained values of the volumes coincide with the values of the lower and upper Darboux sums for the function πf^2 and the partition T: $V(q_T) = S_T^-\bigl(\pi f^2\bigr)$, $V(Q_T) = S_T^+\bigl(\pi f^2\bigr)$.

In addition, a double embedding $q_T \subset \Omega \subset Q_T$ holds for the solids q_T and Q_T.

So, we have constructed two cubable solids q_T and Q_T on the basis of an arbitrary partition T, these solids satisfy the condition $q_T \subset \Omega \subset Q_T$, and the following relations hold: $V(q_T) = S_T^-\bigl(\pi f^2\bigr)$, $V(Q_T) = S_T^+\bigl(\pi f^2\bigr)$.

By the condition of the theorem, the function πf^2 is continuous on $[a, b]$ and therefore is integrable. So, by virtue of the integrability criterion in terms of Darboux sums, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any partition T with $l(T) < \delta$, the estimate $S_T^+\bigl(\pi f^2\bigr) - S_T^-\bigl(\pi f^2\bigr) < \varepsilon$ holds.

Therefore, for a given $\varepsilon > 0$, we can choose a partition T such that, for the cubable solids q_T and Q_T constructed on the basis of this partition and satisfying the condition $q_T \subset \Omega \subset Q_T$, the following relation is fulfilled:

$$V(Q_T) - V(q_T) = S_T^+\bigl(\pi f^2\bigr) - S_T^-\bigl(\pi f^2\bigr) < \varepsilon.$$

We have shown that, for any $\varepsilon > 0$, there exist cubable solids q and Q that satisfy the conditions $q \subset \Omega \subset Q$, $V(Q) - V(q) < \varepsilon$. By virtue of the cubability criterion, this means that the set Ω is a cubable solid.

Formula (17) can be proved by the same reasoning as formula (3) from the theorem on the area of a curvilinear trapezoid if we replace the function f with πf^2. □

Volume of a solid with given cross-sectional areas

2.9B/23:55 (05:47)

In conclusion, we formulate another theorem related to the calculation of volumes, which we accept without proof (a proof of this theorem is given, for example, in [18, Ch. 7, Sec. 37.2]).

Theorem (on the volume of a solid with given cross-sectional areas).

Let the solid Ω be enclosed between planes that are perpendicular to the OX axis and intersect this axis at points $x = a$ and $x = b$ (as usual, we assume that $a < b$). We denote by G_x a cross-section of the solid Ω by the plane perpendicular to the OX axis and passing through the point $x \in [a, b]$ on this axis (Fig. 15).

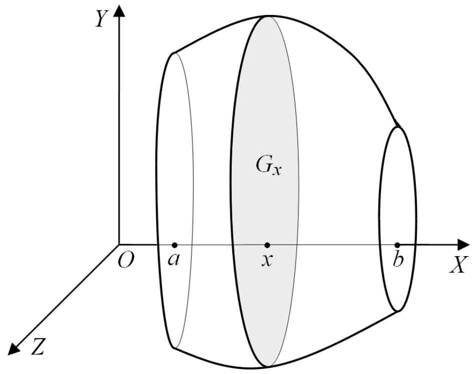

Fig. 15. A solid with given cross-sectional areas

Suppose that, for any point $x \in [a, b]$, the figure G_x is squarable and its area $s(x)$ is a continuous function on the segment $[a, b]$. Suppose, in addition, that, for any $\alpha, \beta \in [a, b]$, the following condition is fulfilled: when projecting the figures G_α and G_β on a plane perpendicular to the OX axis, we get figures, one of which is embedded in the other. Then the solid Ω is cubable and its volume is calculated by the following formula:

$$V(\Omega) = \int_a^b s(x)\, dx.$$

9. Curves and calculating their length

Vector functions and their properties

Vector functions

2.9B/29:42 (04:28)

Definition.

Any map $\overline{r}$ acting from some set $X \subset \mathbb{R}$ into $\mathbb{R}^3$ is called a *vector function*.

When considering vector functions, we always assume that their domain of definition X is some segment $[\alpha, \beta]$.

The name "vector function" means that the value of the vector function $\overline{r}(t)$ can be interpreted not only as some point M in three-dimensional space, but also as a radius vector $\overline{OM}$ with the origin $(0, 0, 0)$ as the starting point and the point M as the ending point.

So, we will interpret the values of the vector function as radius vectors. To emphasize this fact, we will use the overline sign for the notation of vector functions.

If all the values of the vector function lie in one plane, then we assume that it acts in $\mathbb{R}^2$ and write its values in the form of a vector with two coordinates.

The limit of a vector function

2.9B/34:10 (07:39), 2.10A/00:00 (02:14)

Definition.

The vector $\overline{a}$ is called the *limit of the vector function* $\overline{r}(t)$ as $t \to t_0$, $t_0 \in [\alpha, \beta]$, (notation $\lim_{t \to t_0} \overline{r}(t) = \overline{a}$) if $\lim_{t \to t_0} |\overline{r}(t) - \overline{a}| = 0$. Here, $|\overline{r}(t) - \overline{a}|$ denotes the length of the vector $\overline{r}(t) - \overline{a}$.

Thus, the limit of the vector function $\overline{r}(t)$ is defined through the limit of the numerical function $|\overline{r}(t) - \overline{a}|$.

Convergence criterion of a vector function in terms of its coordinate functions

2.10A/02:14 (06:17)

We also introduce the notation of the vector function $\overline{r}(t)$ in the coordinate form: $\overline{r}(t) = \big(x(t), y(t), z(t)\big)$. Here $x(t)$, $y(t)$, $z(t)$ are numerical functions called *coordinate functions*, which are defined, like the vector function $\overline{r}(t)$, on the segment $[\alpha, \beta]$.

We formulate and prove a simple theorem that allows us to reduce the study of the limit of a vector function to the study of the limits of its coordinate functions.

THEOREM (A CRITERION FOR THE CONVERGENCE OF A VECTOR FUNCTION IN TERMS OF ITS COORDINATE FUNCTIONS).

Let $\overline{r}(t) : [\alpha, \beta] \to \mathbb{R}$ be a vector function, $\overline{r}(t) = \big(x(t), y(t), z(t)\big)$, let $\overline{a} \in \mathbb{R}^3$ be a vector, $\overline{a} = (a_1, a_2, a_3)$.

The limit of the vector function $\overline{r}(t)$ is equal to $\overline{a}$, as $t \to t_0$, if and only if the limits of its coordinate functions, as $t \to t_0$, are equal to the corresponding coordinates of the vector $\overline{a}$:

$$\Big(\lim_{t\to t_0} \overline{r}(t) = \overline{a}\Big) \Leftrightarrow \Big(\lim_{t\to t_0} x(t) = a_1, \lim_{t\to t_0} y(t) = a_2, \lim_{t\to t_0} z(t) = a_3\Big).$$

PROOF.

1. Sufficiency. Let us write the expression $|\overline{r}(t) - \overline{a}|$ in coordinate form:

$$|\overline{r}(t) - \overline{a}| = \sqrt{\big(x(t) - a_1\big)^2 + \big(y(t) - a_2\big)^2 + \big(z(t) - a_3\big)^2}\ . \qquad (1)$$

If the limits of the coordinate functions are equal to the coordinates of the vector $\overline{a}$, then each of the differences on the right-hand side of equality (1) approaches 0, therefore, the left-hand side also approaches 0. By definition, this means that the vector function $\overline{r}(t)$ approaches the vector $\overline{a}$. The sufficiency is proved.

2. The necessity. Equation (1) yields the estimates

$$|x(t) - a_1| \le |\overline{r}(t) - \overline{a}|,$$
$$|y(t) - a_2| \le |\overline{r}(t) - \overline{a}|,$$
$$|z(t) - a_3| \le |\overline{r}(t) - \overline{a}|.$$

Thus, if the vector function $\overline{r}(t)$ approaches the vector $\overline{a}$, then the differences indicated on the left-hand side of these estimates also approach 0 (by the theorem on passing to the limit in inequalities) and this is equivalent to the fact that the coordinate functions approach the corresponding coordinates of the vector $\overline{a}$. The necessity is proved. $\square$

Arithmetic properties of the limit of vector functions

2.10A/08:31 (05:53)

Theorem (on arithmetic properties of the limit of vector functions).

Let $\lim_{t\to t_0}\overline{r}_1(t)=\overline{a}$, $\lim_{t\to t_0}\overline{r}_2(t)=\overline{b}$, $\lim_{t\to t_0}f(t)=\alpha$. Then the following relations hold:

$$\lim_{t\to t_0}\bigl(\overline{r}_1(t)+\overline{r}_2(t)\bigr)=\overline{a}+\overline{b},$$

$$\lim_{t\to t_0}f(t)\overline{r}_1(t)=\alpha\overline{a},$$

$$\lim_{t\to t_0}\bigl(\overline{r}_1(t),\overline{r}_2(t)\bigr)=(\overline{a},\overline{b}).$$

In the last equality, $(\overline{a},\overline{b})$ denotes the scalar product of the vectors $\overline{a}$ and $\overline{b}$.

Proof.

We prove the last equality (the other equalities are proved similarly). Let $\overline{r}_i(t)=\bigl(x_i(t),y_i(t),z_i(t)\bigr)$, $i=1,2$, $\overline{a}=(a_1,a_2,a_3)$, $\overline{b}=(b_1,b_2,b_3)$. Then, by virtue of the convergence criterion of a vector function in terms of its coordinate functions, the following limit relations are satisfied:

$$\begin{aligned}&\lim_{t\to t_0}x_1(t)=a_1,\quad \lim_{t\to t_0}y_1(t)=a_2,\quad \lim_{t\to t_0}z_1(t)=a_3,\\ &\lim_{t\to t_0}x_2(t)=b_1,\quad \lim_{t\to t_0}y_2(t)=b_2,\quad \lim_{t\to t_0}z_2(t)=b_3.\end{aligned}\tag{2}$$

Let us write the scalar product in coordinate form:

$$\bigl(\overline{r}_1(t),\overline{r}_2(t)\bigr)=x_1(t)x_2(t)+y_1(t)y_2(t)+z_1(t)z_2(t).$$

Taking into account the limit relations (2) and the theorem on the arithmetic properties of the limit of numerical functions, we obtain that the right-hand side of the resulting equality approaches the expression $a_1b_1+a_2b_2+a_3b_3$. This expression is the coordinate form of the scalar product $(\overline{a},\overline{b})$. □

Differentiable vector functions

Continuity and differentiability of vector functions

2.10A/14:24 (06:37)

Definition.

The vector function $\overline{r}(t)$ is called *continuous* at the point $t_0\in[\alpha,\beta]$ if $\lim_{t\to t_0}\overline{r}(t)=\overline{r}(t_0)$.

Using the criterion for the convergence of a vector function in terms of its coordinate functions, it is easy to show that all the properties previously established for continuous numerical functions remain valid for continuous vector functions.

In particular, arithmetic properties are satisfied for continuous vector functions. One of these properties is the following: if the vector functions $\overline{r}_1(t)$ and $\overline{r}_2(t)$ are continuous at the point t_0, then their scalar product $\big(\overline{r}_1(t), \overline{r}_2(t)\big)$ is a continuous numerical function at the point t_0.

Definition.

If there exists a limit $\lim_{\Delta x \to 0} \frac{1}{\Delta x}\big(\overline{r}(t_0 + \Delta x) - \overline{r}(t_0)\big)$ for the vector function $\overline{r}(t)$, then this limit is called the *derivative* of the vector function $\overline{r}$ at the point t_0 and is denoted by $\overline{r}'(t_0)$.

It follows from the criterion for the convergence of a vector function in terms of its coordinate functions that the derivative of the vector function $\overline{r}(t)$ at t_0 exists if and only if there exist derivatives of its coordinate functions $x(t)$, $y(t)$, $z(t)$; moreover, the following equality holds:

$$\overline{r}'(t_0) = \big(x'(t_0), y'(t_0), z'(t_0)\big).$$

Therefore, we can give the following definition of the differentiability of a vector function: a vector function is called *differentiable* at the point t_0 if all its coordinate functions are differentiable at this point.

Arithmetic properties of differentiable vector functions

2.10A/21:01 (05:13)

Most of the properties previously established for differentiable numerical functions remain valid for differentiable vector functions. In particular, arithmetic properties are fulfilled for them.

Theorem (on arithmetic properties of differentiable vector functions).

Let the vector functions $\overline{r}_1$, $\overline{r}_2$ and the numerical function f be differentiable at the point t_0. Then the vector functions $\overline{r}_1 + \overline{r}_2$, $f\overline{r}_1$ and the numerical function $(\overline{r}_1, \overline{r}_2)$ are also differentiable at the point t_0 and the following relations hold:

$$\big(\overline{r}_1(t_0) + \overline{r}_2(t_0)\big)' = \overline{r}_1'(t_0) + \overline{r}_2'(t_0),$$

$$\big(f(t_0)\overline{r}_1(t_0)\big)' = f'(t_0)\overline{r}_1(t_0) + f(t_0)\overline{r}_1'(t_0),$$

$$\big(\overline{r}_1(t_0), \overline{r}_2(t_0)\big)' = \big(\overline{r}_1'(t_0), \overline{r}_2(t_0)\big) + \big(\overline{r}_1(t_0), \overline{r}_2'(t_0)\big).$$

PROOF.

As in the case of the theorem on the arithmetic properties of the limit of vector functions, it suffices to use the coordinate representations of all expressions and take into account that the differentiability of a vector function is equivalent to the differentiability of its coordinate functions. Let us prove, for example, the last relation using the same notation for coordinate functions as in the theorem on arithmetic properties of the limit of vector functions (in further transformations we omit the argument t_0 for brevity):

$$\begin{aligned}
&\big(\overline{r}_1(t_0), \overline{r}_2(t_0)\big)' = \big(x_1x_2 + y_1y_2 + z_1z_2\big)' = \\
&\quad = \big(x_1x_2\big)' + \big(y_1y_2\big)' + \big(z_1z_2\big)' = \\
&\quad = x_1'x_2 + x_1x_2' + y_1'y_2 + y_1y_2' + z_1'z_2 + z_1z_2' = \\
&\quad = \big(x_1'x_2 + y_1'y_2 + z_1'z_2\big) + \big(x_1x_2' + y_1y_2' + z_1z_2'\big) = \\
&\quad = \big(\overline{r}_1'(t_0), \overline{r}_2(t_0)\big) + \big(\overline{r}_1(t_0)\overline{r}_2'(t_0)\big).
\end{aligned}$$

The remaining relations are proved similarly. □

Lagrange's theorem for vector functions

Violation of the equality from Lagrange's theorem in the case of vector functions

2.10A/26:14 (06:35)

Not all properties of numerical differentiable functions are satisfied in the case of vector functions. In particular, the equality from Lagrange's theorem for numerical functions does not hold for vector functions.

Recall Lagrange's theorem for numerical functions. If the function f is continuous on the segment $[a, b]$ and differentiable on the interval (a, b), then there exists a point $\xi \in (a, b)$ for which the following equality holds:

$$f(b) - f(a) = f'(\xi)(b - a). \tag{3}$$

This equality, generally speaking, does not hold for differentiable vector functions. To do this, it is enough to give an example of a vector function for which this equality is not true.

Consider the vector function $\overline{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. All values of this vector function lie on one plane, therefore, we use two coordinate functions to define it.

The endpoints of the vectors $\overline{r}(t)$ lie on the unit circle with the center at the origin $(0, 0)$. The conditions of Lagrange's theorem related to continuity and differentiability are satisfied, since they are satisfied for the coordinate functions $\cos t$ and $\sin t$.

Further, $\overline{r}(0) = \overline{r}(2\pi) = (1, 0)$, so the difference $\overline{r}(2\pi) - \overline{r}(0)$ is a zero vector. If equality (3) holds for the function $\overline{r}$, then this would mean that there exists a point $\xi \in (0, 2\pi)$ at which the value of the vector function $\overline{r}'(\xi)$ is also equal to the zero vector. But such a point does not exist, since for any $t \in (0, 2\pi)$, we have

$$|\overline{r}'(t)| = |(\cos' t, \sin' t)| = |(-\sin t, \cos t)| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

Thus, $|\overline{r}'(t)| \neq 0$ for any $t \in (0, 2\pi)$, therefore, equality (3) is not true for the function $\overline{r}$.

A version of Lagrange's theorem for vector functions

2.10A/32:49 (10:25), 2.10B/00:00 (03:45)

However, for vector functions, the "weakened" version of Lagrange's theorem holds, which contains the inequality instead of equality (3).

Theorem (Lagrange's theorem for vector functions).

Let the vector function $\overline{r}(t)$ be continuous on the segment $[\alpha, \beta]$ and differentiable on the interval (α, β). Then there exists a point $\xi \in (\alpha, \beta)$ for which the following inequality holds:

$$|\overline{r}(\beta) - \overline{r}(\alpha)| \leq |\overline{r}'(\xi)|\,(\beta - \alpha). \tag{4}$$

Proof.

We introduce the auxiliary numerical function $\varphi(t) = \big(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}(t)\big)$. The function $\varphi(t)$ satisfies all the conditions of Lagrange's theorem for numerical functions on the segment $[\alpha, \beta]$ (this follows from the above properties of continuous and differentiable vector functions). Therefore, there exists a point $\xi \in (\alpha, \beta)$ for which the equality holds:

$$\varphi(\beta) - \varphi(\alpha) = \varphi'(\xi)(\beta - \alpha). \tag{5}$$

Given the definition of the function $\varphi(t)$ and the properties of the scalar product, the left-hand side of the last equality can be represented as follows:

$$\begin{aligned}\varphi(\beta) - \varphi(\alpha) &= \big(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}(\beta)\big) - \big(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}(\alpha)\big) = \\ &= \big(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}(\beta) - \overline{r}(\alpha)\big) = |\overline{r}(\beta) - \overline{r}(\alpha)|^2.\end{aligned}$$

Find the value of $\varphi'(\xi)$ using the formula for the derivative of the scalar product:

$$\begin{aligned}\varphi'(\xi) &= \big(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}(\xi)\big)' = \\ &= \big((\overline{r}(\beta) - \overline{r}(\alpha))', \overline{r}(\xi)\big) + \big(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}'(\xi)\big) = \\ &= \big(\overline{0}, \overline{r}(\xi)\big) + \big(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}'(\xi)\big) = \big(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}'(\xi)\big).\end{aligned}$$

Thus, equality (5) can be rewritten in the following form:

$$|\overline{r}(\beta) - \overline{r}(\alpha)|^2 = \bigl(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}'(\xi)\bigr)(\beta - \alpha). \tag{6}$$

Note that since the left-hand side of the equality is non-negative and the difference $\beta - \alpha$ is positive, the scalar product $\bigl(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}'(\xi)\bigr)$ is also non-negative.

Let us estimate this scalar product from above using the well-known Cauchy–Bunyakovsky inequality, which means that, for any vectors $\overline{a}$ and $\overline{b}$, the estimate $\bigl|(\overline{a}, \overline{b})\bigr| \le |\overline{a}| \cdot |\overline{b}|$ holds:

$$\bigl(\overline{r}(\beta) - \overline{r}(\alpha), \overline{r}'(\xi)\bigr) \le |\overline{r}(\beta) - \overline{r}(\alpha)| \cdot |\overline{r}'(\xi)|.$$

This estimate and equality (6) imply the estimate

$$|\overline{r}(\beta) - \overline{r}(\alpha)|^2 \le |\overline{r}(\beta) - \overline{r}(\alpha)| \cdot |\overline{r}'(\xi)|\,(\beta - \alpha).$$

If the value $|\overline{r}(\beta) - \overline{r}(\alpha)|$ is 0, then the resulting estimate (4) is obviously satisfied. If the value is not equal to 0, then both sides of the last equality can be divided by this value and as a result we obtain the estimate (4). □

Curves in three-dimensional space. Rectifiable curves

Simple curves

2.10B/03:45 (05:36)

Definition.

Let the vector function $\overline{r}(t)$ act from $[\alpha, \beta]$ to $\overline{r}([\alpha, \beta])$ and be continuous and one-to-one. Recall that the one-to-one condition implies that $\overline{r}(t_1) \ne \overline{r}(t_2)$ for $t_1 \ne t_2$.

We denote by $M(t)$ the point that is the endpoint of the radius vector $\overline{r}(t)$: $\overline{r}(t) = \overline{OM}(t)$. Then, due to the continuity of the vector function $\overline{r}(t)$, the set of endpoints of the vector function $\overline{r}(t)$, when t changes from α to β, will be a continuous line starting at the point $M(\alpha)$ and ending at the point $M(\beta)$ and, due to the one-to-one property of the vector function $\overline{r}(t)$, this line will not have self-intersections. This set of points is called an *oriented simple curve* Γ specified by a vector function $\overline{r}$ on the segment $[\alpha, \beta]$ (Fig. 16):

$$\Gamma = \bigl\{M(t) : \overline{r}(t) = \overline{OM}(t),\ t \in [\alpha, \beta]\bigr\}.$$

The point $M(\alpha)$ is called the *starting point* of the curve Γ, the point $M(\beta)$ is called its *ending point*.

As a rule, we will omit the word “oriented”.

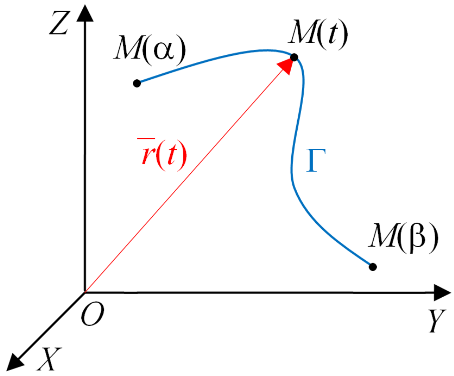

Fig. 16. The oriented simple curve Γ

Remark.

If the vector function $\overline{r}(t)$ is not one-to-one, then the line consisting of points $M(t)$ will have self-intersections. However, if there are a finite set of such intersections, then this line can always be represented as the union of a finite number of oriented simple curves, each of which corresponds to the values of t from some segment $[\alpha_i, \beta_i]$ embedded in $[\alpha, \beta]$. Thus, finding formulas for the length of a simple curve, we can apply these results to a wider class of curves having a finite number of self-intersections.

Rectifiable curve and its length

2.10B/09:21 (07:27)

Consider a simple curve Γ specified by the vector function $\overline{r}(t)$ for $t \in [\alpha, \beta]$. This curve has the starting point A and the ending point B. Let T be a partition of the segment $[\alpha, \beta]$: $\alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta$. The partition T corresponds to points on the curve Γ as follows: $M_0 = M(t_0) = A$, $M_1 = M(t_1)$, ..., $M_{n-1} = M(t_{n-1})$, $M_n = M(t_n) = B$.

Consider a *polyline* (a *polygonal chain*) with vertices at the points M_i, $i = 0, \dots, n$ (see the left-hand part of Fig. 17). The length L_T of this polyline is equal to

$$L_T = \sum_{i=1}^{n} |M_{i-1}M_i| = \sum_{i=1}^{n} |\overline{r}(t_i) - \overline{r}(t_{i-1})|.$$

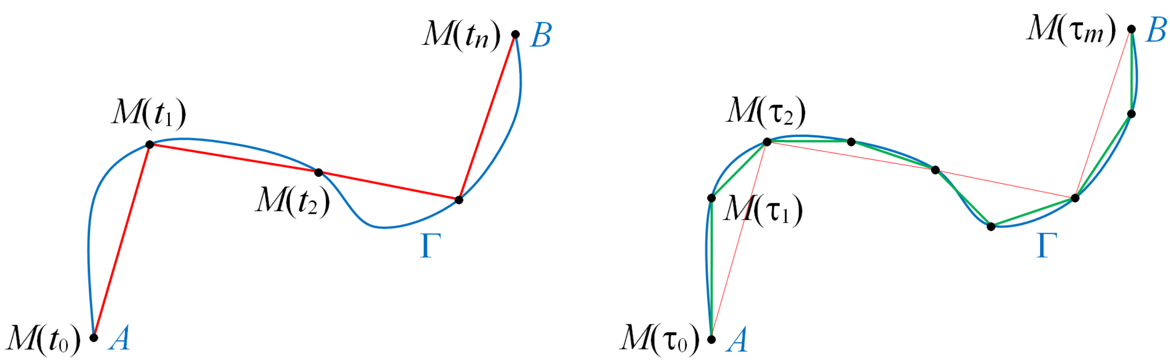

Fig. 17. The simple curve Γ and associated polylines

Note that if we add new points to the partition T, then, due to the triangle inequality, the length of the polyline can either remain unchanged or increase (see the right-hand part of Fig. 17).

DEFINITION.

If the set of L_T values for all possible partitions of T is bounded from above, then the curve Γ is called *rectifiable* and its length $L(\Gamma)$ is defined as follows:

$$L(\Gamma) \stackrel{\text{def}}{=} \sup_T L_T.$$

We accept without proof the following important property of the length of a curve.

THEOREM (ON THE ADDITIVITY OF THE LENGTH OF A CURVE).

If the curve Γ is rectifiable and some point M' divides it into two parts Γ_1 and Γ_2, then the curves Γ_1 and Γ_2 are also rectifiable and the following relation holds:

$$L(\Gamma) = L(\Gamma_1) + L(\Gamma_2).$$

This property is called the *additivity* of the curve length. Its proof is given, for example, in [18, Ch. 4, Sec. 22.5].

Properties of continuously differentiable curves

The theorem on the rectifiability of a continuously differentiable curve

2.10B/16:48 (10:04)

It turns out that it is sufficient for the rectifiability of a curve that the vector function specifying it is continuously differentiable on $[\alpha, \beta]$. Recall that the condition of continuously differentiability means that the vector function has a continuous derivative.

A curve defined by a continuously differentiable vector function is called *continuously differentiable* (or *smooth*).

THEOREM (ON THE RECTIFIABILITY OF A CONTINUOUSLY DIFFERENTIABLE CURVE).

If the curve Γ is specified by the continuously differentiable vector function $\overline{r}(t)$ for $t \in [\alpha, \beta]$, then it is rectifiable and the following estimate holds for its length $L(\Gamma)$:

$$L(\Gamma) \le \max_{t \in [\alpha, \beta]} |\overline{r}'(t)| \cdot (\beta - \alpha). \tag{7}$$

Proof.

It is enough for us to prove that the value on the right-hand side of (7) is an upper bound for the length of any polyline associated with the curve Γ.

Consider some partition T of the segment $[\alpha, \beta]$ and the polyline $M_0 M_1 \ldots M_n$ determined by this partition.

Using Lagrange's theorem for vector functions, we can estimate the length of the segment $M_{i-1}M_i$, $i = 1, \ldots, n$, as follows:

$$|M_{i-1}M_i| = |\overline{r}(t_i) - \overline{r}(t_{i-1})| \leq |\overline{r}'(\xi_i)| \, (t_i - t_{i-1}).$$

Here ξ_i is some point lying on the interval (t_{i-1}, t_i).

Since, by condition, the vector function $\overline{r}(t)$ is continuously differentiable, the vector function $\overline{r}'(t)$ and its absolute value $|\overline{r}'(t)|$ are continuous on $[\alpha, \beta]$. Therefore, the function $|\overline{r}'(t)|$ takes its maximum value on the segment $[\alpha, \beta]$ and, for any point $\xi \in [\alpha, \beta]$, we obtain

$$|\overline{r}'(\xi)| \leq \max_{t \in [\alpha,\beta]} |\overline{r}'(t)|.$$

Thus, the following estimate holds for the length $|M_{i-1}M_i|$:

$$|M_{i-1}M_i| \leq \max_{t \in [\alpha,\beta]} |\overline{r}'(t)| \, (t_i - t_{i-1}).$$

Summarize these inequalities for $i = 1, \ldots, n$:

$$\sum_{i=1}^{n} |M_{i-1}M_i| \leq \max_{t \in [\alpha,\beta]} |\overline{r}'(t)| \sum_{i=1}^{n} (t_i - t_{i-1}).$$

On the left-hand side we got the length L_T of the polyline. If we write the summands of the sum on the right-hand side in the reverse order, then it is easy to verify that only two summands remain:

$$\sum_{i-1}^{n} (t_i - t_{i-1}) = t_n - t_{n-1} + t_{n-1} - t_{n-2} + \cdots + t_1 - t_0 = t_n - t_0 = \beta - \alpha.$$

Consequently, for any partition T, we get the estimate

$$L_T \leq \max_{t \in [\alpha,\beta]} |\overline{r}'(t)| \, (\beta - \alpha).$$

So, we have proved that the set of all values L_T is bounded from above by the indicated quantity, which implies both the rectifiability of the Γ curve and the estimate (7), since the obtained upper bound $\max_{t \in [\alpha,\beta]} |\overline{r}'(t)| \, (\beta - \alpha)$ cannot be less than the least upper bound of $\sup_T L_T$ equal to $L(\Gamma)$. $\square$

Theorem on the derivative for the length of the initial part of a curve

2.10B/26:52 (14:04)

We continue the consideration of continuously differentiable curves. As before, we assume that the curve Γ is specified by a continuously differentiable vector function $\overline{r}(t)$ defined on $[\alpha, \beta]$ and this curve has the starting point A and the ending point B.

For the curve Γ, we introduce an auxiliary function $s(t)$ equal to the length of the part of the curve that starts at A and ends at $M(t)$. This initial part of the original curve is rectifiable by virtue of the additivity theorem for the curve length.

Obviously, $s(\alpha) = 0$, $s(\beta) = L(\Gamma)$. In addition, the function $s(t)$ is increasing.

THEOREM (ON THE DERIVATIVE FOR THE LENGTH OF THE INITIAL PART OF A CURVE).

For a continuously differentiable curve Γ, the function $s(t)$ is also continuously differentiable and, for any point $t \in [\alpha, \beta]$, the formula holds:

$$s'(t) = |\overline{r}'(t)|. \qquad (8)$$

PROOF.

We choose some point $t_0 \in [\alpha, \beta]$ and prove formula (8) for this point. The point t_0 corresponds to the point $M(t_0)$ on the curve Γ. In addition, we choose some nonzero increment Δt (which can be both positive and negative) and consider the point $M(t_0 + \Delta t)$ (Fig. 18).

The part of the curve between the points $M(t_0)$ and $M(t_0 + \Delta t)$ has a length equal to $|s(t_0 + \Delta t) - s(t_0)|$. The difference $s(t_0 + \Delta t) - s(t_0)$ is positive if $\Delta t > 0$ and negative if $\Delta t < 0$.

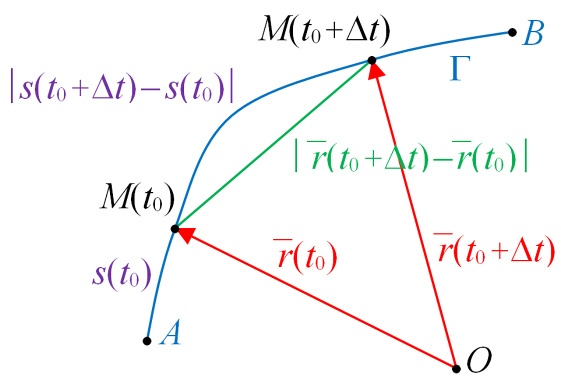

Fig. 18. Points $M(t_0)$, $M(t_0 + \Delta t)$ of the curve Γ

We write the formula for the length of the segment $M(t_0)M(t_0 + \Delta t)$ taking into account that the corresponding vector is the difference of the vectors $\overline{r}(t_0 + \Delta t)$ and $\overline{r}(t_0)$:

$$|M(t_0)M(t_0 + \Delta t)| = |\overline{r}(t_0 + \Delta t) - \overline{r}(t_0)|.$$

Since the length of the curve located between the points $M(t_0)$ and $M(t_0 + \Delta t)$ does not exceed the length of the segment $M(t_0)M(t_0 + \Delta t)$, the following inequality holds:

$$|\overline{r}(t_0 + \Delta t) - \overline{r}(t_0)| \le |s(t_0 + \Delta t) - s(t_0)|.$$

By $P(\Delta t)$, we denote the segment between the points t_0 and $t_0 + \Delta t$: $P(\Delta t) = [t_0, t_0 + \Delta t]$ if $\Delta t > 0$ and $P(\Delta t) = [t_0 + \Delta t, t_0]$ if $\Delta t < 0$.

Using estimate (7) from the previous theorem, we can estimate the value $|s(t_0 + \Delta t) - s(t_0)|$ as follows:

$$|s(t_0 + \Delta t) - s(t_0)| \le \max_{t \in P(\Delta t)} |\overline{r}'(t)| \cdot |\Delta t|.$$

So, we have a double inequality:

$$|\overline{r}(t_0 + \Delta t) - \overline{r}(t_0)| \le |s(t_0 + \Delta t) - s(t_0)| \le \max_{t \in P(\Delta t)} |\overline{r}'(t)| \cdot |\Delta t|.$$

Since $\Delta t \ne 0$, we can divide all parts of this double inequality by $|\Delta t|$:

$$\frac{|\overline{r}(t_0 + \Delta t) - \overline{r}(t_0)|}{|\Delta t|} \le \frac{|s(t_0 + \Delta t) - s(t_0)|}{|\Delta t|} \le \max_{t \in P(\Delta t)} |\overline{r}'(t)|. \qquad (9)$$

In the expression on the left, we can move the number Δt under the sign of absolute value:

$$\frac{|\overline{r}(t_0 + \Delta t) - \overline{r}(t_0)|}{|\Delta t|} = \left|\frac{1}{\Delta t}\big(\overline{r}(t_0 + \Delta t) - \overline{r}(t_0)\big)\right|.$$

Thus, the left-hand expression is the length of the vector $\frac{1}{\Delta t}\big(\overline{r}(t_0 + \Delta t) - \overline{r}(t_0)\big)$.

In the expression in the middle part of estimate (9), we can omit the signs of absolute value, since, as we noted earlier, the expression $s(t_0+\Delta t) - s(t_0)$ and the increment Δt have the same signs and therefore their ratio is positive:

$$\frac{|s(t_0 + \Delta t) - s(t_0)|}{|\Delta t|} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}.$$

Since the function $|\overline{r}'(t)|$ is continuous and therefore, by virtue of the second Weierstrass theorem, it takes its maximum value on the segment $P(\Delta t)$ at some point $\xi \in P(\Delta t)$, the expression on the right-hand side of (9) can be represented as follows:

$$\max_{t \in P(\Delta t)} |\overline{r}'(t)| = |\overline{r}'(\xi)|.$$

Given the indicated transformations, the double inequality (9) takes the following form:

$$\left|\frac{1}{\Delta t}\bigl(\overline{r}(t_0+\Delta t)-\overline{r}(t_0)\bigr)\right| \le \frac{s(t_0+\Delta t)-s(t_0)}{\Delta t} \le |\overline{r}'(\xi)|. \tag{10}$$

If Δt approaches 0, then the left-hand side of the double inequality (10) approaches $|\overline{r}'(t_0)|$ (this follows directly from the definition of the derivative of a vector function). The right-hand side of the double inequality (10) approaches the same limit, since, as $\Delta t \to 0$, the segment $P(\Delta t)$ “contracts” to the point t_0 and therefore the point $\xi \in P(\Delta t)$ also approaches the point t_0.

Thus, both the left-hand side and the right-hand side of the double inequality (10) approach the same value $|\overline{r}'(t_0)|$, therefore, by the theorem on passing to the limit to inequalities, the middle part of inequality also approaches the same value:

$$\lim_{\Delta t\to 0}\frac{s(t_0+\Delta t)-s(t_0)}{\Delta t} = |\overline{r}'(t_0)|.$$

But the limit indicated on the left is equal to the derivative of the function $s(t)$ at the point t_0. So, we simultaneously proved both the differentiability of the function $s(t)$ at an arbitrary point $t_0 \in [\alpha,\beta]$ and the validity of formula (8) for this point.

The fact that the function $s'(t)$ is continuous follows from equality (8), since the function $|\overline{r}'(t)|$ has the same property. $\square$

Versions of the formula for finding the length of a curve

Formula for the length of a curve specified by a vector function

2.10B/40:56 (03:02)

Let us integrate the proved equality (8) from α to β (the integrals exist, since the integrands are continuous):

$$\int_\alpha^\beta s'(t)\,dt = \int_\alpha^\beta |\overline{r}'(t)|\,dt.$$

Since the function $s(t)$ is the antiderivative of the function $s'(t)$, the left-hand side of the last equality can be transformed by the Newton–Leibniz formula as follows:

$$\int_\alpha^\beta s'(t)\,dt = s(\beta)-s(\alpha) = L(\Gamma)-0 = L(\Gamma).$$

Thus, we have obtained the basic formula for the length of the curve Γ specified by the continuously differentiable vector function $\overline{r}(t)$ on the segment $[\alpha,\beta]$:

$$L(\Gamma) = \int_\alpha^\beta |\overline{r}'(t)|\,dt. \tag{11}$$

Formulas for the length of a curve specified in the Cartesian coordinate system

2.11A/00:00 (04:25)

We obtain several versions of formula (11), in which various methods for specifying the vector function $\overline{r}(t)$ are used.

If the vector function $\overline{r}(t)$ is defined by its coordinate functions $\big(x(t), y(t), z(t)\big)$, then its derivative can be obtained by differentiating these coordinate functions: $\overline{r}'(t) = \big(x'(t), y'(t), z'(t)\big)$. Considering the vector length formula, we obtain the following version of formula (11):

$$L(\Gamma) = \int_\alpha^\beta \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}\,dt.$$

Note that if the curve Γ is specified by a set of coordinate functions $\big(x(t), y(t), z(t)\big)$ for $t \in [\alpha, \beta]$, then they say that it is represented in *parametric form* with the parameter t.

If the vector function $\overline{r}(t)$ takes values on the plane, then two coordinate functions are enough to define it: $\overline{r}(t) = \big(x(t), y(t)\big)$ (in this case, we can assume that the points of the curve lie on the plane OXY and their third coordinate is 0). Therefore, for the length of plane curves represented in parametric form, we obtain the following formula:

$$L(\Gamma) = \int_\alpha^\beta \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\,dt. \tag{12}$$

A continuously differentiable curve Γ on a plane can also be defined as a graph of some continuously differentiable function: $y = f(x)$, $x \in [\alpha, \beta]$. This graph consists of points $\big(x, f(x)\big)$, so the vector function $\overline{r}(t)$ that specifies the curve Γ can be defined as follows: $\overline{r}(t) = \big(t, f(t)\big)$, $t \in [\alpha, \beta]$. Since $x(t) = t$, $y(t) = f(t)$, the integrand in formula (12) takes the form

$$\sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2} = \sqrt{(t')^2 + \big(f'(t)\big)^2} = \sqrt{1 + \big(f'(t)\big)^2}.$$

Substituting this expression into formula (12), we obtain a version of the formula for the length of a plane curve represented in the form of a graph $y = f(x)$. In this version, it is convenient to use the variable x as an integration parameter:

$$L(\Gamma) = \int_\alpha^\beta \sqrt{1 + \big(f'(x)\big)^2}\,dx.$$

Formula for the length of a curve specified in the polar coordinate system

2.11A/04:25 (06:33)

Let us consider one more way of specifying a curve Γ: when it is represented as a graph of a function in the polar coordinate system. In this case, the equation of the graph of the function has the form $\rho = f(\varphi)$, $\varphi \in [\alpha, \beta]$, where ρ is the distance from the origin, φ is the angle from the coordinate OX axis, and the function $f(\varphi)$ is continuously differentiable on $[\alpha, \beta]$.

We start by finding the parametric representation of the Γ curve in the Cartesian coordinate system, i. e., by finding its coordinate functions $x(t)$, $y(t)$. To do this, we use the relation between the polar (ρ, φ) and the Cartesian (x, y) coordinates: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. If we take the polar angle φ as the parameter, then we get

$$x(\varphi) = \rho\cos\varphi = f(\varphi)\cos\varphi, \quad y(\varphi) = \rho\sin\varphi = f(\varphi)\sin\varphi. \tag{13}$$

In formula (12), the expression $\bigl(x'(t)\bigr)^2 + \bigl(y'(t)\bigr)^2$ is under the root sign. Replace the argument t with φ in this expression, substitute the values $x(\varphi)$ and $y(\varphi)$ defined by formulas (13), and transform the resulting expression by differentiating the products and applying the formula $(a+b)^2 = a^2+2ab+b^2$:

$$\begin{aligned}
\bigl(x'(\varphi)\bigr)^2 + \bigl(y'(\varphi)\bigr)^2 &= \bigl((f(\varphi)\cos\varphi)'\bigr)^2 + \bigl((f(\varphi)\sin\varphi)'\bigr)^2 = \\
&= \bigl(f'(\varphi)\cos\varphi + f(\varphi)(\cos\varphi)'\bigr)^2 + \bigl(f'(\varphi)\sin\varphi + f(\varphi)(\sin\varphi)'\bigr)^2 = \\
&= \bigl(f'(\varphi)\cos\varphi - f(\varphi)\sin\varphi\bigr)^2 + \bigl(f'(\varphi)\sin\varphi + f(\varphi)\cos\varphi\bigr)^2 = \\
&= \bigl(f'(\varphi)\bigr)^2\cos^2\varphi - 2f(\varphi)f'(\varphi)\cos\varphi\sin\varphi + f^2(\varphi)\sin^2\varphi + \\
&\quad + \bigl(f'(\varphi)\bigr)^2\sin^2\varphi + 2f(\varphi)f'(\varphi)\cos\varphi\sin\varphi + f^2(\varphi)\cos^2\varphi = \\
&= \bigl(f'(\varphi)\bigr)^2 + f^2(\varphi).
\end{aligned}$$

At the final stage of the transformations, we twice used the Pythagorean trigonometric identity $\sin^2\varphi + \cos^2\varphi = 1$.

Substituting the transformed expression into formula (12), we obtain the formula for the length of the curve represented in the form of a graph $\rho = f(\varphi)$ in the polar coordinate system:

$$L(\Gamma) = \int_\alpha^\beta \sqrt{\bigl(f'(\varphi)\bigr)^2 + f^2(\varphi)}\,d\varphi.$$

10. Improper integrals: definition and properties

Tasks leading to the notion of an improper integral

3.8B/00:00 (04:13)

Starting to study a definite integral, we considered the problem of finding the area of a curvilinear trapezoid defined using some continuous function f on the segment $[a, b]$. We further proved that to solve this problem, it is necessary to calculate the integral $\int_a^b f(x)\,dx$.

Now suppose that the function f is defined and continuous on the entire positive semiaxis OX and it is positive and decreasing on this semiaxis. Is it possible to determine the area of the *infinite* region D bounded by the positive semiaxis OX, the line $x = 0$, and the graph $y = f(x)$?

Let us choose some point $c > 0$ and consider the part of the region D located to the left of the line $x = c$. This part is a curvilinear trapezoid defined on the segment $[0, c]$ and its area is $\Phi(c) = \int_0^c f(x)\,dx$.

As the value of c increases, the area of $\Phi(c)$ will increase too. If there exists a limit $\Phi(c)$ as $c \to +\infty$, then it is natural to consider this limit as the area of an infinite region D.

Consider another example. Suppose now that the function f is defined and continuous on the half-interval $(0, b]$, takes positive values on it and increases unlimitedly as $x \to +0$.

In this case, we get an infinite region D bounded by the segment $[0, b]$ of the axis OX, the lines $x = 0$ and $x = b$, and the graph $y = f(x)$. To determine the area of the region D, we can choose the point $c \in (0, b)$ and consider the part of the region D bounded by the vertical lines $x = c$ and $x = b$. This part is a curvilinear trapezoid and its area is $\Phi(c) = \int_c^b f(x)\,dx$.

If there exists a limit $\Phi(c)$ as $c \to +0$, then this limit can be considered as the area of the infinite region D.

These examples show that improper integrals can be of two types: integrals over an infinite integration interval of a bounded function and integrals over a finite interval, but of a function that is unbounded on a given interval. In any of these cases, the passing to limit is used to determine the improper integral.

Definitions of an improper integral

Improper integral over a semi-infinite interval

3.8B/04:13 (10:02)

DEFINITION 1 (DEFINITION OF AN IMPROPER INTEGRAL OVER A SEMI-INFINITE INTERVAL).

Let a function f be defined on the set $[a, +\infty)$ and integrable on any segment $[a, c]$, $c > a$. If there exists a finite limit of the integral $\int_a^c f(x)\,dx$ as $c \to +\infty$, then they say that there exists an *improper integral* $\int_a^{+\infty} f(x)\,dx$ and its value is assumed to be equal to this limit:

$$\int_a^{+\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{c\to+\infty} \int_a^c f(x)\,dx.$$

In this case, they say that the improper integral $\int_a^{+\infty} f(x)\,dx$ *converges*.

If the limit $\lim_{c\to+\infty} \int_a^c f(x)\,dx$ does not exist or is equal to infinity, then they say that the improper integral $\int_a^{+\infty} f(x)\,dx$ *diverges*.

An improper integral over a semi-infinite interval of the form $(-\infty, b]$ is defined in a similar way.

EXAMPLES.

1. Consider the integral $\int_1^{+\infty} \frac{dx}{x^\alpha}$, $\alpha \in \mathbb{R}$.

We choose the value $c > 1$ and find the integral over a finite segment:

$$\int_1^c \frac{dx}{x^\alpha} = \begin{cases} \left.\dfrac{x^{-\alpha+1}}{-\alpha+1}\right|_1^c = \dfrac{c^{-\alpha+1}}{-\alpha+1} - \dfrac{1}{-\alpha+1}, & \alpha \ne 1, \\ \ln x|_1^c = \ln c, & \alpha = 1. \end{cases}$$

The function $\ln c$ approaches infinity as $c \to +\infty$. The function $c^{-\alpha+1}$ approaches infinity as $c \to +\infty$ if $\alpha < 1$ and approaches 0 if $\alpha > 1$. Consequently, the initial improper integral diverges for $\alpha \le 1$ and converges for $\alpha > 1$ and, for the converging integral, the formula holds:

$$\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{c\to+\infty} \left(\frac{c^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1} \right) = \frac{1}{\alpha-1}, \quad \alpha > 1.$$

2. Consider the integral $\int_0^{+\infty} e^{-x}\,dx$. In this case, for the segment $[0, c]$, we have

$$\int_0^c e^{-x}\,dx = -e^{-x}\big|_0^c = -e^{-c} + 1.$$

Hence,

$$\int_0^{+\infty} e^{-x}\,dx = \lim_{c\to+\infty} (-e^{-c} + 1) = 1.$$

Improper integral for an unbounded function and the definition of an improper integral in the general case 3.8B/14:15 (09:12)

Definition 2 (definition of an improper integral for an unbounded function).

Let the function f be defined on the half-interval $[a, b)$ and integrable on any segment $[a, c]$, $a < c < b$. If there exists a finite limit of the integral $\int_a^c f(x)\,dx$ as $c \to b - 0$, then they say that there exists an improper integral $\int_a^b f(x)\,dx$ and its value is assumed to be equal to this limit:

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{c \to b-0} \int_a^c f(x)\,dx.$$

In this case, they also say that the improper integral $\int_a^b f(x)\,dx$ converges.

If the limit $\lim_{c \to b-0} \int_a^c f(x)\,dx$ does not exist or is equal to infinity, then they say that the improper integral $\int_a^b f(x)\,dx$ diverges.

The improper integral for the function defined on the half-interval $(a, b]$ is defined in a similar way.

Example.

Consider the integral $\int_0^1 \frac{dx}{x^\alpha}$, $\alpha \in \mathbb{R}$. Obviously, for $\alpha \le 0$, this integral is an usual (proper) integral, since the function $\frac{1}{x^\alpha}$ in this case is defined and continuous on the entire segment $[0, 1]$. The value of the integral for $\alpha < 0$ is equal to

$$\int_0^1 \frac{dx}{x^\alpha} = \frac{x^{-\alpha+1}}{-\alpha+1}\bigg|_0^1 = \frac{1}{1-\alpha}.$$

The formula $\int_0^1 \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}$ is also valid for the case $\alpha = 0$.

For $\alpha > 0$, we have an improper integral, since the function $\frac{1}{x^\alpha}$ is unbounded in a neighborhood of the point 0. So, we choose the value $c \in (0, 1)$ and find the integral over the finite segment:

$$\int_c^1 \frac{dx}{x^\alpha} = \begin{cases} \dfrac{x^{-\alpha+1}}{-\alpha+1}\bigg|_c^1 = \dfrac{1}{-\alpha+1} - \dfrac{c^{-\alpha+1}}{-\alpha+1}, & \alpha \ne 1, \\ \ln x\big|_c^1 = -\ln c, & \alpha = 1. \end{cases}$$

The function $-\ln$ c approaches infinity as $c \to +0$. The function $c^{-\alpha+1}$ approaches infinity as $c \to +0$ if $\alpha > 1$ and approaches 0 if $\alpha < 1$. Consequently, the initial improper integral diverges for $\alpha \ge 1$ and converges for $0 < \alpha < 1$ and the formula holds for the converging integral:

$$\int_0^1 \frac{dx}{x^\alpha} = \lim_{c \to +0}\left(\frac{1}{-\alpha+1} - \frac{c^{-\alpha+1}}{-\alpha+1}\right) = \frac{1}{1-\alpha}, \quad 0 < \alpha < 1.$$

So, the integral $\int_0^1 \frac{dx}{x^\alpha}$ exists for $\alpha < 1$, it is equal to $\frac{1}{1-\alpha}$, and, for $0 < \alpha < 1$, it must be understood in an improper sense.

In the future, it will be convenient for us to simultaneously consider improper integrals over semi-infinite intervals and improper integrals of unbounded functions. So, let us give a general definition of an improper integral.

Definition 3 (the definition of an improper integral in the general case).

Let the function f be defined on the half-interval $[a, b)$ and integrable on any segment $[a, c]$, $a < c < b$. The point b is either finite or equal to $+\infty$. If there exists a finite limit of the integral $\int_a^c f(x)\,dx$ as $c \to b - 0$, then they say that there exists an improper integral $\int_a^b f(x)\,dx$ and its value is assumed to be equal to this limit:

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{c \to b-0} \int_a^c f(x)\,dx.$$

In this case, they also say that the improper integral $\int_a^b f(x)\,dx$ converges.

If the limit $\lim_{c \to b-0} \int_a^c f(x)\,dx$ does not exist or is equal to infinity, then they say that the improper integral $\int_a^b f(x)\,dx$ diverges.

An improper integral with a singularity at the left endpoint a of the integration interval is defined in a similar way; the left endpoint may be equal to $-\infty$.

If an improper integral has a singularity at both endpoints of the integration interval (a, b), then it is considered as the sum of the integrals over the intervals $(a, d]$ and $[d, b)$ for some point $d \in (a, b)$ and is convergent if and only if improper integrals converge over each of the intervals $(a, d]$ and $[d, b)$. We return to the discussion of integrals with several singularities at the end of the next chapter.

Properties of improper integrals

Linearity of the improper integral with respect to the integrand

3.8B/23:27 (05:12)

Theorem 1 (on the linearity of an improper integral with respect to integrand).

Let the functions f and g be defined on $[a, b)$, $\alpha, \beta \in \mathbb{R}$. Let there exist improper integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$. Then there exists an improper integral $\int_a^b \bigl(\alpha f(x) + \beta g(x)\bigr)\,dx$ and the following formula holds:

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx. \qquad (1)$$

PROOF.

Let $c \in (a, b)$. From the definition of the improper integral, we obtain that there exist integrals $\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$. Then, due to the linearity of the usual (proper) definite integral with respect to integrands, the integral $\int_a^c \big(\alpha f(x) + \beta g(x)\big)\,dx$ also exists and the equality holds:

$$\int_a^c \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int_a^c f(x)\,dx + \beta \int_a^c g(x)\,dx.$$

In the resulting equality, we pass to the limit as $c \to b - 0$. By the definition of an improper integral, the limits of $\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$ exist and are equal to $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$, respectively. Using the arithmetic properties of the limit, we obtain that the limit on the left-hand side also exists and is equal to $\alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$.

Thus, we proved that the integral $\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx$ converges and we also proved formula (1). □

Additivity of an improper integral with respect to the integration interval and change of variables in an improper integral

3.8B/28:39 (05:47)

THEOREM 2 (ON THE ADDITIVITY OF AN IMPROPER INTEGRAL WITH RESPECT TO THE INTEGRATION INTERVAL).

Let the function f be defined on $[a, b)$ and there exists an improper integral $\int_a^b f(x)\,dx$. Then, for any point $d \in (a, b)$, the improper integral $\int_d^b f(x)\,dx$ converges and the equality holds:

$$\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx.$$

The proof of this theorem is carried out similarly to the proof of Theorem 1, using the additivity property of the usual definite integral with respect to the integration segment and arithmetic properties of the limit. □

THEOREM 3 (ON THE CHANGE OF VARIABLES IN AN IMPROPER INTEGRAL).

Let the function f be defined on $[a, b)$ and there exists an improper integral $\int_a^b f(x)\,dx$. Let the function φ act from $[\alpha, \beta)$ on $[a, b)$, be continuously differentiable on $[\alpha, \beta)$, $\varphi'(t) > 0$ for $t \in [\alpha, \beta)$, $\varphi(\alpha) = a$ and $\lim_{t\to\beta-0} \varphi(t) = b$. Then there exists an improper integral $\int_\alpha^\beta f\big(\varphi(t)\big)\varphi'(t)\,dt$ and the equality holds:

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt. \tag{2}$$

PROOF[1].

Let $\gamma \in (\alpha, \beta)$. Then, by the theorem on the change of variables in a usual (proper) definite integral, the equality holds:

$$\int_a^{\varphi(\gamma)} f(x)\,dx = \int_\alpha^\gamma f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt. \tag{3}$$

The left-hand side of equality (3) can be represented as a superposition, where the external function has the argument c and the internal function has the argument γ:

$$\int_a^{\varphi(\gamma)} f(x)\,dx = \Bigl(\int_a^c f(x)\,dx\Bigr) \circ \varphi(\gamma).$$

The conditions of the theorem imply the following limit equalities: $\lim_{\gamma\to\beta-0} \varphi(\gamma) = b$, $\lim_{c\to b-0} \int_a^c f(x)\,dx = \int_a^b f(x)\,dx$, and $\varphi(t) \neq b$ when $t \in [\alpha, \beta)$. Thus, all the conditions of the superposition limit theorem are satisfied and, by virtue of this theorem, the limit of the superposition $\int_a^{\varphi(\gamma)} f(x)\,dx$ as $\gamma \to \beta - 0$ is equal to the limit of the external function:

$$\lim_{\gamma\to\beta-0} \int_a^{\varphi(\gamma)} f(x)\,dx = \lim_{c\to b-0} \int_a^c f(x)\,dx = \int_a^b f(x)\,dx.$$

Therefore, the limit of the right-hand side of equality (3) as $\gamma \to \beta - 0$ also exists and is equal to $\int_a^b f(x)\,dx$.

Thus, we proved that the improper integral $\int_\alpha^\beta f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt$ converges and we also obtained formula (2). $\square$

Integration formula by parts for improper integrals. Theorem on the coincidence of the integral in the proper and improper sense

3.8B/34:26 (07:19)

THEOREM 4 (ON INTEGRATION BY PARTS OF AN IMPROPER INTEGRAL).

Let the functions u and v be defined and continuously differentiable on $[a, b)$ and there exists a limit $\lim_{x\to b} u(x)v(x)$. Then the improper integrals $\int_a^b uv'\,dx$ and $\int_a^b u'v\,dx$ either both converge or both diverge, and if they converge, then the following relation holds, which is called the *integration formula by parts* for improper integrals:

[1]There is no proof of this theorem in video lectures.

$$\int_a^b uv'\,dx = (uv)\big|_a^b - \int_a^b u'v\,dx.$$

In this formula, the notation $(uv)\big|_a^b$ means the following difference: $\lim_{x\to b} u(x)v(x) - u(a)v(a)$.

PROOF.

The proof of this theorem is carried out similarly to the proof of Theorem 1, using the integration formula by parts for the usual definite integral and arithmetic properties of the limit. □

THEOREM 5 (ON THE COINCIDENCE OF THE INTEGRAL IN THE PROPER AND IMPROPER SENSE).

If the function f is defined and integrable on the interval $[a, b]$, then the following equality holds, where the usual (proper) integral is indicated on the left-hand side:

$$\int_a^b f(x)\,dx = \lim_{c\to b-0} \int_a^c f(x)\,dx. \tag{4}$$

Thus, if a proper integral exists, then it coincides with the corresponding integral understood in an improper sense (i. e., determined by passing to the limit).

PROOF.

The integral $\int_a^c f(x)\,dx$, which is considered as a function $\Phi(c)$ of the argument c, is an integral with a variable upper limit:

$$\Phi(c) = \int_a^c f(x)\,dx.$$

Since, by condition, the function f is integrable over the segment $[a, b]$, we obtain, by virtue of the properties of the integral with a variable upper limit, that the function $\Phi(c)$ is continuous on this segment. Hence,

$$\lim_{c\to b-0} \Phi(c) = \Phi(b).$$

Taking into account the definition of the function $\Phi(c)$, we obtain the relation (4). □

11. Absolute and conditional convergence of improper integrals

Cauchy criterion for the convergence of an improper integral

3.9A/00:00 (10:08)

Theorem (Cauchy criterion for the convergence of an improper integral).

Let the function f be defined on the interval $[a,b)$ and there exists the integral $\int_a^c f(x)\,dx$ for any point $c \in (a,b)$. The improper integral $\int_a^b f(x)\,dx$ converges if and only if the following condition is satisfied:

$$\forall\,\varepsilon > 0 \quad \exists\,B \in (a,b) \quad \forall\,c', c'', B < c' < c'' < b,$$
$$\Bigl|\int_{c'}^{c''} f(x)\,dx\Bigr| < \varepsilon. \tag{1}$$

Proof.

Let us introduce the auxiliary function $\Phi(c) = \int_a^c f(x)\,dx$. According to the definition of an improper integral, the convergence of the integral $\int_a^b f(x)\,dx$ is equivalent to the existence of the limit of the function $\Phi(c)$ as $c \to b-0$.

By virtue of the Cauchy criterion for the existence of a function limit, the limit of $\Phi(c)$ as $c \to b-0$ exists if and only if the following condition is satisfied:

$$\forall\,\varepsilon > 0 \quad \exists\,B \in (a,b) \quad \forall\,c', c'', B < c' < c'' < b,$$
$$|\Phi(c'') - \Phi(c')| < \varepsilon. \tag{2}$$

Let us transform the difference $\Phi(c'') - \Phi(c')$:

$$\Phi(c'') - \Phi(c') = \int_a^{c''} f(x)\,dx - \int_a^{c'} f(x)\,dx =$$
$$= \int_a^{c'} f(x)\,dx + \int_{c'}^{c''} f(x)\,dx - \int_a^{c'} f(x)\,dx = \int_{c'}^{c''} f(x)\,dx.$$

After substituting the found expression for the difference $\Phi(c'') - \Phi(c')$ into condition (2), we obtain condition (1).

Thus, condition (1) is equivalent to the existence of the limit $\lim_{c\to b-0}\Phi(c)$ and the existence of this limit is equivalent to the convergence of the integral $\int_a^b f(x)\,dx$, therefore the condition (1) is necessary and sufficient for the convergence of this integral. □

Absolute convergence of improper integrals

3.9A/10:08 (06:46)

DEFINITION.

Let the function f be defined on the interval $[a,b)$. The improper integral $\int_a^b f(x)\,dx$ is called *absolutely convergent* if the integral $\int_a^b |f(x)|\,dx$ converges.

THEOREM (ON THE CONVERGENCE OF AN ABSOLUTELY CONVERGENT INTEGRAL).

If the improper integral $\int_a^b f(x)\,dx$ absolutely converges, then it converges.

REMARK.

The converse is not true: we will show later that a convergent improper integral is not necessarily absolutely convergent. Thus, the property of absolute convergence is stronger than the property of usual convergence.

PROOF.

We are given that the integral $\int_a^b |f(x)|\,dx$ converges, and we need to prove that the integral $\int_a^b f(x)\,dx$ converges.

Since the integral $\int_a^b |f(x)|\,dx$ converges, by the necessary condition of the Cauchy criterion for improper integrals, we get

$$\forall\,\varepsilon>0\quad\exists\,B\in(a,b)\quad\forall\,c',c'',B<c'<c''<b,$$
$$\left|\int_{c'}^{c''}|f(x)|\,dx\right|<\varepsilon. \tag{3}$$

Since $c'<c''$, the last inequality in condition (3) can be rewritten without specifying the external absolute value sign on the left-hand side:

$$\int_{c'}^{c''}|f(x)|\,dx<\varepsilon. \tag{4}$$

Recall the property of the integral of the absolute value of a function:

$$\left|\int_{c'}^{c''}f(x)\,dx\right|\le\int_{c'}^{c''}|f(x)|\,dx. \tag{5}$$

Estimates (4) and (5) imply the following estimate:

$$\left|\int_{c'}^{c''}f(x)\,dx\right|<\varepsilon. \tag{6}$$

Thus, we can use (6) as the last inequality in condition (3):

$$\forall\,\varepsilon > 0 \quad \exists\, B \in (a, b) \quad \forall\, c', c'', B < c' < c'' < b, \qquad \left| \int_{c'}^{c''} f(x)\, dx \right| < \varepsilon.$$

This means, due to the sufficient condition of the Cauchy criterion for improper integrals, that the integral $\int_a^b f(x)\, dx$ converges. □

Properties of improper integrals of non-negative functions

Criterion for the convergence of improper integrals of non-negative functions

3.9A/16:54 (10:00)

In this section, we consider improper integrals of non-negative functions. Since the absolute value of the function is non-negative, all the results obtained in this section can also be used to study the absolute convergence of improper integrals of functions taking both negative and positive values.

Theorem (criterion for the convergence of improper integrals of non-negative functions).

Let a function f be defined on $[a, b)$ and $f(x) \ge 0$ for any value $x \in [a, b)$. Suppose that, for any $c \in [a, b)$, there exists an integral $\int_a^c f(x)\, dx$. Then the improper integral $\int_a^b f(x)\, dx$ converges if and only if the set of values of all integrals $\int_a^c f(x)\, dx$ is bounded from above:

$$\exists\, M > 0 \quad \forall\, c \in [a, b) \quad \int_a^c f(x)\, dx \le M. \tag{7}$$

Proof.

We introduce an auxiliary function $\Phi(c) = \int_a^c f(x)\, dx$.

The inequality $f(x) \ge 0$, which holds, by condition, for all $x \in [a, b)$, implies the inequality $\int_{c'}^{c''} f(x)\, dx \ge 0$ for any $c', c'' \in [a, b)$ such that $c' < c''$. Therefore, for $c' < c''$, we have

$$\Phi(c'') = \int_a^{c''} f(x)\, dx = \int_a^{c'} f(x)\, dx + \int_{c'}^{c''} f(x)\, dx =$$

$$= \Phi(c') + \int_{c'}^{c''} f(x)\, dx \ge \Phi(c').$$

We obtain that, for all $c' < c''$, the estimate $\Phi(c') \le \Phi(c'')$ is true. This means that the function $\Phi(c)$ is non-decreasing on the interval $[a, b)$.

1. Sufficiency. Given: condition (7) is satisfied. Prove: the integral $\int_a^b f(x)\, dx$ converges.

Condition (7) means that the function $\Phi(c)$ is bounded from above on the interval $[a,b)$. Thus, the function $\Phi(c)$ is non-decreasing and bounded from above on the interval $[a,b)$, therefore, by virtue of the theorem on the limit of a monotonous and upper-bounded function, there exists a limit of the function $\Phi(c)$ as $c \to b-0$ (equal to $\sup_{c\in[a,b)} \Phi(c)$). It remains to note that the existence of this limit is equivalent to the convergence of the improper integral $\int_a^b f(x)\,dx$. The sufficiency is proven.

2. Necessity. Given: the integral $\int_a^b f(x)\,dx$ converges. Prove: condition (7) is satisfied.

As noted above, if the integral $\int_a^b f(x)\,dx$ converges, then there exists a limit $\lim_{c\to b-0} \Phi(c) = M$. Since the function $\Phi(c)$ is non-decreasing, we obtain that, for all $c', c'' \in [a,b)$ such that $c' < c''$, the inequality holds:

$$\Phi(c') \le \Phi(c'').$$

In this inequality, we pass to the limit as $c'' \to b-0$. By virtue of the theorem on passing to the limit in the inequalities, the following inequality holds for any $c' \in [a,b)$:

$$\Phi(c') \le M.$$

Substituting the definition of the function Φ into this inequality, we obtain the inequality from condition (7). $\square$

The comparison test

3.9A/26:54 (10:35)

THEOREM (THE COMPARISON TEST FOR IMPROPER INTEGRALS OF NON-NEGATIVE FUNCTIONS).

Let the functions f and g be defined on the interval $[a,b)$ and the double inequality $0 \le f(x) \le g(x)$ holds for any $x \in [a,b)$. Suppose that, for any $c \in [a,b)$, there exist integrals $\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$. Then the following two statements are true.

1. If the improper integral $\int_a^b g(x)\,dx$ converges, then the integral $\int_a^b f(x)\,dx$ also converges.

2. If the improper integral $\int_a^b f(x)\,dx$ diverges, then the integral $\int_a^b g(x)\,dx$ also diverges.

PROOF.

1. Let the integral $\int_a^b g(x)\,dx$ converge. Then, by the necessary condition of the previous criterion, we obtain

$$\exists\, M > 0 \quad \forall\, c \in [a,b) \quad \int_a^c g(x)\,dx \le M. \tag{8}$$

Since $f(x) \le g(x)$ for all $x \in [a, b)$, a similar inequality holds for the proper integrals:

$$\int_a^c f(x)\,dx \le \int_a^c g(x)\,dx. \tag{9}$$

Combining estimates (8) and (9), we obtain

$$\int_a^c f(x)\,dx \le \int_a^c g(x)\,dx \le M.$$

Thus, condition (8) is also satisfied for the integral $\int_a^c f(x)\,dx$. Therefore, by virtue of a sufficient part of the previous criterion, the integral $\int_a^b f(x)\,dx$ converges.

2. Let the integral $\int_a^b f(x)\,dx$ diverge.

If we assume that the integral $\int_a^b g(x)\,dx$ converges, then by the already proved statement 1, the integral $\int_a^b f(x)\,dx$ should also converge. But this contradicts the condition. Therefore, the assumption made is false and the integral $\int_a^b g(x)\,dx$ diverges. □

REMARK.

Obviously, the theorem remains valid if the functions f and g satisfy the double inequality $0 \le f(x) \le Cg(x)$ with some constant $C > 0$ on the interval $[a, b)$.

Corollary of the comparison test

3.9A/37:29 (04:25)

COROLLARY.

Let the functions f and g be defined on the interval $[a, b)$ and be non-negative on this interval. Let $f(x) \sim g(x)$ as $x \to b - 0$. Then the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.

PROOF[2].

The equivalence of the functions f and g as $x \to b-0$ means that in some left-hand neighborhood U_b^- of the point b, the relation $f(x) = \alpha(x)g(x)$ holds, where $\alpha(x) \to 1$ as $x \to b - 0$.

Since $\lim_{x\to b-0} \alpha(x) = 1$, we can choose a neighborhood $V_b^- \subset U_b^-$, in which the double inequality holds for the function $\alpha(x)$:

$$1 - \frac{1}{2} < \alpha(x) < 1 + \frac{1}{2}.$$

We multiply all parts of this inequality by $g(x)$ and take into account that the equality $f(x) = \alpha(x)g(x)$ is true in the neighborhood V_b^-:

[2] There is no proof of the corollary in video lectures.

$$\Big(1-\frac{1}{2}\Big)g(x) < \alpha(x)g(x) < \Big(1+\frac{1}{2}\Big)g(x),$$
$$\frac{1}{2}\,g(x) < f(x) < \frac{3}{2}\,g(x). \tag{10}$$

Choosing some value $B \in V_b^-$, we obtain that inequality (10) holds for all $x \in [B, b)$.

Suppose that the integral $\int_B^b f(x)\,dx$ converges. Then, taking into account statement 1 of the previous theorem, the remark, and the estimate $g(x) < 2f(x)$, which follows from the left-hand side of (10), we obtain that the integral $\int_B^b g(x)\,dx$ also converges. If we assume that the integral $\int_B^b g(x)\,dx$ converges, then from the right-hand side of (10) (i. e., $f(x) < \frac{3}{2}\,g(x)$), it follows that the integral $\int_B^b f(x)\,dx$ also converges.

On the other hand, if we assume that the integral $\int_B^b g(x)\,dx$ diverges, then, taking into account statement 2 of the previous theorem, the remark, and the estimate $g(x) < 2f(x)$, which follows from the left-hand side of (10), we obtain that the integral $\int_B^b f(x)\,dx$ also diverges, and if we assume that the integral $\int_B^b f(x)\,dx$ diverges, then it follows from the right-hand side of (10) that the integral $\int_B^b g(x)\,dx$ also diverges.

So, we have proved that the improper integrals $\int_B^b f(x)\,dx$ and $\int_B^b g(x)\,dx$ either both converge or both diverge. Taking into account the theorem on the additivity of an improper integral with respect to the integration interval, we obtain that the same statement holds for the initial integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$. □

Examples of using the comparison test `3.9B/00:00 (08:23)`

1. Consider the integral $\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$.

For the absolute value of the integrand, the following estimate holds:

$$\left|\frac{\sin x}{x^2}\right| \le \frac{1}{x^2}.$$

Earlier, we proved that the integral $\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges. Therefore, the integral $\int_1^{+\infty} \left|\frac{\sin x}{x^2}\right| dx$ also converges by the comparison test. And this, in turn, means that the initial integral converges absolutely.

In a similar way, one can prove that absolute convergence holds for the integral $\int_1^{+\infty} \frac{\sin x}{x^\alpha}\,dx$ for any $\alpha > 1$.

2. Consider the integral $\int_2^{+\infty} \frac{1}{\ln x}\,dx$.

For any $x > 1$, the double estimate $0 < \ln x < x$ is valid. It follows that $\frac{1}{x} < \frac{1}{\ln x}$. Earlier, we proved that the integral $\int_1^{+\infty} \frac{1}{x}\,dx$ diverges. Obviously, the integral $\int_2^{+\infty} \frac{1}{x}\,dx$ also diverges. Then the initial integral $\int_2^{+\infty} \frac{1}{\ln x}\,dx$ also diverges by the comparison test.

3. Consider the integral $\int_1^{+\infty} \frac{1}{x^\alpha+\sin x}\,dx$, $\alpha > 1$.

Let us show that the integrand is equivalent to the function $\frac{1}{x^\alpha}$ as $x \to +\infty$:

$$\lim_{x\to+\infty} \frac{\frac{1}{x^\alpha+\sin x}}{\frac{1}{x^\alpha}} = \lim_{x\to+\infty} \frac{x^\alpha}{x^\alpha+\sin x} = \lim_{x\to+\infty} \frac{1}{1+\frac{\sin x}{x^\alpha}} = 1.$$

So, we have proved that $\frac{1}{x^\alpha+\sin x} \sim \frac{1}{x^\alpha}$, $x \to +\infty$.

Since the integral $\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ converges for $\alpha > 1$, we obtain from the corollary of the comparison test that the initial integral also converges.

Conditional convergence of improper integrals

3.9B/08:23 (15:56)

Definition.

The improper integral $\int_a^b f(x)\,dx$ is called *conditionally convergent* if this integral converges and the integral $\int_a^b |f(x)|\,dx$ diverges. In other words, the integral converges conditionally if it converges, but it is not absolutely convergent.

It is clear that conditional convergence may hold only for integrals whose integrands change sign.

Example.

Consider the integral $\int_1^{+\infty} \frac{\sin x}{x}\,dx$ and show that it converges conditionally.

We begin by proving the convergence of this integral and consider the proper integral with integration limits from 1 to c, where $c > 1$. We use the integration formula by parts, setting $u = \frac{1}{x}$, $dv = \sin x\,dx$ (in this case, $v = -\cos x$, $du = -\frac{1}{x^2}\,dx$):

$$\int_1^c \frac{\sin x}{x}\,dx = -\frac{\cos x}{x}\bigg|_1^c - \int_1^c \frac{\cos x}{x^2}\,dx. \tag{11}$$

The integral $\int_1^c \frac{\cos x}{x^2}\,dx$ converges (moreover, it absolutely converges). This can be proved by the comparison test (see example 1 from the previous section). Thus, the second term on the right-hand side of (11) has a finite limit as $c \to +\infty$. Let us transform the first term:

$$-\frac{\cos x}{x}\Big|_1^c = -\frac{\cos c}{c} + \frac{\cos 1}{1}.$$

The limit of this expression as $c \to +\infty$ also exists and is equal to $\cos 1$.

Since the right-hand side of equality (11) has a finite limit as $c \to +\infty$, we conclude that the left-hand side also has a finite limit. We have proved that the integral $\int_1^{+\infty} \frac{\sin x}{x}\,dx$ converges.

It remains for us to show that the integral $\int_1^{+\infty}\left|\frac{\sin x}{x}\right|dx$ diverges. We can move the absolute value sign to the numerator, since the denominator of the integrand is positive:

$$\int_1^{+\infty}\left|\frac{\sin x}{x}\right|dx = \int_1^{+\infty} \frac{|\sin x|}{x}\,dx.$$

The function $|\sin x|$ can be estimated from below by the function $\sin^2 x$: $|\sin x| \geq \sin^2 x$ for any $x \in \mathbb{R}$. Therefore, for any $x \geq 1$, the estimate holds:

$$\frac{\sin^2 x}{x} \leq \frac{|\sin x|}{x}.$$

If we prove that the integral $\int_1^{+\infty} \frac{\sin^2 x}{x}\,dx$ diverges, then the integral $\int_1^{+\infty}\left|\frac{\sin x}{x}\right|dx$ will diverge as well.

So, it remains for us to prove the divergence of the integral $\int_1^{+\infty} \frac{\sin^2 x}{x}\,dx$. We consider the proper integral with integration limits from 1 to c, where $c > 1$, and transform it using the formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$:

$$\int_1^c \frac{\sin^2 x}{x}\,dx = \frac{1}{2}\int_1^c \frac{1-\cos 2x}{x}\,dx = \frac{1}{2}\int_1^c \frac{1}{x}\,dx - \frac{1}{2}\int_1^c \frac{\cos 2x}{x}\,dx.$$

The second integral on the right-hand side of the last equality converges to a finite limit as $c \to +\infty$. This can be proved using the same method of integration by parts, which we previously applied in the study of the integral $\int_1^c \frac{\sin x}{x}\,dx$.

The first integral on the right-hand side equals $\ln|c|$ and therefore it approaches $+\infty$ as $c \to +\infty$. Thus, the limit of the right-hand side is $+\infty$, so the limit of the left-hand side is also $+\infty$. We have proved that the improper integral $\int_1^{+\infty} \frac{\sin^2 x}{x}\,dx$ diverges.

Therefore, the integral $\int_1^{+\infty}\left|\frac{\sin x}{x}\right|dx$ also diverges by the comparison test. The conditional convergence of the initial integral $\int_1^{+\infty} \frac{\sin x}{x}\,dx$ is proved.

Dirichlet's test for conditional convergence of an improper integral

Formulation of Dirichlet's test

3.9B/24:19 (06:36)

Theorem (Dirichlet's test for conditional convergence of an improper integral).

Let the functions f and g be defined on the interval $[a, b)$ and satisfy the following conditions:

1) the function f is continuous on $[a, b)$, and the integral $\int_a^c f(x)\,dx$ is uniformly bounded for all $c \in (a, b)$, i.e.,

$$\exists\, M > 0 \quad \forall\, c \in (a, b) \quad \left|\int_a^c f(x)\,dx\right| \le M;$$

2) the function g is continuously differentiable on $[a, b)$, and $g(c)$ monotonously approaches 0 as $c \to b - 0$ (the monotonicity condition means that $g'(c)$ preserves the sign for all $c \in (a, b)$).

Then the improper integral $\int_a^b f(x)g(x)\,dx$ converges (generally speaking, conditionally).

Proof of Dirichlet's test

3.9B/30:55 (13:14)

We introduce an auxiliary function $\Phi(c) = \int_a^c f(x)\,dx$. By condition 1, this function is uniformly bounded on (a, b):

$$\exists\, M > 0 \quad \forall\, c \in (a, b) \quad |\Phi(c)| \le M. \tag{12}$$

In addition, the function $\Phi(c)$ is differentiable on (a, b) as an integral with a variable upper limit and the continuous integrand f, and the equality $\Phi'(c) = f(c)$ holds.

Therefore, the proper integral $\int_a^c f(x)g(x)\,dx$ can be represented in the following form:

$$\int_a^c f(x)g(x)\,dx = \int_a^c \Phi'(x)g(x)\,dx.$$

The resulting integral can be transformed by the integration formula by parts, setting $u = g(x)$, $dv = \Phi'(x)\,dx$, whence $v = \Phi(x)$:

$$\int_a^c \Phi'(x)g(x)\,dx = \Phi(x)g(x)\Big|_a^c - \int_a^c \Phi(x)g'(x)\,dx =$$

$$= \Phi(c)g(c) - \Phi(a)g(a) - \int_a^c \Phi(x)g'(x)\,dx. \tag{13}$$

Since the function $\Phi(c)$ is bounded (see (12)) and the function $g(c)$ approaches 0 as $c \to b-0$ by condition 2, we obtain that the first term $\Phi(c)g(c)$ of the right-hand side approaches 0 as $c \to b-0$. The second term $\Phi(a)g(a)$ does not depend on the parameter c.

It remains to show that the integral $\int_a^c \Phi(x)g'(x)\,dx$ also has a finite limit as $c \to b-0$, i. e., that the improper integral $\int_a^b \Phi(x)g'(x)\,dx$ converges.

We prove the convergence of the integral $\int_a^b \Phi(x)g'(x)\,dx$ using the Cauchy criterion for the convergence of the improper integral.

However, we first use the Cauchy criterion for the existence of a function limit. By condition 2, the function $g(c)$ has a limit as $c \to b-0$. By virtue of the necessary part of the Cauchy criterion for the existence of a function limit, this means the following:

$$\forall\,\varepsilon > 0 \quad \exists\, B \in (a,b) \quad \forall\, c', c'', B < c' < c'' < b,$$
$$|g(c'') - g(c')| < \frac{\varepsilon}{M}. \tag{14}$$

In inequality (14), we used the constant M from estimate (12).

According to the Cauchy criterion, to prove the convergence of the integral $\int_a^b \Phi(x)g'(x)\,dx$, it suffices to establish the following fact:

$$\forall\,\varepsilon > 0 \quad \exists\, B \in (a,b) \quad \forall\, c', c'', B < c' < c'' < b,$$
$$\left|\int_{c'}^{c''} \Phi(x)g'(x)\,dx\right| < \varepsilon. \tag{15}$$

We choose an arbitrary value $\varepsilon > 0$, get the value $B \in (a,b)$ from condition (14), and show that estimate (15) holds for this value B. To do this, we transform the integral $\left|\int_{c'}^{c''} \Phi(x)g'(x)\,dx\right|$ using the theorem on the integral of the absolute value of a function and estimate (12):

$$\left|\int_{c'}^{c''} \Phi(x)g'(x)\,dx\right| \le \int_{c'}^{c''} |\Phi(x)|\cdot|g'(x)|\,dx \le M \int_{c'}^{c''} |g'(x)|\,dx. \tag{16}$$

Since, by condition 2, the derivative $g'(x)$ preserves the sign on (a,b), we can move the absolute value sign outside the integral sign in the integral $\int_{c'}^{c''} |g'(x)|\,dx$:

$$\int_{c'}^{c''} |g'(x)|\,dx = \left|\int_{c'}^{c''} g'(x)\,dx\right|.$$

Indeed, if the derivative $g'(x)$ is always positive, then the absolute value can be omitted, and if the derivative $g'(x)$ is always negative, then the minus sign can be taken out of the integral sign, the positive function remains under the integral sign, and the external minus can be removed using the external operation of taking the absolute value.

The integral on the right-hand side of the last equality can be transformed according to the Newton–Leibniz formula:

$$\int_{c'}^{c''} g'(x)\,dx = g(c'') - g(c').$$

Thus, the chain of inequalities (16) can be continued as follows:

$$M\int_{c'}^{c''} |g'(x)|\,dx \le M|g(c'') - g(c')|.$$

Since the values of c' and c'' are chosen so that condition (14) is satisfied, the expression $M|g(c'') - g(c')|$ is estimated by ε. Taking into account that the transformations in the chain of inequalities (16) started with the integral $\bigl|\int_{c'}^{c''} \Phi(x)g'(x)\,dx\bigr|$, we finally obtain the estimate

$$\left|\int_{c'}^{c''} \Phi(x)g'(x)\,dx\right| < \varepsilon.$$

Thus, condition (15) is satisfied. Therefore, by virtue of a sufficient part of the Cauchy criterion for the convergence of the improper integral, the integral $\int_a^b \Phi(x)g'(x)\,dx$ converges.

We have proved that all terms on the right-hand side of equality (13) have a finite limit as $c \to b - 0$. This means that the integral $\int_a^c f(x)g(x)\,dx$ also has a finite limit and therefore the initial improper integral $\int_a^b f(x)g(x)\,dx$ converges. $\square$

Integrals with several singularities 3.10A/00:00 (11:35)

Let the function f be defined on the interval (a, b) and either the endpoints of this interval are points at infinity or the function is unbounded in a neighborhood of these endpoints (or a combination of these situations takes place). Then the improper integral $\int_a^b f(x)\,dx$, which has singularities at both endpoints of the integration interval, can be represented as the sum of the improper integrals considered above with unique singularity:

$$\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx. \tag{17}$$

Here d is some point belonging to the interval (a, b). If both integrals on the right-hand side of equality (17) converge, then the initial integral $\int_a^b f(x)\,dx$ is also convergent and its value is equal to the sum of the values of the integrals on the right-hand side. If at least one integral on the right-hand side diverges, then the initial integral is divergent too.

The same can be done if the singularity arises at some internal point of the integration interval. Let the function f be defined on the set $[a,b]\setminus\{d\}$ and be unbounded in a neighborhood of the point d. Then the integral $\int_a^b f(x)\,dx$ must be understood as an improper integral defined by the same relation (17), in which the improper integrals on the right-hand side have unique singularity at the point d.

In this case, given the definition of an improper integral, the value of the integral $\int_a^b f(x)\,dx$ (provided that it converges) will be equal to the sum of the following limits:

$$\int_a^b f(x)\,dx = \lim_{c\to d-0}\int_a^c f(x)\,dx + \lim_{c'\to d+0}\int_{c'}^b f(x)\,dx.$$

The rate at which the point c approaches d from the left and the rate at which the point c' approaches d from the right are not related in any way: these limits must be considered independently of each other.

If we define an improper integral with a singularity at one internal point in this way, then the integral $\int_{-1}^1 \frac{1}{x}\,dx$ (with a singularity at 0) will diverge, since the integral $\int_0^1 \frac{1}{x}\,dx$ diverges:

$$\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to+0}\int_\varepsilon^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to+0}(\ln|x|)\Big|_\varepsilon^1 = \lim_{\varepsilon\to+0}(-\ln|\varepsilon|) = +\infty.$$

Similarly, we can establish that the integral $\int_{-1}^0 \frac{1}{x}\,dx$ also diverges:

$$\int_{-1}^0 \frac{1}{x}\,dx = \lim_{\varepsilon\to-0}\int_{-1}^\varepsilon \frac{1}{x}\,dx = \lim_{\varepsilon\to-0}(\ln|x|)\Big|_{-1}^\varepsilon = \lim_{\varepsilon\to-0}\ln|\varepsilon| = -\infty.$$

However, it is easy to see that we would get a finite limit value if this limit was calculated not for two integrals separately, but simultaneously for the sum of the integrals $\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_\varepsilon^1 \frac{1}{x}\,dx$ as $\varepsilon\to+0$:

$$\lim_{\varepsilon\to+0}\Big(\int_{-1}^{-\varepsilon}\frac{1}{x}\,dx + \int_\varepsilon^1 \frac{1}{x}\,dx\Big) = \lim_{\varepsilon\to+0}(\ln|-\varepsilon| - \ln|\varepsilon|) = \lim_{\varepsilon\to+0} 0 = 0.$$

The main feature here is that the parameter ε approaches a singular point on the left and right at the same rate, which allows us to eliminate two infinitely growing terms.

Such a type of convergence of an improper integral with a singularity at an internal point is called *convergence in the sense of the principal value*. We give a general definition of this type of convergence.

Definition.

Let the function f be defined on the set $[a, b] \setminus \{d\}$ and be unbounded in a neighborhood of the point d. The improper integral $\int_a^b f(x)\,dx$ is said to *converge in the sense of the principal value* (or *in the sense of the Cauchy principal value*) if there exists a finite limit on the sum of the integrals $\int_a^{d-\varepsilon} f(x)\,dx + \int_{d+\varepsilon}^b f(x)\,dx$ as $\varepsilon \to +0$. For this limit, the notation (v. p.) $\int_a^b f(x)\,dx$ is used:

$$\text{(v. p.)} \int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{\varepsilon \to +0} \Bigl(\int_a^{d-\varepsilon} f(x)\,dx + \int_{d+\varepsilon}^b f(x)\,dx \Bigr).$$

Thus, the previously obtained result for the integral of the function $\frac{1}{x}$ on the segment $[-1, 1]$ can be written as follows:

$$\text{(v. p.)} \int_{-1}^1 \frac{1}{x}\,dx = 0.$$

There is an extensive theory related to the convergence of improper integrals in the sense of the Cauchy principal value, but we will not study this type of convergence in this book.

12. Numerical series

Numerical series: definition and examples

Definition of a numerical series

3.10A/11:35 (05:39)

Recall how the finite sum of terms is written using the summation symbol $\sum$:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

If the symbol ∞ is indicated in the notation of the sum instead of the finite number n, then this notation can be considered as a formal notation of the sum of an infinite number of terms (such a construction is called a *formal sum*):

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_k + \dots$$

The expression $\sum_{k=1}^{\infty} a_k$ is called a *numerical series*, and the value a_k is called a *common term of the series*. Thus, the series of numbers $\sum_{k=1}^{\infty} a_k$ is the formal sum of all elements of the sequence $\{a_k\}$ (the elements are taken in ascending order of their indices).

Under additional conditions, a specific numerical value (called the *sum of a series*) can be associated with a numerical series. Consider the finite sum

$$S_n = \sum_{k=1}^{n} a_k.$$

This sum is called the *partial sum of the series* $\sum_{k=1}^{\infty} a_k$; it exists for any number $n \in \mathbb{R}$. Thus, we get a sequence of partial sums $\{S_n\}$.

If there exists a finite limit S of the sequence $\{S_n\}$ as $n \to \infty$, then the numerical series $\sum_{k=1}^{\infty} a_k$ is called *convergent* and the limit S is called the *sum* of this numerical series. If the series converges, then its notation $\sum_{k=1}^{\infty} a_k$ usually means the value of its sum, i. e., the limit S (just as the notation of an improper integral means the limit value of usual proper integrals):

$$\sum_{k=1}^{\infty} a_k \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

If the sequence of partial sums $\{S_n\}$ has no limit or has an infinite limit, then the series $\sum_{k=1}^{\infty} a_k$ is called *divergent*; in this case, the sum of the series is not defined (as well as the value of the divergent improper integral).

We emphasize that, in any case, the notation $\sum_{k=1}^{\infty} a_k$ can be considered as a formal sum of an infinite number of terms, regardless of whether this formal notation corresponds to some numerical value or not.

As a summation parameter, the symbols i and j are often used along with the symbol k.

The initial value of the summation parameter does not have to be 1. Series with a summation parameter starting with 0 are often considered. Obviously, if the series $\sum_{k=1}^{\infty} a_k$ converges, then the series $\sum_{k=n_0}^{\infty} a_k$ also converges for any $n_0 \in \mathbb{N}$.

Example of a numerical series: the sum of the elements of a geometric progression 3.10A/17:14 (10:21)

Let $q \neq 0$ be an arbitrary real number. Consider a series with the common term q^k:

$$\sum_{k=0}^{\infty} q^k = 1 + q + q^2 + \dots + q^k + \dots$$

This series is the formal sum of all terms of the geometric progression with 1 as the first term and q as the ratio.

Recall the formula for the sum of the initial terms of such a geometric progression (provided that $q \neq 1$):

$$S_n = \sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}.$$

In this case, S_n denotes the sum of $(n+1)$ initial terms of the geometric progression. It is clear that if $q = 1$, then $S_n = n + 1$.

If $|q| < 1$, then $\lim_{n\to\infty} S_n = \frac{1}{1-q}$. If $|q| \geq 1$, then the limit of the sequence $\{S_n\}$ as $n \to \infty$ is either infinite or (for $q = -1$) does not exist (since, for $q = -1$, the sequence $\{q^n\}$ has the form $\{1, -1, 1, -1, \dots\}$ and therefore the sequence $\{S_n\}$ is equal to $\{1, 0, 1, 0, \dots\}$).

So, if $|q| \geq 1$, then the series $\sum_{k=0}^{\infty} q^k$ diverges, and if $|q| < 1$, then the series $\sum_{k=0}^{\infty} q^k$ converges and its sum is $\frac{1}{1-q}$:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \quad |q| < 1, q \neq 0.$$

This formula is called the *formula of the sum of an infinitely decreasing geometric progression*.

Cauchy criterion for the convergence of a numerical series and a necessary condition for its convergence

Cauchy criterion for the convergence of a numerical series

3.10A/27:35 (07:57)

THEOREM (CAUCHY CRITERION FOR THE CONVERGENCE OF A NUMERICAL SERIES).

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if the following condition is satisfied:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, p \in \mathbb{N} \quad \left| \sum_{k=m+1}^{m+p} a_k \right| < \varepsilon. \tag{1}$$

PROOF.

Let $S_n = \sum_{k=1}^{n} a_k$ be a partial sum of the initial series. A series converges if and only if the sequence of partial sums $\{S_n\}$ is convergent.

For the sequence $\{S_n\}$, we write the condition from the Cauchy criterion for the convergence of a sequence:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m_1, m_2 > N \quad |S_{m_2} - S_{m_1}| < \varepsilon.$$

If we put $m_1 = m$, $m_2 = m + p$ for some $p \in \mathbb{N}$, then the last condition can be rewritten in the following form:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, p \in \mathbb{N} \quad |S_{m+p} - S_m| < \varepsilon. \tag{2}$$

Let us transform the difference $S_{m+p} - S_m$ taking into account the formula for partial sums:

$$S_{m+p} - S_m = \sum_{k=1}^{m+p} a_k - \sum_{k=1}^{m} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{m+p} a_k - \sum_{k=1}^{m} a_k = \sum_{k=m+1}^{m+p} a_k.$$

Substituting the obtained expression for the difference $S_{m+p} - S_m$ into condition (2), we obtain condition (1).

So, we have shown that condition (1) is necessary and sufficient for the convergence of the sequence $\{S_n\}$, and the convergence of this sequence takes place if and only if the initial series converges. □

A necessary condition for the convergence of a numerical series

3.10A/35:32 (04:58)

COROLLARY (A NECESSARY CONDITION FOR THE CONVERGENCE OF A NUMERICAL SERIES).

If the series $\sum_{k=1}^{\infty} a_k$ converges, then its common term a_k approaches zero:

$$\lim_{k\to\infty} a_k = 0.$$

REMARKS.

1. This condition means that if the common term of a series does not approach 0, then the series is not convergent. Thus, it makes it easy to prove the divergence of many series. However, it should be emphasized that this condition is not a sufficient condition for convergence: from the fact that the common term of a series approaches 0, it does not follow that the series converges (we will give the corresponding examples later).

2. A similar condition for improper integrals over a semi-infinite interval, generally speaking, does not hold. There exist conditionally convergent improper integrals of the form $\int_a^{+\infty} f(x)\,dx$ for which the integrand $f(x)$ does not approach zero as $x \to +\infty$. An example of such an integral is $\int_1^{+\infty} \sin e^x\,dx$. It is easy to prove the convergence of this integral by changing the variable $t = e^x$, since, as a result of this changing, the integral will take the form $\int_e^{+\infty} \frac{\sin t}{t}\,dt$. At the same time, if the function $f(x)$ is non-negative and non-increasing on the interval $[a, +\infty)$, then the convergence of the integral $\int_a^{+\infty} f(x)\,dx$ implies that $\lim_{x\to+\infty} f(x) = 0$ (this fact follows from the integral convergence test considered in the next chapter).

PROOF.

Since the initial series converges, condition (1) of the Cauchy criterion for the convergence of a numerical series is fulfilled for it. We put $p = 1$ in this condition (this can be done, since it is allowed to take any $p \in \mathbb{N}$ in condition (1)):

$$\forall\,\varepsilon > 0 \quad \exists\,N \in \mathbb{N} \quad \forall\,m > N \quad \Bigl|\sum_{k=m+1}^{m+1} a_k\Bigr| < \varepsilon. \tag{3}$$

Since $\sum_{k=m+1}^{m+1} a_k = a_{m+1}$, the last inequality takes the form $|a_{m+1}| < \varepsilon$.

Thus, condition (3) coincides with the definition (in the language ε–N) of a convergent sequence $\{a_k\}$ in the case when its limit is 0. □

Absolutely convergent numerical series and arithmetic properties of convergent numerical series

Absolutely convergent numerical series

3.10A/40:30 (01:44), 3.10B/00:00 (03:09)

DEFINITION.

The series $\sum_{k=1}^{\infty} a_k$ *absolutely converges* if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Theorem (on the convergence of an absolutely convergent numerical series).

If the series absolutely converges, then it is convergent.

Proof.

Let the series $\sum_{k=1}^{\infty} a_k$ absolutely converge. This means that the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Therefore, by virtue of the necessary part of the Cauchy criterion for the convergence of a numerical series, condition (1) is satisfied:

$$\forall\,\varepsilon > 0 \quad \exists\,N \in \mathbb{N} \quad \forall\,m > N \quad \forall\,p \in \mathbb{N} \quad \sum_{k=m+1}^{m+p} |a_k| < \varepsilon.$$

The sum $\sum_{k=m+1}^{m+p} |a_k|$ can be estimated from below using the following absolute value property (which is a generalization of the triangle inequality for the case of the sum of n terms):

$$\Bigl|\sum_{k=m+1}^{m+p} a_k\Bigr| \le \sum_{k=m+1}^{m+p} |a_k|.$$

Since the right-hand side of the last inequality is bounded from above by ε, the same estimate holds for the left-hand side of the inequality:

$$\Bigl|\sum_{k=m+1}^{m+p} a_k\Bigr| < \varepsilon.$$

This inequality coincides with condition (1) of the Cauchy criterion for the convergence of the numerical series $\sum_{k=1}^{\infty} a_k$. Therefore, by virtue of a sufficient part of the Cauchy criterion, this series converges. ☐

Arithmetic properties of convergent numerical series

3.10B/03:09 (08:19)

Theorem (on arithmetic properties of convergent numerical series).

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be convergent series with sums S_a and S_b, respectively. Let $\alpha, \beta \in \mathbb{R}$.

Then the series $\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k)$ also converges and its sum is $\alpha S_a + \beta S_b$.

Thus, for convergent series, the same arithmetic transformations can be used as for finite sums:

$$\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k. \tag{4}$$

In addition, if the initial series converge absolutely, then the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ also converges absolutely.

Proof.

Let us introduce partial sums:

$$S'_n = \sum_{k=1}^{n} a_k, \quad S''_n = \sum_{k=1}^{n} b_k, \quad S_n = \sum_{k=1}^{n}(\alpha a_k + \beta b_k).$$

Obviously, for these finite sums, the equality holds:

$$S_n = \sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k = \alpha S'_n + \beta S''_n.$$

Since, by condition, $\lim_{n\to\infty} S'_n = S_a$, $\lim_{n\to\infty} S''_n = S_b$, we obtain, by arithmetic properties of the limit of a sequence, that the limit S_n as $n \to \infty$ exists and is equal to $\alpha S_a + \beta S_b$. Thus, we simultaneously proved the convergence of the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ and formula (4).

To prove the absolute convergence of the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ in the case when the initial series absolutely converge, we use the Cauchy criterion.

For $\alpha = \beta = 0$, the statement is obvious; therefore, we will assume that $|\alpha| + |\beta| \neq 0$. Let us choose the value $\varepsilon > 0$. For absolutely convergent initial series, by virtue of the Cauchy criterion, the following conditions are satisfied:

$$\exists\, N_1 \in \mathbb{N} \quad \forall\, m > N_1 \quad \forall\, p \in \mathbb{N} \quad \sum_{k=m+1}^{m+p} |a_k| < \frac{\varepsilon}{|\alpha| + |\beta|},$$

$$\exists\, N_2 \in \mathbb{N} \quad \forall\, m > N_2 \quad \forall\, p \in \mathbb{N} \quad \sum_{k=m+1}^{m+p} |b_k| < \frac{\varepsilon}{|\alpha| + |\beta|}.$$

If we put $N = \max\{N_1, N_2\}$, then the following estimate will be true for any $m > N$ and $p \in \mathbb{N}$:

$$\sum_{k=m+1}^{m+p} |\alpha a_k + \beta b_k| \le \sum_{k=m+1}^{m+p} (|\alpha| \cdot |a_k| + |\beta| \cdot |b_k|) =$$

$$= |\alpha| \sum_{k=m+1}^{m+p} |a_k| + |\beta| \sum_{k=m+1}^{m+p} |b_k| < |\alpha| \cdot \frac{\varepsilon}{|\alpha| + |\beta|} + |\beta| \cdot \frac{\varepsilon}{|\alpha| + |\beta|} = \varepsilon.$$

So, we have proved that, for the series $\sum_{k=1}^{\infty} |\alpha a_k + \beta b_k|$, condition (1) of the Cauchy criterion is fulfilled. Therefore, this series converges, which means that the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ converges absolutely. $\square$

13. Convergence tests for numerical series with non-negative terms

Comparison test

Criterion for convergence of numerical series with non-negative terms

3.10B/11:28 (07:21)

THEOREM (CRITERION FOR CONVERGENCE OF NUMERICAL SERIES WITH NON-NEGATIVE TERMS).

Let all terms of the series $\sum_{k=1}^{\infty} a_k$ be non-negative:

$$\forall\, k \in \mathbb{N} \quad a_k \geq 0.$$

Then this series converges if and only if the set of values of its partial sums $\sum_{k=1}^{n} a_k$ is bounded from above:

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N} \quad \sum_{k=1}^{n} a_k \leq M. \tag{1}$$

PROOF.

Consider the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$. Since all the terms a_k are non-negative, we obtain that this sequence is non-decreasing:

$$\forall\, n \in \mathbb{N} \quad S_{n+1} = \sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1} \geq \sum_{k=1}^{n} a_k = S_n.$$

When studying the limit of a sequence, we proved that a non-decreasing sequence converges if and only if it is bounded from above.

Thus, the condition (1), which means that the partial sums S_n (with non-negative terms) are bounded from above, is equivalent to the convergence of the sequence $\{S_n\}$, and the convergence of this sequence is equivalent to the convergence of the numerical series. $\square$

Comparison test for numerical series

3.10B/18:49 (06:49)

Theorem (comparison test for numerical series).

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series for which the following condition holds:

$$\exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad 0 \leq a_k \leq b_k.$$

Then two statements are valid.

1. If the series $\sum_{k=1}^{\infty} b_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ also converges.
2. If the series $\sum_{k=1}^{\infty} a_k$ diverges, then the series $\sum_{k=1}^{\infty} b_k$ also diverges.

Proof.

Since the convergence of the series $\sum_{k=1}^{\infty} a_k$ is equivalent to the convergence of the series $\sum_{k=m}^{\infty} a_k$ and the same fact is true for series with terms b_k, it is enough to prove the theorem for the series $\sum_{k=m}^{\infty} a_k$ and $\sum_{k=m}^{\infty} b_k$, all terms of which are non-negative and satisfy the inequality $a_k \leq b_k$.

1. If the series $\sum_{k=m}^{\infty} b_k$ converges, then, by the criterion for the convergence of numerical series with non-negative terms, we have

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N}, n \geq m, \quad \sum_{k=m}^{n} b_k \leq M.$$

Then, due to the inequality $a_k \leq b_k$, we obtain that a similar estimate is also valid for partial sums of the series $\sum_{k=m}^{\infty} a_k$:

$$\sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} b_k \leq M.$$

Therefore, by the same criterion, the series $\sum_{k=m}^{\infty} a_k$ converges.

2. Let the series $\sum_{k=m}^{\infty} a_k$ diverge.

If we assume that the series $\sum_{k=m}^{\infty} b_k$ converges, then, by already proved statement 1, the series $\sum_{k=m}^{\infty} a_k$ should also converge. But this fact contradicts the condition. Therefore, the assumption made is false and the series $\sum_{k=m}^{\infty} b_k$ diverges. $\square$

Remark.

From the comparison test for numerical series, one can obtain a corollary similar to the corollary from the comparison test for improper integrals: if, for all $k \in \mathbb{N}$ starting from some m, the estimates $a_k > 0$, $b_k > 0$ are fulfilled and the limit relation $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$ holds, then the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or both diverge.

Integral test of convergence

Formulation of the integral test of convergence

3.10B/25:38 (03:55)

Theorem (integral test of convergence).

Let the function f be defined on the set $[1, +\infty)$, be non-negative and non-increasing, and $\lim_{x\to+\infty} f(x) = 0$.

Then the improper integral $\int_1^{+\infty} f(x)\,dx$ and the series $\sum_{k=1}^{\infty} f(k)$ either both converge or both diverge.

Initial stage of the proof

3.10B/29:33 (07:07)

We choose the points $k, k+1 \in \mathbb{N}$ and assume that the point $x \in \mathbb{R}$ is between k and $k+1$: $k \le x \le k+1$. Since the function f is non-increasing, the following double inequality holds:

$$f(k+1) \le f(x) \le f(k).$$

Let us integrate all the terms of the resulting double inequality from k to $k+1$; this operation will not change the sign of inequality:

$$f(k+1)\int_k^{k+1} dx \le \int_k^{k+1} f(x)\,dx \le f(k+1)\int_k^{k+1} dx.$$

The integrals on the left-hand and right-hand sides of this double inequality are equal to 1:

$$\int_k^{k+1} dx = x\Big|_k^{k+1} = k+1-k = 1.$$

Thus, the double inequality takes the form

$$f(k+1) \le \int_k^{k+1} f(x)\,dx \le f(k).$$

Now we summarize the inequalities obtained for $k = 1, \dots, n$:

$$\sum_{k=1}^{n} f(k+1) \le \sum_{k=1}^{n}\int_k^{k+1} f(x)\,dx \le \sum_{k=1}^{n} f(k).$$

Given the property of additivity of the integral with respect to the integration interval, the resulting double inequality can be rewritten as follows:

$$\sum_{k=1}^{n} f(k+1) \le \int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k).$$

Let us introduce the notation for the partial sum of the series: $S_n = \sum_{k=1}^{n} f(k)$. Using this notation, we finally obtain

$$S_{n+1} - f(1) \le \int_1^{n+1} f(x)\,dx \le S_n. \tag{2}$$

The final stage of the proof

3.10B/36:40 (07:13)

Now we consider various situations related to the convergence or divergence of the initial integral and series.

1. Let the improper integral $\int_1^{+\infty} f(x)\,dx$ converge. Then, by virtue of the criterion for the convergence of improper integrals of non-negative functions, we have

$$\exists\, M > 0 \quad \forall\, c > 1 \quad \int_1^c f(x)\,dx \le M.$$

Using this estimate for the integral and the left-hand side of estimate (2), we obtain

$$S_{n+1} - f(1) \le \int_1^{n+1} f(x)\,dx \le M.$$

Thus, we have proved that the partial sums of S_n are uniformly bounded:

$$\forall\, n \in \mathbb{N} \quad S_{n+1} \le M + f(1).$$

Therefore, by the criterion for the convergence of a numerical series with non-negative terms, the series $\sum_{k=1}^{\infty} f(k)$ converges.

2. Let the series $\sum_{k=1}^{\infty} f(k)$ converge. Then, by virtue of the criterion for the convergence of a numerical series with non-negative terms, we have

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N} \quad S_n \le M.$$

Choose an arbitrary real number $c > 1$ and consider the integral $\int_1^c f(x)\,dx$. For any number $c > 1$, there exists an integer n such that $c < n + 1$. Since the function $f(x)$ is non-negative, the estimate holds:

$$\int_1^c f(x)\,dx \le \int_1^{n+1} f(x)\,dx.$$

Using this estimate and the right-hand side of estimate (2), we obtain

$$\int_1^c f(x)\,dx \le \int_1^{n+1} f(x)\,dx \le S_n \le M.$$

We have proved that the integrals $\int_1^c f(x)\,dx$ are uniformly bounded:

$$\forall\, c > 1 \quad \int_1^c f(x)\,dx \le M.$$

Therefore, by the criterion for the convergence of improper integrals of non-negative functions, the integral $\int_1^{+\infty} f(x)\,dx$ converges.

3. Let the series $\sum_{k=1}^{\infty} f(k)$ diverge. Assuming that the integral $\int_1^{+\infty} f(x)\,dx$ converges, we obtain that, by the result already proved in section 1, the series $\sum_{k=1}^{\infty} f(k)$ should also converge, but this contradicts the condition. Therefore, the integral diverges.

4. Let the integral $\int_1^{+\infty} f(x)\,dx$ diverge. If we assume that the series $\sum_{k=1}^{\infty} f(k)$ converges, then, by the result already proved in section 2, the integral $\int_1^{+\infty} f(x)\,dx$ must also converge, but this contradicts the condition. Therefore, the series diverges. □

REMARK.

The limit relation $\lim_{x\to+\infty} f(x) = 0$ was not used in the proof. It is required in order to ensure that the necessary condition for the convergence of the series $\sum_{k=1}^{\infty} f(k)$ is satisfied, since if this condition is violated, the series will necessarily diverge (and, as follows from the proof, the integral $\int_1^{+\infty} f(x)\,dx$ will also diverge).

An example of applying the integral test of convergence

3.11A/00:00 (04:49)

Earlier, we found that the improper integral $\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ converges for $\alpha > 1$ and diverges for $\alpha \le 1$. Now we can extend this result to the corresponding series. For $\alpha > 0$, the function $f(x) = \frac{1}{x^\alpha}$ satisfies all the conditions of the previous theorem (it is non-negative and monotonously approaches 0 as $x \to +\infty$), therefore, by virtue of of the previous theorem, the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges for $\alpha > 1$ and diverges for $\alpha \in (0, 1]$. For $\alpha \le 0$, the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ also diverges, since, in this case, its common term $\frac{1}{k^\alpha}$ does not approach 0 as $k \to \infty$ and therefore the necessary convergence condition is not satisfied for the series. Thus, we have proved the following statement.

THEOREM (ON THE CONVERGENCE OF NUMERICAL SERIES WITH COMMON TERMS THAT ARE POWER FUNCTIONS).

The numerical series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges for $\alpha > 1$ and diverges for $\alpha \le 1$. In particular, the series $\sum_{k=1}^{\infty} \frac{1}{k}$, called the *harmonic series*, diverges.

D'Alembert's test and Cauchy's test for convergence of a numerical series

Formulation of D'Alembert's test

3.11A/04:49 (04:01)

The tests considered in this section have no analogues for improper integrals.

Theorem (D'Alembert's test for convergence of a numerical series).

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms: $\forall\, k \in \mathbb{N}\ \ a_k > 0$.

1. Let the following condition be satisfied:

$$\exists\, q \in (0,1) \quad \exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad \frac{a_{k+1}}{a_k} \leq q.$$

Then the series $\sum_{k=1}^{\infty} a_k$ converges.

2. Let the following condition be satisfied:

$$\exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad \frac{a_{k+1}}{a_k} \geq 1.$$

Then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Proof of D'Alembert's test

3.11A/08:50 (08:54)

1. Consider the terms of the initial series, starting with $k = m$. By condition, $\frac{a_{m+1}}{a_m} \leq q$, whence

$$a_{m+1} \leq q a_m.$$

The same inequality holds for the term a_{m+2}: $a_{m+2} \leq q a_{m+1}$. Given the previous inequality, we obtain

$$a_{m+2} \leq q a_{m+1} \leq q^2 a_m.$$

Obviously, for the terms a_{m+k}, $k \in \mathbb{N}$, the following estimate holds (which can be rigorously proved by mathematical induction):

$$a_{m+k} \leq q^k a_m. \tag{3}$$

Consider the series $\sum_{k=1}^{\infty} a_{m+k}$ and $\sum_{k=1}^{\infty} q^k a_m$.

The first series can be rewritten in the form $\sum_{k=m+1}^{\infty} a_k$, therefore, it coincides with the initial series, from which m first terms are removed. So, if the series $\sum_{k=1}^{\infty} a_{m+k}$ converges, then the initial series also converges, since the presence or absence of a finite number of initial terms of the series does not affect its convergence.

The second series can be transformed as follows: $\sum_{k=1}^{\infty} q^k a_m = a_m \sum_{k=1}^{\infty} q^k$. Since, by condition, $q \in (0,1)$, we obtain, by virtue of the formula for the sum of infinite geometric progression, that the series $\sum_{k=1}^{\infty} q^k$ converges.

Considering estimate (3) and applying the comparison test for numerical series, we obtain that the series $\sum_{k=1}^{\infty} a_{m+k}$ also converges and therefore the initial series $\sum_{k=1}^{\infty} a_k$ converges too.

2. As in the proof of section 1, we consider the terms of the initial series, starting with $k = m$. By condition, $\frac{a_{m+1}}{a_m} \geq 1$, whence

$$a_{m+1} \geq a_m.$$

Similarly, we obtain the estimate $a_{m+2} \geq a_{m+1} \geq a_m$. The same estimate will be valid for all terms a_{m+k} for $k \in \mathbb{N}$:

$$a_{m+k} \geq a_m.$$

We have obtained that the terms of the initial series, starting with a_m, are bounded from below by the positive value a_m. This means that the sequence $\{a_k\}$ cannot approach 0 as $k \to \infty$. Indeed, choosing the number $\varepsilon > 0$ equal to the minimum of a finite set of positive numbers $a_1, a_2, \ldots, a_m$, we get that the ε-neighborhood of zero does not contain any element of the sequence $\{a_k\}$. But, by the definition of the limit equal to A, any neighborhood of the point A should contain all elements of the sequence except, perhaps, a finite number of its initial elements.

Since the necessary convergence condition is not satisfied for the series $\sum_{k=1}^{\infty} a_k$, this series diverges. □

The limit D'Alembert test

3.11A/17:44 (07:23)

Corollary (the limit D'Alembert test).

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms: $\forall\, k \in \mathbb{N} \quad a_k > 0$. Suppose that there exists a limit $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = q$. If $q < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges; if $q > 1$, then the series diverges.

Proof.

Using the limit definition in the language ε–N, we can write

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, k > N \quad \left| \frac{a_{k+1}}{a_k} - q \right| < \varepsilon.$$

1. If $q < 1$, then choosing $\varepsilon = \frac{1-q}{2} > 0$, we get that, for all $k > N$, the inequality $\frac{a_{k+1}}{a_k} - q < \frac{1-q}{2}$ holds, from which the estimate follows:

$$\frac{a_{k+1}}{a_k} < q + \frac{1-q}{2} = \frac{1+q}{2} = q'.$$

Since $q < 1$, we obtain that $q' < 1$, therefore the condition of statement 1 of D'Alembert's test is satisfied for the initial series. Consequently, the series converges.

2. If $q > 1$, then choosing $\varepsilon = \frac{q-1}{2} > 0$, we get that, for all $k > N$, the inequality $\frac{a_{k+1}}{a_k} - q > -\frac{q-1}{2}$ holds, from which the estimate follows:

$$\frac{a_{k+1}}{a_k} > q - \frac{q-1}{2} = \frac{q+1}{2} > 1.$$

Thus, for the initial series, the condition of statement 2 of D'Alembert's test is satisfied, therefore the series diverges. $\square$

REMARKS.

1. If the limit $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ is 1, then nothing can be said about the convergence or divergence of the series and further investigation is required.

2. If the limit $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ is equal to $+\infty$, then, by similar reasoning, we can prove that the series diverges.

An example of applying D'Alembert's test 3.11A/25:07 (02:48)

Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Recall that, by definition, it is supposed that $0! = 1$. Here x is an arbitrary real number. Denote $a_k = \frac{x^k}{k!}$ and consider the following limit:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} = \lim_{k\to\infty} \frac{x^{k+1}k!}{x^k(k+1)!} = \lim_{k\to\infty} \frac{x}{k+1} = 0.$$

The limit exists and its value is less than 1, therefore, due to the limit D'Alembert test, this series converges for any value of the parameter $x \in \mathbb{R}$.

REMARK.

In what follows, we prove that the sum of the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is equal to e^x.

Cauchy's test 3.11A/27:55 (07:22)

THEOREM (CAUCHY'S TEST FOR CONVERGENCE OF A NUMERICAL SERIES).

Let $\sum_{k=1}^{\infty} a_k$ be a series with non-negative terms: $\forall\, k \in \mathbb{N}\ \ a_k \geq 0$.

1. Let the following condition be satisfied:

$$\exists\, q \in (0,1) \quad \exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad \sqrt[k]{a_k} \leq q.$$

Then the series $\sum_{k=1}^{\infty} a_k$ converges.

2. Let the following condition be satisfied:

$$\exists\, m \in \mathbb{N} \quad \forall\, k \geq m \quad \sqrt[k]{a_k} \geq 1.$$

Then the series $\sum_{k=1}^{\infty} a_k$ diverges.

PROOF.

1. Consider the terms of the initial series, starting with $k = m$. By condition, $\sqrt[k]{a_k} \leq q$; let us raise both sides of this inequality to the power of k:

$$a_k \leq q^k. \tag{4}$$

Estimate (4) is valid for terms of the series $\sum_{k=m}^{\infty} a_k$ and $\sum_{k=m}^{\infty} q^k$. Since, by condition, $q \in (0,1)$, we obtain, by virtue of the formula for the sum of infinite geometric progression, that the series $\sum_{k=m}^{\infty} q^k$ converges.

Taking into account estimate (4) and applying the comparison test for numerical series, we obtain that the series $\sum_{k=m}^{\infty} a_k$ also converges and therefore the original series $\sum_{k=1}^{\infty} a_k$ converges too.

2. As in the proof of section 1, we consider the terms of the initial series, starting with $k = m$. By condition, $\sqrt[k]{a_k} \geq 1$. We raise both sides of this inequality to the power of k:

$$a_k \geq 1.$$

Arguing in the same way as in the proof of section 2 of D'Alembert's test, we obtain that the sequence $\{a_k\}$ cannot approach 0 as $k \to \infty$, and therefore the necessary convergence condition is not satisfied for the series $\sum_{k=1}^{\infty} a_k$. So, this series diverges. □

Corollary (the limit Cauchy test).

Let $\sum_{k=1}^{\infty} a_k$ be a series with non-negative terms: $\forall\, k \in \mathbb{N}\ \ a_k \geq 0$. Suppose that there exists a limit $\lim_{k\to\infty} \sqrt[k]{a_k} = q$. If $q < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges; if $q > 1$, then the series diverges.

The proof is carried out in the same way as the proof of the limit D'Alembert test. □

Remarks.

1. If the limit $\lim_{k\to\infty} \sqrt[k]{a_k}$ is 1, then nothing can be said about the convergence or divergence of the series and further investigation is required.

2. If the limit $\lim_{k\to\infty} \sqrt[k]{a_k}$ is equal to $+\infty$, then we can prove that the series diverges.

An example of applying Cauchy's test 3.11A/35:17 (06:06)

Consider the series $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k^2}$. Denote $a_k = \left(1+\frac{1}{k}\right)^{-k^2}$ and consider the following limit:

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left(1+\frac{1}{k}\right)^{-k^2}} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{-k} = \frac{1}{e}.$$

In the last step, we used the second remarkable limit $\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{k} = e$. Thus, the limit $\lim_{k\to\infty} \sqrt[k]{a_k}$ exists and its value $\frac{1}{e}$ is less than 1. Therefore, by virtue of the limit Cauchy test, this series converges.

Note that the series $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k}$ diverges, since its common term $\left(1+\frac{1}{k}\right)^{-k}$ does not approach 0 as $k \to \infty$ (as shown above, the limit of the common term is $\frac{1}{e}$).

14. Alternating series and conditional convergence

Alternating series

Definition of conditional convergence, alternating series, and the Leibniz series

3.11B/00:00 (04:22)

Definition 1.

The series $\sum_{k=1}^{\infty} a_k$ is called *conditionally convergent* if it converges and the series $\sum_{k=1}^{\infty} |a_k|$ diverges. Thus, a convergent series is called conditionally convergent if it does not converge absolutely.

Such a situation is possible only when the terms of a series have different signs.

Definition 2.

A series of the form $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ is called an *alternating series*, if all elements of the sequence $\{a_k\}$ have the same sign.

Definition 3.

An alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ is called the *Leibniz series*, if the sequence $\{a_k\}$ monotonously approaches zero as $k \to \infty$.

Remarks.

1. When studying Leibniz series of the form $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$, we assume, for definiteness, that $a_k > 0$, $k \in \mathbb{N}$ (in this case, the sequence $\{a_k\}$ is a *non-increasing* sequence approaching zero).

2. The "Leibniz series" notion is also referred to the alternating series of a special form $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$, which was studied by G. W. Leibniz (he proved that the sum of this series is equal to $\frac{\pi}{4}$).

Theorem on the convergence of the Leibniz series

3.11B/04:22 (11:11)

Theorem (on the convergence of the Leibniz series).

The Leibniz series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

Proof.

Consider the partial sums of the Leibniz series with an even number of terms:

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n}. \qquad (1)$$

We place parentheses on the right-hand side of equality (1) as follows:

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}).$$

Since the sequence $\{a_k\}$ is non-increasing, we obtain that each expression in parentheses is non-negative: $a_{2k-1} - a_{2k} \geq 0$, $k = 1, 2, \ldots$ Hence,

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}) \geq S_{2n}.$$

This estimate means that the sequence of partial sums $\{S_{2n}\}$ is non-decreasing.

Now we put parentheses in (1) in another way:

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}.$$

Since, as before, each expression in parentheses is non-negative, we obtain that the sum S_{2n} is estimated from above by the value a_1:

$$S_{2n} \leq a_1.$$

Thus, the sequence $\{S_{2n}\}$ is not only non-decreasing, but also bounded from above. Then, by virtue of the convergence theorem for monotone bounded sequences, the sequence $\{S_{2n}\}$ has a finite limit S:

$$\lim_{n\to\infty} S_{2n} = S.$$

Consider the partial sums of the Leibniz series with an odd number of terms: S_{2n+1}. For them, the following equality holds:

$$S_{2n+1} = S_{2n} + a_{2n+1}. \qquad (2)$$

We have already proved that $S_{2n} \to S$ as $n \to \infty$. In addition, $a_{2n+1} \to 0$ as $n \to \infty$, since by condition $a_k \to 0$ as $k \to \infty$ and thus the subsequence $\{a_{2n+1}\}$ of the sequence $\{a_k\}$ must also converge to this limit by the theorem on the limit of subsequences of a converging sequence.

Therefore, the right-hand side of equality (2) has a limit S, so the left-hand side approaches the same limit.

So, we have proved that $S_{2n} \to S$ as $n \to \infty$ and $S_{2n+1} \to S$ as $n \to \infty$. This means that the entire sequence $\{S_n\}$ converges to the limit S, since any neighborhood of the point S contains all elements of the sequence $\{S_n\}$ (with even and odd indices), with the possible exception of some finite number of its initial elements.

The convergence of the sequence of partial sums $\{S_n\}$ to a finite limit means that the corresponding series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges. □

Remark.

The theorem on the convergence of the Leibniz series guarantees only its conditional convergence. For example, the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ is a Leibniz series, however, we previously established that the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$, consisting of absolute values of terms of the initial series, is divergent. In what follows, we will prove that the sum of the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ is equal to $\ln 2$.

Estimation of the Leibniz series in terms of its partial sums

3.11B/15:33 (14:19)

Theorem (on the estimation of the Leibniz series in terms of its partial sums).

Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = S$ be the Leibniz series and $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ be its partial sums. Then, for any $k \in \mathbb{N}$, the following estimate holds:

$$|S - S_k| \le a_{k+1}. \tag{3}$$

Proof.

In the proof of the previous theorem, we established that the sequence $\{S_{2n}\}$ is non-decreasing and has a limit S. This means that the following equality holds for all $n \in \mathbb{N}$:

$$S_{2n} \le S. \tag{4}$$

On the other hand, the sequence $\{S_{2n+1}\}$ is non-increasing since

$$S_{2n+1} = a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1}) - (a_{2n} - a_{2n+1}) \ge$$
$$\ge a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1}) = S_{2n-1}.$$

In addition, its limit is also equal to S. Therefore, the equality holds for all $n \in \mathbb{N}$:

$$S \le S_{2n+1}. \tag{5}$$

Let us subtract S_{2n} from both sides of inequality (5):

$$S - S_{2n} \le S_{2n+1} - S_{2n} = a_{2n+1}. \tag{6}$$

It follows from inequality (4) that $S - S_{2n} \ge 0$. Therefore, inequality (6) can be rewritten in the form

$$|S - S_{2n}| \leq a_{2n+1}.$$

We have obtained estimate (3) for the case of even k.

Now we turn to inequality (4) and subtract S_{2n-1} from both its parts:

$$S_{2n} - S_{2n-1} \leq S - S_{2n-1}.$$

Since $S_{2n} - S_{2n-1} = -a_{2n}$, this inequality can be transformed as follows:

$$S_{2n-1} - S \leq a_{2n}. \tag{7}$$

It follows from inequality (5) that $S_{2n-1} - S \geq 0$. Therefore, inequality (7) can be rewritten in the form

$$|S_{2n-1} - S| \leq a_{2n}.$$

We have obtained estimate (3) for the case of odd k.

Thus, estimate (3) is proved for all positive integers k. □

Dirichlet's test and Abel's test for conditional convergence of a numerical series

Dirichlet's test for conditional convergence of a numerical series 3.11B/29:52 (04:29), 3.12A/00:00 (03:18)

Theorem (Dirichlet's test for conditional convergence of a numerical series).

Let the following conditions be satisfied for the series $\sum_{k=1}^{\infty} a_k b_k$:

1) $\exists\, M \quad \forall\, n \in \mathbb{N} \quad \left|\sum_{k=1}^{n} a_k\right| \leq M$;

2) $b_k \to 0$ as $k \to \infty$, $\{b_k\}$ is monotone.

Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges (generally speaking, conditionally).

Proof[3].

Let us show that, for the series $\sum_{k=1}^{\infty} a_k b_k$, the condition for the Cauchy criterion for the convergence of a numerical series is fulfilled. For this, we will obtain an estimate for the sum $\left|\sum_{k=m+1}^{m+p} a_k b_k\right|$ when $m, p \in \mathbb{N}$.

First, let us transform the sum $\sum_{k=m+1}^{m+p} a_k b_k$ using the auxiliary notation $A_n = \sum_{k=1}^{n} a_k$:

$$\sum_{k=m+1}^{m+p} a_k b_k = \sum_{k=m+1}^{m+p} (A_k - A_{k-1}) b_k = \sum_{k=m+1}^{m+p} A_k b_k - \sum_{k=m+1}^{m+p} A_{k-1} b_k =$$

$$= \sum_{k=m+2}^{m+p+1} A_{k-1} b_{k-1} - \sum_{k=m+1}^{m+p} A_{k-1} b_k =$$

[3]There is no proof of this theorem in video lectures.

$$= A_{m+p}b_{m+p} + \sum_{k=m+2}^{m+p} A_{k-1}b_{k-1} - \sum_{k=m+2}^{m+p} A_{k-1}b_k - A_m b_{m+1} =$$

$$= A_{m+p}b_{m+p} + \sum_{k=m+2}^{m+p} A_{k-1}(b_{k-1} - b_k) - A_m b_{m+1}.$$

Let us estimate the value $\left|\sum_{k=m+1}^{m+p} a_k b_k\right|$ using condition 1 of the theorem, from which it follows that $|A_k| \le M$ for $k \in \mathbb{N}$:

$$\Bigl|\sum_{k=m+1}^{m+p} a_k b_k\Bigr| = \Bigl|A_{m+p}b_{m+p} + \sum_{k=m+2}^{m+p} A_{k-1}(b_{k-1} - b_k) - A_m b_{m+1}\Bigr| \le$$

$$\le M|b_{m+p}| + M \sum_{k=m+2}^{m+p} |b_{k-1} - b_k| + M|b_{m+1}|. \tag{8}$$

Since, by condition 2 of the theorem, the sequence $\{b_k\}$ monotonously approaches 0, we obtain that all the differences $b_{k-1} - b_k$ have the same sign. Therefore, in the sum $\sum_{k=m+2}^{m+p} |b_{k-1} - b_k|$, the absolute value sign can be moved outside the sum sign:

$$\sum_{k=m+2}^{m+p} |b_{k-1} - b_k| = \Bigl|\sum_{k=m+2}^{m+p} (b_{k-1} - b_k)\Bigr| =$$

$$= |(b_{m+1} - b_{m+2}) + (b_{m+2} - b_{m+3}) + \dots + (b_{m+p-1} - b_{m+p})| =$$

$$= |b_{m+1} - b_{m+p}| \le |b_{m+1}| + |b_{m+p}|.$$

Now we substitute the estimate for $\sum_{k=m+2}^{m+p} |b_{k-1} - b_k|$ into inequality (8):

$$\Bigl|\sum_{k=m+1}^{m+p} a_k b_k\Bigr| \le M|b_{m+p}| + M(|b_{m+1}| + |b_{m+p}|) + M|b_{m+1}| =$$

$$= 2M(|b_{m+1}| + |b_{m+p}|).$$

It remains to use the condition $b_k \to 0$ as $k \to \infty$, which can be written as follows:

$$\forall\,\varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, p \in \mathbb{N} \quad |b_{m+p}| < \frac{\varepsilon}{4M}.$$

For $\left|\sum_{k=m+1}^{m+p} a_k b_k\right|$, we finally get

$$\Bigl|\sum_{k=m+1}^{m+p} a_k b_k\Bigr| \le 2M(|b_{m+1}| + |b_{m+p}|) < 2M\Bigl(\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\Bigr) = \varepsilon.$$

We have proved that the Cauchy criterion condition is satisfied for the initial series:

$$\forall\,\varepsilon > 0 \quad \exists\,N \in \mathbb{N} \quad \forall\,m > N \quad \forall\,p \in \mathbb{N} \quad \left|\sum_{k=m+1}^{m+p} a_k b_k\right| < \varepsilon.$$

Therefore, the series $\sum_{k=1}^{\infty} a_k b_k$ converges. $\square$

Examples of applying Dirichlet's test 3.12A/03:18 (11:41)

1. Once again, let us turn to the Leibniz series and write it in the following form: $\sum_{k=1}^{\infty}(-1)^{k+1}b_k$. By the definition of the Leibniz series, two conditions are satisfied for the sequence $\{b_k\}$: $b_k \to 0$ as $k \to \infty$, $\{b_k\}$ is monotone. Thus, the condition 2 of Dirichlet's test is satisfied for $\{b_k\}$. Also we can take the sequence $\left\{(-1)^{k+1}\right\}$ as the sequence $\{a_k\}$. Obviously, this sequence satisfies condition 1 of Dirichlet's test:

$$\forall\,n \in \mathbb{N} \quad \left|\sum_{k=1}^{n} a_k\right| = |1 - 1 + 1 - 1 + \ldots| \le 1.$$

Thus, the convergence of the Leibniz series follows directly from the Dirichlet's test.

2. Consider the following series: $\sum_{k=1}^{\infty} \frac{\sin kx}{k^{\alpha}}$, $x \in \mathbb{R}$, $\alpha > 0$. If $\alpha > 1$, then this series converges absolutely for any $x \in \mathbb{R}$, since, in this case, the absolute value of its common term can be estimated as follows:

$$\left|\frac{\sin kx}{k^{\alpha}}\right| \le \frac{1}{k^{\alpha}}.$$

Earlier, when discussing the integral convergence test, we established that the series $\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$ converges for $\alpha > 1$. Therefore, using the comparison test, we obtain that the series $\sum_{k=1}^{\infty}\left|\frac{\sin kx}{k^{\alpha}}\right|$ also converges, which means that the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^{\alpha}}$ converges absolutely.

Consider the case $\alpha \in (0, 1]$ and show that in this case all conditions of Dirichlet's test are satisfied for the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^{\alpha}}$.

First, we discard the case of $x = 2\pi m$, $m \in \mathbb{Z}$, since in this case all terms of the series turn to 0 and therefore the sum of the series is also 0.

We take $\frac{1}{k^{\alpha}}$ as b_k, since it is obvious that the sequence $\left\{\frac{1}{k^{\alpha}}\right\}$ is monotone (decreasing) and approaches zero as $k \to \infty$. We take $\sin kx$ as a_k and show that condition 1 of Dirichlet's test is satisfied for partial sum $\sum_{k=1}^{n} \sin kx$. To do this, we transform this partial sum by multiplying and dividing the common term by $2\sin\frac{x}{2}$ (this factor is not equal to 0, since we assume that $x \ne 2\pi m$, $m \in \mathbb{Z}$):

$$\sum_{k=1}^{n} \frac{2\sin kx \sin\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} 2\sin kx \sin\frac{x}{2}. \tag{9}$$

Let us transform the product of the sines $\sin kx \sin\frac{x}{2}$ according to the formula $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$:

$$\sum_{k=1}^{n} 2\sin kx \sin\frac{x}{2} = \sum_{k=1}^{n}\Bigl(\cos\Bigl(kx-\frac{x}{2}\Bigr) - \cos\Bigl(kx+\frac{x}{2}\Bigr)\Bigr) =$$
$$= \cos\frac{x}{2} - \cos\frac{3x}{2} + \cos\frac{3x}{2} - \cos\frac{5x}{2} + \dots +$$
$$+\cos\frac{(2n-1)x}{2} - \cos\frac{(2n+1)x}{2} = \cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}.$$

Now let us transform the last difference using the formula $\cos\alpha - \cos\beta = 2\sin\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}$:

$$\cos\frac{x}{2} - \cos\frac{(2n+1)x}{2} = 2\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}.$$

Substituting the resulting expression into the right-hand side of (9), we finally obtain

$$\sum_{k=1}^{n}\sin kx = \frac{1}{2\sin\frac{x}{2}}\cdot 2\sin\frac{(n+1)x}{2}\sin\frac{nx}{2} = \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

This implies the following estimate for partial sum $\sum_{k=1}^{n}\sin kx$, $n\in\mathbb{N}$:

$$\Bigl|\sum_{k=1}^{n}\sin kx\Bigr| \le \frac{1}{\bigl|\sin\frac{x}{2}\bigr|}.$$

Thus, condition 1 of Dirichlet's test is also satisfied, and the series $\sum_{k=1}^{\infty}\frac{\sin kx}{k^\alpha}$ is convergent for $\alpha\in(0,1]$. However, for these values of α, convergence is conditional.

The proof of the absence of absolute convergence

3.12A/14:59 (06:03)

The fact that the series $\sum_{k=1}^{\infty}\frac{\sin kx}{k^\alpha}$ is not absolutely convergent for $\alpha\in(0,1]$ is proved in the same way as a similar fact for the improper integral $\int_1^{+\infty}\frac{\sin x}{x}\,dx$. First of all, recall the estimate for the function $\frac{\sin kx}{k^\alpha}$; this estimate is valid for all k and x:

$$\Bigl|\frac{\sin kx}{k^\alpha}\Bigr| \ge \frac{\sin^2 kx}{k^\alpha}. \tag{10}$$

Let us prove that the series $\sum_{k=1}^{\infty}\frac{\sin^2 kx}{k^\alpha}$ diverges. To do this, consider its partial sum and transform it as follows:

$$\sum_{k=1}^{n}\frac{\sin^2 kx}{k^\alpha} = \sum_{k=1}^{n}\frac{1-\cos 2kx}{2k^\alpha} = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k^\alpha} - \frac{1}{2}\sum_{k=1}^{n}\frac{\cos 2kx}{k^\alpha}\,. \tag{11}$$

The second term on the right-hand side of (11) has a finite limit as $n \to \infty$, since the series $\sum_{k=1}^{\infty} \frac{\cos 2kx}{k^\alpha}$ converges (this fact can be proved in the same way as the convergence of the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^\alpha}$). The first term on the right-hand side of (11) approaches infinity as $n \to \infty$, since the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ diverges for $\alpha \in (0,1]$.

Therefore, the right-hand side of equality (11) has an infinite limit as $n \to \infty$, this is also true for the left-hand side, so the series $\sum_{k=1}^{\infty} \frac{\sin^2 kx}{k^\alpha}$ diverges. Using the comparison test, we obtain from estimate (10) that the series $\sum_{k=1}^{\infty} \left|\frac{\sin kx}{k^\alpha}\right|$ also diverges. So, for $\alpha \in (0,1]$, the initial series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^\alpha}$ converges conditionally.

Abel's test for conditional convergence of a numerical series

3.12A/21:02 (06:58)

Theorem (Abel's test for conditional convergence of a numerical series).

Let the following conditions be satisfied for a series $\sum_{k=1}^{\infty} a_k b_k$:

1) the series $\sum_{k=1}^{\infty} a_k$ converges;

2) the sequence $\{b_k\}$ is monotone and bounded.

Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges (generally speaking, conditionally).

Remark.

If we compare Dirichlet's test and Abel' test, then it can be noted that in Abel's test, condition 1 is stronger (since the convergence of the corresponding series is required instead of uniformly boundedness of its partial sums) and condition 2 is weaker (since it is not necessary that the sequence $\{b_k\}$ had a zero limit).

Proof.

By virtue of the theorem on monotone and bounded sequences, the sequence $\{b_k\}$ has a finite limit: $b_k \to c$ as $k \to \infty$.

We transform the partial sum of the initial series as follows:

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k (b_k - c + c) = \sum_{k=1}^{n} a_k (b_k - c) + c \sum_{k=1}^{n} a_k. \tag{12}$$

The second term on the right-hand side of (12) has a finite limit as $n \to \infty$, since, by condition 1, the series $\sum_{k=1}^{\infty} a_k$ converges.

The first term on the right-hand side of (12) is a partial sum of the series $\sum_{k=1}^{\infty} a_k (b_k - c)$, which converges according to Dirichlet's test. Indeed, condition 1 of Dirichlet's test follows from condition 1 of Abel's test, since if the series $\sum_{k=1}^{\infty} a_k$ converges, then its partial sums are uniformly bounded. Condition 2 of Dirichlet's test follows from condition 2 of Abel's test and the fact that $\lim_{k\to\infty} b_k = c$, since in this case the sequence $\{b_k - c\}$ monotonously approaches zero as $k \to \infty$. So, the first term on the right-hand side of (12) also has a finite limit.

Therefore, the partial sums $\sum_{k=1}^{n} a_k b_k$ also have a finite limit, and the initial series converges. $\square$

Additional remarks on absolutely and conditionally convergent series

3.12A/28:00 (07:07)

The question arises: will the sum of the convergent series $\sum_{k=1}^{\infty} a_k$ change if the order of its terms is changed? For example, it is possible to organize the summation, for which, after each term a_k of the initial series with an odd index $(a_1, a_3, a_5, \dots)$, *several* terms with even indices will follow, and their amount will increase by 1 each time $(a_1+a_2+a_3+a_4+a_6+a_5+a_8+a_{10}+a_{12}+a_7+\dots)$ or it will double each time $(a_1+a_2+a_3+a_4+a_6+a_5+a_8+a_{10}+a_{12}+a_{14}+a_7+\dots)$.

It turns out that, for an absolutely convergent series, its sum does not change with any change in the order of its terms. However, for a conditionally convergent series, this statement is false.

Moreover, if the series conditionally converges, then, by rearranging its terms, it can be achieved that the resulting series converges to any pre-selected number $A \in \mathbb{R}$ or diverges. This fact is called the *Riemann theorem on conditionally convergent series* (its proof is given, for example, in [18, Ch. 8, Sec. 41.4]).

15. Functional sequences and series

Pointwise and uniform convergence of a functional sequence and a functional series

Functional sequence and functional series, their pointwise convergence

3.12A/35:07 (10:52)

Definition.

A *functional sequence* $\{f_n(x)\}$ is a sequence of functions $f_n(x)$ defined on a set E.

If we choose some point $x_0 \in E$ and substitute it in all the functions $f_n(x)$, then we get the numerical sequence $\{f_n(x_0)\}$. It is said that the functional sequence $\{f_n(x)\}$ *converges at the point* $x_0 \in E$ if the numerical sequence $\{f_n(x_0)\}$ converges.

It is said that the functional sequence $\{f_n(x)\}$ *converges on the set* E if it converges at all points $x_0 \in E$ (notation $f_n(x) \overset{E}{\to} f(x)$, $n \to \infty$). Thus, the limit of a functional sequence converging on the set E is some function defined on this set.

A *functional series* is a series $\sum_{k=1}^{\infty} u_k(x)$ with terms that are functions defined on a set E.

If the functional sequence of partial sums $S_n(x) = \sum_{k=1}^{n} u_k(x)$ converges on the set E to the function $S(x)$, then they say that the functional series $\sum_{k=1}^{\infty} u_k(x)$ *converges on the set* E. Moreover, the function $S(x)$ is called the *sum of the convergent functional series* $\sum_{k=1}^{\infty} u_k(x)$, and in this case the notation $\sum_{k=1}^{\infty} u_k(x)$ usually means the sum $S(x)$ of the series:

$$\sum_{k=1}^{\infty} u_k(x) = S(x).$$

For functional series, as well as for numerical ones, the concepts of absolute and conditional convergence can be introduced: the series $\sum_{k=1}^{\infty} u_k(x)$ *absolutely converges on* E if the series $\sum_{k=1}^{\infty} |u_k(x)|$ converges on this set; the series $\sum_{k=1}^{\infty} u_k(x)$ *conditionally converges on* E if it converges on this set but its convergence is not absolute.

The considered type of convergence of functional sequences and series on some set is called *pointwise convergence*. The definition of pointwise convergence of the functional sequence $\{f_n(x)\}$ on the set E to the function $f(x)$ can be written as follows, using the language ε–N:

$$\forall x \in E \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |f_n(x) - f(x)| < \varepsilon. \tag{1}$$

However, another type of convergence can be defined for functional sequences and series. This type of convergence allows a more detailed study of the properties of the limits of functional sequences and sums of functional series.

Uniform convergence of the functional sequence

3.12B/00:00 (03:57)

Definition.

It is said that the functional sequence $\{f_n(x)\}$ *uniformly converges* on the set E to the function $f(x)$ (notation $f_n(x) \overset{E}{\rightrightarrows} f(x)$, $n \to \infty$) if the following condition is true:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad \forall x \in E \quad |f_n(x) - f(x)| < \varepsilon. \tag{2}$$

Thus, with uniform convergence, the number N is selected only by the value of ε and *does not depend* on the choice of the point $x \in E$. Note that the same differences hold for the concepts of continuity and uniform continuity of a function on a set (see [1, Ch. 13]).

Criterion for uniform convergence of a functional sequence in terms of the supremum limit

3.12B/03:57 (10:28)

Theorem (criterion for uniform convergence of a functional sequence in terms of the supremum limit).

The functional sequence $\{f_n(x)\}$ converges uniformly on the set E to the function $f(x)$ if and only if the following limit relation holds:

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0. \tag{3}$$

Remark.

Relation (3) can be verified if the limit function $f(x)$ has already been found. Thus, this relation makes it relatively easy to establish both the presence and absence of uniform convergence, provided that pointwise convergence is already established.

PROOF.

1. Necessity. Given: condition (2) is satisfied. Prove: the limit relation (3) holds.

We rewrite condition (2) with a slight change in the last inequality:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad \forall\, x \in E \quad |f_n(x) - f(x)| < \frac{\varepsilon}{2}. \tag{4}$$

Since the condition is satisfied for all $x \in E$, we obtain that the set of all values $|f_n(x) - f(x)|$ is bounded from above and $\frac{\varepsilon}{2}$ is its upper bound. Since the supremum is the least upper bound, the following estimate holds:

$$\sup_{x\in E} |f_n(x) - f(x)| \le \frac{\varepsilon}{2} < \varepsilon.$$

Thus, condition (4) implies the following condition:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon. \tag{5}$$

We obtained a definition in the language ε - N of the fact that the limit of the sequence $\{\sup_{x\in E} |f_n(x) - f(x)|\}$ as $n \to \infty$ is 0. The necessity is proven.

2. Sufficiency. Given: the limit relation (3) holds. Prove: condition (2) is satisfied.

Relation (3) can be written in the language ε–N in the form (5).

The condition $\sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$, by the definition of supremum, implies an estimate that holds for all $x \in E$:

$$|f_n(x) - f(x)| \le \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon.$$

Replacing in condition (5) the estimate $\sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$ with the resulting estimate $\forall\, x \in E \ |f_n(x) - f(x)| < \varepsilon$, we obtain (2), i. e., the definition of the uniform convergence of the sequence $\{f_n(x)\}$. $\square$

Examples of applying the criterion for uniform convergence of a functional sequence 3.12B/14:25 (14:47)

Consider the sequence of functions $f_n(x) = x^n$ on the set $E = [0, 1]$.

Obviously, for any point $x \in [0, 1)$, there exists a limit $\lim_{n\to\infty} x^n = 0$, and the limit is 1 for $x = 1$: $\lim_{n\to\infty} 1^n = 1$ (Fig. 19).

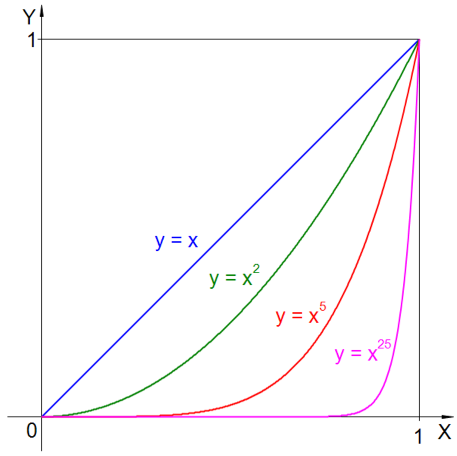

Fig. 19. Graphs $y = x^n$, $n = 1, 2, 5, 25$

Thus, the functional sequence $\{x^n\}$ converges on the set $[0, 1]$ to the limit function

$$f(x) = \begin{cases} 0, & x \in [0, 1), \\ 1, & x = 1. \end{cases}$$

Let us study the question of the uniform convergence of this sequence. To do this, we use the previously proven criterion and find the limit of the expression $\sup_{x\in E} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |x^n - f(x)|$. At the point $x = 1$, the value of difference $|x^n - f(x)|$ is 0 since $f(1) = 1$. Therefore, it suffices to find the value of the supremum on the half-interval $[0, 1)$, where $f(x) = 0$. Thus, we get the following chain of equalities:

$$\sup_{x\in[0,1]} |x^n - f(x)| = \sup_{x\in[0,1)} |x^n - 0| = \sup_{x\in[0,1)} x^n.$$

For any $n \in \mathbb{N}$, the function x^n has the least upper bound 1 on the half-interval $[0, 1)$, although this value is not reached. This fact follows from the limit relation $\lim_{x\to 1} x^n = 1$, which means that the function x^n takes values arbitrarily close to 1 on the half-interval $[0, 1)$. So, we have proved that the following relation holds for any $n \in \mathbb{N}$:

$$\sup_{x\in[0,1]} |x^n - f(x)| = 1.$$

When passing to the limit as $n \to \infty$, this result will not change:

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |x^n - f(x)| = 1.$$

Thus, the limit is not equal to 0; therefore, by virtue of the criterion, the convergence of the sequence $\{x^n\}$ on the segment $[0, 1]$ is not uniform.

Remarks.

1. In what follows, we prove that if a sequence of functions continuous on the set E converges uniformly on this set to some function $f(x)$, then

the limit function $f(x)$ is also continuous. This fact immediately implies the absence of uniform convergence for the considered sequence of functions x^n continuous on the set $[0,1]$, since its limit function is not continuous (it has a discontinuity of the first kind at point 1).

2. If we consider the half-interval $E=[0,1)$, then in this case the sequence $\{x^n\}$ will approach the function $f(x)$, which is identically equal to 0. Thus, the limit function is continuous on the set E. Nevertheless, the convergence on the half-interval $[0,1)$ is not uniform either, since we have already established that $\sup_{x\in[0,1)}|x^n-f(x)|=\sup_{x\in[0,1)}x^n=1$ and therefore $\lim_{n\to\infty}\sup_{x\in[0,1)}|x^n-f(x)|=1\neq 0$.

3. If we consider the segment $[0,q]$ for $q<1$ as the set E, then the sequence $\{x^n\}$ will converge uniformly on this segment to the function $f(x)\equiv 0$. Indeed, in this case we have

$$\sup_{x\in[0,q])}|x^n-f(x)|=\sup_{x\in[0,q]}x^n=q^n.$$

Since $q^n\to 0$ as $n\to 0$, we obtain that condition (3) of the criterion is satisfied, so the convergence of the sequence $\{x^n\}$ is uniform. Convergence remains uniform even in the case of the half-interval $[0,q)$ for $q<1$.

Uniform convergence of a functional series and a criterion for uniform convergence of a series in terms of the supremum limit 3.12B/29:12 (05:20)

Definition.

It is said that the functional series $\sum_{k=1}^{\infty}u_k(x)$ *converges uniformly* on the set E to the function $S(x)$ (notation $\sum_{k=1}^{\infty}u_k(x)\overset{E}{\rightrightarrows}S(x)$) if the sequence of partial sums $\left\{\sum_{k=1}^{n}u_k(x)\right\}$ converges uniformly to $S(x)$: $\sum_{k=1}^{n}u_k(x)\overset{E}{\rightrightarrows}S(x)$, $n\to\infty$.

In the language ε–N, the definition of uniform convergence of a functional series is as follows. The series $\sum_{k=1}^{\infty}u_k(x)$ converges uniformly on the set E to the function $S(x)$ if the condition holds:

$$\forall\,\varepsilon>0\quad\exists\,N\in\mathbb{N}\quad\forall\,n>N\quad\forall\,x\in E\quad\left|\sum_{k=1}^{n}u_k(x)-S(x)\right|<\varepsilon.$$

Using the previously proved criterion for uniform convergence of a functional sequence, we can immediately obtain a similar criterion for uniform convergence of a functional series.

Theorem (criterion for uniform convergence of a functional series in terms of the supremum limit).

The functional series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on the set E to the function $S(x)$ if and only if the following limit relation holds:

$$\lim_{n\to\infty} \sup_{x\in E} \left| \sum_{k=1}^{n} u_k(x) - S(x) \right| = 0.$$

Cauchy criterion for the uniform convergence of a functional sequence and a functional series

Formulation of the Cauchy criterion for uniform convergence of a functional sequence

3.12B/34:32 (04:54)

Theorem (Cauchy criterion for uniform convergence of a functional sequence).

The functional sequence $\{f_n(x)\}$ converges uniformly on the set E if and only if the following condition is satisfied:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, p \in \mathbb{N} \quad \forall\, x \in E$$
$$|f_m(x) - f_{m+p}(x)| < \varepsilon. \tag{6}$$

Proof of the Cauchy criterion for uniform convergence of a functional sequence

3.13A/00:00 (10:53)

1. Necessity. Given: the functional sequence $\{f_n(x)\}$ converges uniformly on the set E. Prove: condition (6) is satisfied.

Let the sequence $\{f_n(x)\}$ converge uniformly to the function $f(x)$. We rewrite the definition of uniform convergence (2) with a slight change in the last inequality (note that the same version of condition (2) was also used to prove the necessity for the criterion for uniform convergence in terms of the supremum limit):

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad \forall\, x \in E \quad |f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

Then, for any $m > N$, $p \in \mathbb{N}$, $x \in E$, we have

$$|f_m(x) - f(x)| < \frac{\varepsilon}{2},$$
$$|f_{m+p}(x) - f(x)| < \frac{\varepsilon}{2}.$$

Let us transform the difference $|f_m(x) - f_{m+p}(x)|$ using the last two estimates:

$$|f_m(x) - f_{m+p}(x)| = |f_m(x) - f(x) + f(x) - f_{m+p}(x)| \leq$$
$$\leq |f_m(x) - f(x)| + |f(x) - f_{m+p}(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, condition (6) is satisfied and the necessity is proven.

2. Sufficiency. Given: condition (6) is satisfied. Prove: the functional sequence $\{f_n(x)\}$ converges uniformly on the set E.

First, we prove that the sequence $\{f_n(x)\}$ converges pointwise on the set E. We choose some value $x_0 \in E$. From condition (6), we obtain:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, p \in \mathbb{N} \quad |f_m(x_0) - f_{m+p}(x_0)| < \varepsilon$$

This condition coincides with the Cauchy criterion condition for the convergence of the numerical sequence $\{f_n(x_0)\}$. It follows from this criterion that the numerical sequence $\{f_n(x_0)\}$ has a limit; denote it by $f(x_0)$. Since the choice of $x_0 \in E$ is arbitrary, we obtain that the functional sequence $\{f_n(x_0)\}$ converges on the set E to some function $f(x)$. It remains to prove that the convergence to the function $f(x)$ is uniform.

We rewrite condition (6) with a slight change in the last inequality:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, p \in \mathbb{N} \quad \forall\, x \in E$$
$$|f_m(x) - f_{m+p}(x)| < \frac{\varepsilon}{2}. \tag{7}$$

Since this condition is satisfied for any $p \in \mathbb{N}$, we can pass to the limit as $p \to \infty$. Taking into account the pointwise convergence of the sequence $\{f_n(x)\}$ already proved, we get that $f_{m+p}(x) \to f(x)$. Therefore, the inequality $|f_m(x) - f_{m+p}(x)| < \frac{\varepsilon}{2}$, by virtue of the theorem on passing to the limit in inequalities, takes the following form:

$$|f_m(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus, as a result of passing to the limit as $p \to \infty$, condition (7) is transformed as follows:

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, x \in E \quad |f_m(x) - f(x)| < \varepsilon.$$

We obtained a version of condition (2) from the definition of uniform convergence (its only difference from (2) is the use of the symbol m instead of the symbol n). Therefore, the sequence $\{f_n(x)\}$ converges uniformly on E to $f(x)$. $\square$

Cauchy criterion for uniform convergence of a functional series

3.13A/10:53 (03:41)

Corollary (Cauchy criterion for uniform convergence of a functional series).

The functional series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on the set E if and only if the following condition is satisfied:

$$\forall\,\varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, p \in \mathbb{N} \quad \forall\, x \in E$$

$$\left| \sum_{k=m+1}^{m+p} u_k(x) \right| < \varepsilon. \tag{8}$$

Proof.

Consider the sequence of partial sums of the initial series: $\{S_n(x)\} = \left\{\sum_{k=1}^{n} u_k(x)\right\}$. The difference $|S_{m+p}(x) - S_m(x)|$ for this sequence can be written as follows:

$$|S_{m+p}(x) - S_m(x)| = \left| \sum_{k=1}^{m+p} u_k(x) - \sum_{k=1}^{m} u_k(x) \right| = \left| \sum_{k=m+1}^{m+p} u_k(x) \right|.$$

Thus, condition (6) of the Cauchy criterion for uniform convergence of the functional sequence $\{S_n(x)\}$ coincides with condition (8). Therefore, condition (8) holds if and only if the sequence $\{S_n(x)\}$ converges uniformly on E. It remains to note that the uniform convergence of the series $\sum_{k=1}^{\infty} u_k(x)$ is equivalent to the uniform convergence of the sequence of its partial sums $\{S_n(x)\}$. $\square$

Tests of uniform convergence of functional series

Weierstrass test

3.13A/14:34 (12:01)

Theorem (Weierstrass test for uniform convergence of a functional series).

Let the functional series $\sum_{k=1}^{\infty} u_k(x)$ be defined on the set E and the following condition holds for its terms:

$$\exists\, N' \in \mathbb{N} \quad \forall\, k > N' \quad \forall\, x \in E \quad |u_k(x)| \le a_k. \tag{9}$$

If the numerical series $\sum_{k=1}^{\infty} a_k$ converges, then the initial functional series $\sum_{k=1}^{\infty} u_k(x)$ converges absolutely and uniformly on E. In English, this test is also called the Weierstrass M-test.

Proof.

Let us write the condition for the Cauchy criterion for a convergent numerical series $\sum_{k=1}^{\infty} a_k$:

$$\forall\, \varepsilon > 0 \quad \exists\, N'' \in \mathbb{N} \quad \forall\, m > N'' \quad \forall\, p \in \mathbb{N} \quad \sum_{k=m+1}^{m+p} a_k < \varepsilon. \tag{10}$$

In the last inequality, we do not use the sign of the absolute value operation, since, by virtue of (9), all terms a_k are non-negative.

If we now put $N = \max\{N', N''\}$ and use condition (9), then, for any $m > N$, $p \in \mathbb{N}$, and $x \in E$, we get

$$\sum_{k=m+1}^{m+p} |u_k(x)| \le \sum_{k=m+1}^{m+p} a_k < \varepsilon.$$

So, we have obtained that the following condition follows from conditions (9) and (10):

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m > N \quad \forall\, p \in \mathbb{N} \quad \forall\, x \in E$$

$$\sum_{k=m+1}^{m+p} |u_k(x)| < \varepsilon.$$

By the Cauchy criterion for uniform convergence of a functional series, this condition is sufficient for the uniform convergence of the series $\sum_{k=1}^{\infty} |u_k(x)|$. We proved that the initial series $\sum_{k=1}^{\infty} u_k(x)$ converges absolutely on E.

It remains to note that the convergence of the initial series is uniform. Indeed, the condition of the Cauchy criterion of uniform convergence is also satisfied for the initial series, since the following estimate holds:

$$\Big|\sum_{k=m+1}^{m+p} u_k(x)\Big| \le \sum_{k=m+1}^{m+p} |u_k(x)| < \varepsilon. \ \square$$

Example.

Consider the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$. For its common term, the following estimate is true:

$$\forall\, k \in \mathbb{N} \quad \forall\, x \in \mathbb{R} \quad \Big|\frac{\sin kx}{k^2}\Big| \le \frac{1}{k^2}.$$

Since the numerical series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, we conclude, by the Weierstrass test, that the functional series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ converges absolutely and uniformly on the entire real axis $\mathbb{R}$.

Dirichlet's test and Abel's test

3.13A/26:35 (07:50)

Theorem (Dirichlet's test for uniform convergence of a functional series).

Let the functional series $\sum_{k=1}^{\infty} u_k(x)v_k(x)$ be defined on E and the following conditions are satisfied for it:

1) $\exists M \quad \forall n \in \mathbb{N} \quad \forall x \in E \quad \left|\sum_{k=1}^{n} u_k(x)\right| \le M$;

2) $v_k(x) \overset{E}{\rightrightarrows} 0$ as $k \to \infty$, the numerical sequence $\{v_k(x)\}$ is monotone for any $x \in E$.

Then the series $\sum_{k=1}^{\infty} u_k(x)v_k(x)$ converges uniformly on E (the convergence is, generally speaking, conditional).

Proof[4].

Introducing the auxiliary notation $U_n(x) = \sum_{k=1}^{n} u_k(x)$ and carrying out the same reasoning as at the beginning of the proof of Dirichlet's test for the convergence of a numerical series, we can obtain the following estimate, which holds for any $m, p \in \mathbb{N}$ and $x \in E$:

$$\left|\sum_{k=m+1}^{m+p} u_k(x)v_k(x)\right| \le 2M(|v_{m+1}(x)| + |v_{m+p}(x)|).$$

Now we use the condition that the sequence $\{v_k(x)\}$ converges uniformly on the set E to 0, and write this condition in the following form:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N \quad \forall p \in \mathbb{N} \quad \forall x \in E \quad |v_{m+p}(x)| < \frac{\varepsilon}{4M}.$$

For $\left|\sum_{k=m+1}^{m+p} u_k(x)v_k(x)\right|$, we finally get

$$\left|\sum_{k=m+1}^{m+p} u_k(x)v_k(x)\right| \le 2M(|v_{m+1}(x)| + |v_{m+p}(x)|) <$$
$$< 2M\left(\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\right) = \varepsilon.$$

Thus, we have proved that the condition for uniform convergence on E from the Cauchy criterion is satisfied for the initial series:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N \quad \forall p \in \mathbb{N} \quad \forall x \in E$$
$$\left|\sum_{k=m+1}^{m+p} u_k(x)v_k(x)\right| < \varepsilon. \ \square$$

We also give the formulation of Abel's test for the uniform convergence of a functional series (for the proof, see, for example, [18, Ch. 9, Sec. 42.4]).

[4]There is no proof of this theorem in video lectures.

Theorem (Abel's test for uniform convergence of a functional series).

Let the functional series $\sum_{k=1}^{\infty} u_k(x)v_k(x)$ be defined on E and the following conditions are satisfied for it:

1) the series $\sum_{k=1}^{n} u_k(x)$ converges uniformly on E;

2) the sequence $\{v_k(x)\}$ is monotone for any $x \in E$ and is also uniformly bounded:

$$\exists\, M > 0 \quad \forall\, k \in \mathbb{N} \quad \forall\, x \in E \quad |v_k(x)| \leq M.$$

Then the series $\sum_{k=1}^{\infty} u_k(x)v_k(x)$ converges uniformly on E (the convergence is, generally speaking, conditional).

16. Properties of uniformly converging sequences and series

Continuity of the uniform limit

Formulation of the theorem on the continuity of the uniform limit

3.13A/34:25 (09:02)

Theorem (on the continuity of the uniform limit of a functional sequence with continuous elements).

Let the functional sequence $\{f_n(x)\}$ converge uniformly on the segment $[a, b]$ to the function $f(x)$. Suppose that the function $f_n(x)$ is continuous on $[a, b]$ for any $n \in \mathbb{N}$. Then the limit function $f(x)$ is also continuous on $[a, b]$.

Remarks.

1. If the convergence is not uniform, then the statement of the theorem may not hold. Earlier we gave an example of a sequence of continuous functions $\{x^n\}$. This sequence is not uniformly convergent on the segment $[0, 1]$, but converges pointwise on this segment. The limit of this (not uniformly converging) sequence is a function that has a discontinuity of the first kind at point 1.

2. This theorem means that, for a uniformly converging sequence of continuous functions, we can swap two limit operations: as $n \to \infty$ and as $x \to x_0$, where x_0 is some point of the segment $[a, b]$. Indeed, due to the continuity of the functions $f_n(x)$ and the limit function $f(x)$, we have $\lim_{x \to x_0} f_n(x) = f_n(x_0)$, $\lim_{x \to x_0} f(x) = f(x_0)$; due to the convergence of the functional sequence, we have $\lim_{n \to \infty} f_n(x) = f(x)$, therefore the following chain of equalities holds:

$$\lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = \lim_{n \to \infty} f_n(x_0) = f(x_0) = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} \lim_{n \to \infty} f_n(x).$$

We emphasize that the indicated inversion of the limits can be performed only if the initial sequence of continuous functions is uniformly convergent, since only uniform convergence ensures the continuity of the limit function.

Proof of the theorem on the continuity of the uniform limit

3.13B/00:00 (13:55)

It is enough for us to prove that the limit function $f(x)$ is continuous at some arbitrarily chosen point $x_0 \in [a, b]$.

Let us give the definition of the function $f(x)$ continuous at the point x_0 in the language ε–δ:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in [a, b], |x - x_0| < \delta, \quad |f(x) - f(x_0)| < \varepsilon. \tag{1}$$

Now we write down the conditions that the initial sequence $\{f_n(x)\}$ satisfies. Firstly, this sequence converges uniformly on the set $[a, b]$ to the function $f(x)$. We write this fact in the language ε–N for the previously selected value ε:

$$\exists\, N \in \mathbb{N} \quad \forall\, n > N \quad \forall\, x \in [a, b] \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3}. \tag{2}$$

We choose some value $n_0 > N$. For this value n_0 and for any $x \in [a, b]$, by virtue of (2), the estimate holds:

$$|f_{n_0}(x) - f(x)| < \frac{\varepsilon}{3}. \tag{3}$$

Since the previously selected point x_0 also belongs to the segment $[a, b]$, a similar estimate is fulfilled for it:

$$|f_{n_0}(x_0) - f(x_0)| < \frac{\varepsilon}{3}. \tag{4}$$

Secondly, by condition, the functions f_n, $n \in \mathbb{N}$, are continuous at the point x_0. Therefore, the function f_{n_0} is also continuous at the point x_0. We write this fact in the language ε–δ for the previously selected value ε:

$$\exists\, \delta > 0 \quad \forall\, x \in [a, b], |x - x_0| < \delta, \quad |f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}. \tag{5}$$

So, we got that, for the chosen value $\varepsilon > 0$, there exists a value $\delta > 0$ (specified in condition (5)) such that estimates (3), (4), and (5) are simultaneously fulfilled for all $x \in [a, b]$, $|x - x_0| < \delta$.

We transform the expression $|f(x) - f(x_0)|$ using these three estimates:

$$\begin{aligned} &|f(x) - f(x_0)| = \\ &\quad = |f(x) - f_{n_0}(x) + f_{n_0}(x) - f_{n_0}(x_0) + f_{n_0}(x_0) - f(x_0)| \le \\ &\quad \le |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| < \\ &\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, we have shown that condition (1) is satisfied. Therefore, the limit function f is continuous at an arbitrary point x_0 of the segment $[a, b]$. □

Corollary for functional series

3.13B/13:55 (09:55)

Corollary (on the continuity of the sum of a uniformly converging functional series with continuous terms).

Let the series $\sum_{k=1}^{\infty} u_k(x)$ uniformly converge on the segment $[a, b]$ to the function $S(x)$. Let all terms of the series $u_k(x)$ be continuous functions on the segment $[a, b]$. Then the sum of the series $S(x)$ is also a continuous function on this segment.

Remark.

This corollary can also be interpreted in terms of a swap of two limit operations. Namely, we choose the point $x_0 \in [a, b]$ and consider the following limit: $\lim_{x\to x_0} \sum_{k=1}^{\infty} u_k(x)$. This limit is understood as the limit of the sum of a converging series: $\lim_{x\to x_0} S(x)$.

Since, according to the corollary, the sum $S(x)$ is a continuous function on $[a, b]$, the last limit is equal to the value of the function S at the point x_0. But the value of $S(x_0)$ is the sum of the series $\sum_{k=1}^{\infty} u_k(x_0)$. If we now use the continuity of the functions u_k, the last series can be written in the form $\sum_{k=1}^{\infty} \lim_{x\to x_0} u_k(x)$.

Removing the intermediate transformations, we obtain the following relation:

$$\lim_{x\to x_0} \sum_{k=1}^{\infty} u_k(x) = \sum_{k=1}^{\infty} \lim_{x\to x_0} u_k(x). \tag{6}$$

This relation means that in the case of a uniformly converging series with continuous terms, the limit as $x \to x_0$ can be moved under the infinite sum sign or taken out of it. This makes it possible in some cases to more simply calculate the limit of the sum of a uniformly converging series: instead of finding an expression for the sum of the initial functional series and passing to the limit in this expression (i. e., performing the actions in the left-hand side of (6)), it often turns out to be easier to find the limit for the terms of this series and then find the sum of the resulting numerical series (i. e., perform the actions in the right-hand side of (6)).

Note also that a special justification of reversing the limit sign and the summation sign is required only in the case of infinite sums. If we replace the series with a finite sum of the form $\sum_{k=1}^{n} u_k(x)$ in relation (6), then the validity of this relation will immediately follow from the fact that the finite sum of continuous functions is a continuous function.

Proof.

By definition, the function $S(x)$ is the limit of a sequence of partial sums $S_n(x) = \sum_{k=1}^{n} u_k(x)$. The uniform convergence of the series means that the sequence $\{S_n(x)\}$ converges to $S(x)$ uniformly on $[a, b]$.

In addition, the partial sum $S_n(x)$ is a continuous function as a finite sum of continuous functions $u_k(x)$.

So, all the conditions of the previous theorem are satisfied for the sequence $\{S_n(x)\}$: it converges uniformly on $[a, b]$ to $S(x)$ and its elements are continuous functions on this segment. Applying this theorem, we obtain that the limit function $S(x)$ is also continuous on $[a, b]$. $\square$

Integration of functional sequences and series

Formulation of the theorem on the integration of a functional sequence

3.13B/23:50 (03:57)

Theorem (on the integration of a uniformly converging functional sequence).

Let the functional sequence $\{f_n(x)\}$ converge uniformly on the segment $[a, b]$ to the function $f(x)$. Let the function $f_n(x)$ be continuous on $[a, b]$ for any $n \in \mathbb{N}$. Let $x_0 \in [a, b]$.

Then the functional sequence $\left\{ \int_{x_0}^{x} f_n(t)\, dt \right\}$ is defined on the segment $[a, b]$ and converges uniformly on this segment to the function $\int_{x_0}^{x} f(t)\, dt$:

$$\int_{x_0}^{x} f_n(t)\, dt \overset{[a,b]}{\rightrightarrows} \int_{x_0}^{x} f(t)\, dt, \quad n \to \infty. \tag{7}$$

Remarks.

1. All the integrals mentioned in the theorem exist, since the integrands are continuous: the functions $f_n(x)$ are continuous by condition, the limit function $f(x)$ is continuous by virtue of the previous theorem.

2. The result of the theorem can be reformulated in terms of reversing the limit operation and the integration operation:

$$\lim_{n\to\infty} \int_{x_0}^{x} f_n(t)\, dt = \int_{x_0}^{x} f(t)\, dt = \int_{x_0}^{x} \lim_{n\to\infty} f_n(t)\, dt.$$

Proof of the theorem on the integration of a functional sequence

3.13B/27:47 (06:14)

Let us write the condition for uniform convergence of the initial sequence $\{f_n(x)\}$ in the language ε–N:

$$\forall\,\varepsilon > 0 \quad \exists\,N \in \mathbb{N} \quad \forall\,n > N \quad \forall\,x \in [a,b]$$
$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}. \tag{8}$$

For definiteness, we assume that the estimate $x_0 < x$ holds.

Consider the difference $\left|\int_{x_0}^{x} f_n(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right|$ and transform it as follows:

$$\left|\int_{x_0}^{x} f_n(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right| = \left|\int_{x_0}^{x} \bigl(f_n(t) - f(t)\bigr)\,dt\right| \le$$
$$\le \int_{x_0}^{x} |f_n(t) - f(t)|\,dt.$$

We use estimate (8), which is valid for any t, since $t \in [x_0, x] \subset [a,b]$, and the corollary of the theorem on integration of a positive continuous function:

$$\int_{x_0}^{x} |f_n(t) - f(t)|\,dt < \frac{\varepsilon}{b-a}\cdot(x - x_0) \le \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon.$$

So, we have proved that the following condition is satisfied for the sequence $\left\{\int_{x_0}^{x} f_n(t)\,dt\right\}$:

$$\forall\,\varepsilon > 0 \quad \exists\,N \in \mathbb{N} \quad \forall\,n > N \quad \forall\,x \in [a,b]$$
$$\left|\int_{x_0}^{x} f_n(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right| < \varepsilon.$$

This means that the sequence $\left\{\int_{x_0}^{x} f_n(t)\,dt\right\}$ converges uniformly on $[a,b]$ and the limit relation (7) holds. □

Formulation of the corollary on the integration of a functional series

3.14A/00:00 (08:28)

Corollary (on the integration of a uniformly converging functional series).

Let the series $\sum_{k=1}^{\infty} u_k(x)$ converge uniformly on the segment $[a,b]$ to the function $S(x)$. Let all terms of the series $u_k(x)$ be continuous functions on the segment $[a,b]$. Then the series $\sum_{k=1}^{\infty} \int_{x_0}^{x} u_k(t)\,dt$ converges uniformly on $[a,b]$ to the integral $\int_{x_0}^{x} S(t)\,dt$ for any point $x_0 \in [a,b]$.

Remarks.

1. All the integrals mentioned in the corollary exist because the integrands are continuous: the functions $u_k(x)$ are continuous by condition, the limit function $S(x)$ is continuous by virtue of the corollary of the theorem on the continuity of the uniform limit.

2. The result of the corollary can be reformulated in terms of reversing the operation of infinite summation and the integration operation. If we start with the integral of the sum of the series $\int_{x_0}^{x}\bigl(\sum_{k=1}^{\infty} u_k(t)\bigr)\,dt$ and take into account that the sum of the series is a function $S(t)$, then this integral will take the form $\int_{x_0}^{x} S(t)\,dt$. But by virtue of the corollary, it is equal to the sum of the series $\sum_{k=1}^{\infty}\int_{x_0}^{x} u_k(t)\,dt$. Thus, we obtain the following equality:

$$\int_{x_0}^{x}\Bigl(\sum_{k=1}^{\infty} u_k(t)\Bigr)\,dt = \sum_{k=1}^{\infty}\int_{x_0}^{x} u_k(t)\,dt. \tag{9}$$

This equality means that the sign of the integral can be taken out of the sign of an infinite sum (or moved under its sign) if the series converges uniformly. Note that in the case of a finite sum, this property immediately follows from the additivity property of a definite integral with respect to the integrand.

Relation (9) allows, in some cases, to simplify finding the integral of the sum of the series (the left-hand side of equality (9)) if it is easier to integrate the terms of the initial series at first and then find the sum of the resulting series. Sometimes relation (9) simplifies finding the sum of a series consisting of integrals (the right-hand side of (9)) if it is easier to find the sum of a series containing integrands, and then integrate the found sum.

Proof of the corollary on the integration of a functional series

3.14A/08:28 (04:38)

By definition, the function $S(x)$ is the limit of a sequence of partial sums $S_n(x) = \sum_{k=1}^{n} u_k(x)$. The uniform convergence of the series means that the sequence $\{S_n(x)\}$ converges to $S(x)$ uniformly on $[a,b]$. In addition, the partial sums of $S_n(x)$ are continuous on $[a,b]$ as finite sums of continuous functions and, by the corollary of the theorem on the continuity of the uniform limit, the function $S(x)$ is also continuous.

Then the sequence of partial sums $\sigma_n(x) = \sum_{k=1}^{n}\int_{x_0}^{x} u_k(t)\,dt$ can be transformed as follows, using the additivity property of a definite integral with respect to the integrand:

$$\sigma_n(x) = \sum_{k=1}^{n}\int_{x_0}^{x} u_k(t)\,dt = \int_{x_0}^{x}\Bigl(\sum_{k=1}^{n} u_k(t)\Bigr)\,dt = \int_{x_0}^{x} S_n(t)\,dt. \tag{10}$$

Since the sequence $\{S_n(x)\}$ satisfies all the conditions of the theorem on integrating a uniformly converging functional sequence, we obtain that the sequence $\Bigl\{\int_{x_0}^{x} S_n(t)\,dt\Bigr\}$ is also uniformly convergent on $[a,b]$ and its limit is equal to the function $\int_{x_0}^{x} S(t)\,dt$. Then the same is true for the sequence $\{\sigma_n(x)\}$ from the left-hand side of equality (10).

Thus, we have proved that the sequence of partial sums $\sigma_n(x) = \sum_{k=1}^{n} \int_{x_0}^{x} u_k(t)\,dt$ converges uniformly on $[a, b]$ to the function $\int_{x_0}^{x} S(t)\,dt$. By definition, this means that the series $\sum_{k=1}^{\infty} \int_{x_0}^{x} u_k(t)\,dt$ converges uniformly on $[a, b]$ to the same function. □

Differentiation of functional sequences and series

Formulation of the theorem on the differentiation of a functional sequence

3.14A/13:06 (06:39)

THEOREM (ON DIFFERENTIATION OF A FUNCTIONAL SEQUENCE).

Let the functional sequence $\{f_n(x)\}$ contain continuously differentiable functions on $[a, b]$.

Let the sequence of derivatives of these functions $\{f_n'(x)\}$ converge uniformly on $[a, b]$ to some function $\varphi(x)$.

Suppose, in addition, that there exists a point $x_0 \in [a, b]$ such that the numerical sequence $\{f_n(x_0)\}$ converges to some number f_0.

Then the functional sequence $\{f_n(x)\}$ converges uniformly on the segment $[a, b]$ to the function $f(x)$, the function $f(x)$ is continuously differentiable on $[a, b]$, and the equality holds:

$$f'(x) = \varphi(x). \tag{11}$$

REMARK.

The result of the theorem can be reformulated in terms of reversing the limit operation and the differentiation operation:

$$\Bigl(\lim_{n\to\infty} f_n(x)\Bigr)' = f'(x) = \varphi(x) = \lim_{n\to\infty} f_n'(x).$$

Such a transformation is valid only if the sequence $\{f_n'(x)\}$ is uniformly convergent.

Proof of the theorem on the differentiation of a functional sequence

3.14A/19:45 (07:29)

Since, by condition, the sequence $\{f_n'(x)\}$ converges uniformly to the function $\varphi(x)$ and the functions $f_n'(x)$ are continuous, we can apply the theorem on the integration of a uniformly converging functional sequence and obtain, as a result, the following limit relation:

$$\int_{x_0}^{x} f_n'(t)\,dt \overset{[a,b]}{\rightrightarrows} \int_{x_0}^{x} \varphi(t)\,dt, \quad n \to \infty. \tag{12}$$

Here the point x_0 is chosen so that the sequence $\{f_n(x_0)\}$ is convergent (this point exists by condition).

Using the Newton–Leibniz formula, the left-hand side of the limit relation (12) can be transformed as follows:

$$\int_{x_0}^{x} f_n'(t)\,dt = f_n(x) - f_n(x_0).$$

Thus, the limit relation (12) can be rewritten in the form

$$\lim_{n\to\infty} \bigl(f_n(x) - f_n(x_0)\bigr) = \int_{x_0}^{x} \varphi(t)\,dt. \tag{13}$$

Moreover, the convergence is uniform on $[a, b]$.

We represent the left-hand side of equality (13) in the form of a difference of limits:

$$\lim_{n\to\infty} \bigl(f_n(x) - f_n(x_0)\bigr) = \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(x_0). \tag{14}$$

Since the limit on the left-hand side of (14) exists (and is equal to $\int_{x_0}^{x} \varphi(t)\,dt$) and, by condition, the second limit on the right-hand side of (14) also exists (and is equal to f_0), we conclude, due to the arithmetic properties of the limits, that the first limit on the right-hand side also exists and the equality holds:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \bigl(f_n(x) - f_n(x_0)\bigr) + \lim_{n\to\infty} f_n(x_0) = \int_{x_0}^{x} \varphi(t)\,dt + f_0.$$

In addition, since the sequence $\{f_n(x) - f_n(x_0)\}$ converges uniformly on $[a, b]$ and the sequence $\{f_n(x_0)\}$ is a numerical sequence, we obtain that the sequence $\{f_n(x)\}$ also converges uniformly on $[a, b]$.

So, we have proved the validity of the following limit relation:

$$f_n(x) \overset{[a,b]}{\rightrightarrows} \int_{x_0}^{x} \varphi(t)\,dt + f_0, \quad n \to \infty.$$

It remains for us to prove that the limit function $f(x) = \int_{x_0}^{x} \varphi(t)\,dt + f_0$ is continuously differentiable on $[a, b]$ and equality (11) holds for it. These facts follow from the properties of an integral with a variable upper limit. Indeed, the function $\varphi(x)$ is continuous on $[a, b]$ as the uniform limit of the sequence of continuous functions $f_n'(x)$, therefore, the integral with a variable upper limit of the function $\varphi(x)$ is a continuously differentiable function and its derivative is equal to the integrand $\varphi(x)$:

$$f'(x) = \Big(\int_{x_0}^{x} \varphi(t)\,dt + f_0\Big)' = \varphi(x) + 0 = \varphi(x). \ \square$$

Formulation of the corollary on the differentiation of a functional series

3.14A/27:14 (04:20)

Corollary (on the differentiation of a functional series).

Let all the terms of the functional series $\sum_{k=1}^{\infty} u_k(x)$ be continuously differentiable functions on the segment $[a, b]$.

Let the series $\sum_{k=1}^{\infty} u_k'(x)$ converge uniformly on $[a, b]$ to the function $\sigma(x)$.

Suppose that there exists a point $x_0 \in [a, b]$ such that the numerical series $\sum_{k=1}^{\infty} u_k(x_0)$ converges to some number S_0.

Then the series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ to the function $S(x)$, the sum $S(x)$ is a continuously differentiable function, and the equality holds:

$$S'(x) = \sigma(x). \tag{15}$$

Remark.

The result of the corollary can be reformulated in terms of reversing the operation of infinite summation and the differentiation operation. If we start with a series $\sum_{k=1}^{\infty} u_k'(t)$ containing derivatives and take into account that the sum of this series is a function $\sigma(x)$, which, by virtue of (15) is equal to $S'(x)$, i. e., the derivative of the sum of the series $\sum_{k=1}^{\infty} u_k(t)$, then, omitting the intermediate transformations, we obtain the following equality:

$$\sum_{k=1}^{\infty} u_k'(t) = \Big(\sum_{k=1}^{\infty} u_k(t)\Big)'. \tag{16}$$

This equality means that the sign of the derivative can be taken out of the sign of an infinite sum (or moved under its sign) if the series containing the derivatives converges uniformly. Note that in the case of a finite sum, this property immediately follows from the arithmetic properties of derivatives.

Relation (16) allows, in some cases, to simplify finding the sum of a series containing derivatives (the left-hand side of equality (16)) if it is easier to find the sum of a series containing the initial functions, and then differentiate this sum. Sometimes this relation simplifies the differentiation of the sum of a series (the right-hand side of (16)) if it is easier to differentiate the terms of the initial series at first and then find the sum of the series containing derivatives.

Proof of the corollary on the differentiation of a functional series

3.14A/31:34 (06:26)

By the definition of convergence of a series, the function $\sigma(x)$ is the limit of a sequence of partial sums $\sigma_n(x) = \sum_{k=1}^{n} u_k'(x)$. The uniform convergence of the series means that the sequence $\{\sigma_n(x)\}$ converges to $\sigma(x)$ uniformly on $[a, b]$. In addition, the partial sums $\sigma_n(x)$ are continuous on $[a, b]$ as finite sums of continuous functions.

If we additionally define a sequence of partial sums for the initial series $S_n(x) = \sum_{k=1}^{n} u_k(x)$, then, firstly, these partial sums will be continuously differentiable functions on $[a, b]$ as finite sums of continuously differentiable functions, secondly, the relation $S_n'(x) = \sigma_n(x)$ will be satisfied, by the arithmetic properties of derivatives, and thirdly, for a point $x_0 \in [a, b]$, the numerical sequence $\{S_n(x_0)\}$ will converge to S_0 by condition.

Thus, all the conditions of the theorem on the differentiation of functional sequences are satisfied for the sequence $\{S_n(x)\}$ (in particular, the sequence $\{S_n'(x)\}$ converges uniformly to $\sigma(x)$). Therefore, the sequence $\{S_n(x)\}$ converges uniformly on $[a, b]$ to a continuously differentiable function $S(x)$ and relation (15) holds for this function. By the definition of a uniformly converging functional series, we obtain that the initial series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ to the function $S(x)$. $\square$

17. Power series

Power series: definition and Abel's theorems on its convergence

Definition of a power series

3.14A/38:00 (03:19)

Definition.

A *power series* is a functional series of the form

$$\sum_{k=0}^{\infty} c_k (x - x_0)^k. \tag{1}$$

The point $x_0 \in \mathbb{R}$ is called the *center* of the series (1), the numbers $c_k \in \mathbb{R}$ for $k = 0, 1, 2, \ldots$ are called the *coefficients* of the series (1), and x is a variable.

Thus, a power series is a functional series whose terms are power functions. The partial sums $S_n(x)$ of the power series are polynomials of formal degree n:

$$S_n(x) = \sum_{k=0}^{n} c_k (x - x_0)^k.$$

By changing the variables $t = x - x_0$ (i. e., by performing a shift by x_0), we can transform series (1) to the following form (with the original variable name x):

$$\sum_{k=0}^{\infty} c_k x^k.$$

The properties of a series centered at point 0 and a series of general form (1) are similar, so, in what follows, we will mainly consider power series centered at zero.

Formulation of the first Abel theorem

3.14B/00:00 (04:18)

Theorem (the first Abel theorem on the convergence of a power series).

If the power series $\sum_{k=0}^{\infty} c_k x^k$ converges at $x = x_0$, then it converges absolutely on the interval $(-|x_0|, |x_0|)$ and converges uniformly on any segment $[-|x_0| + \delta, |x_0| - \delta]$ for $\delta > 0$.

Proof of the first Abel theorem

3.14B/04:18 (12:36)

Since the series $\sum_{k=0}^{\infty} c_k x_0^k$ converges, we obtain, by the necessary condition for the convergence of a numerical series, that $c_k x_0^k \to 0$ as $k \to \infty$.

Since the sequence $\{c_k x_0^k\}$ converges, it is bounded:

$$\exists\, M > 0 \quad \forall\, k \in \mathbb{N} \quad |c_k x_0^k| \le M. \tag{2}$$

Using estimate (2), let us transform the expression $|c_k x^k|$ as follows:

$$|c_k x^k| = \left|c_k\left(\frac{x}{x_0}\right)^k x_0^k\right| = |c_k x_0^k| \cdot \left|\frac{x}{x_0}\right|^k \le M\left|\frac{x}{x_0}\right|^k.$$

Denote $q = \left|\frac{x}{x_0}\right|$. We finally get the estimate

$$|c_k x^k| \le M q^k. \tag{3}$$

Let $|x| < |x_0|$. Then $q \in (0,1)$ and therefore the series $\sum_{k=0}^{\infty} M q^k$ is convergent. Then, by the comparison criterion for numerical series, the series $\sum_{k=0}^{\infty} |c_k x^k|$ also converges. So, we have proved that the initial series $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely at x when $|x| < |x_0|$.

Note that we cannot state that the series $\sum_{k=0}^{\infty} |c_k x^k|$ is uniformly convergent for $|x| < |x_0|$ by the Weierstrass convergence criterion, since the value of q in the expression Mq^k from (3) depends on x and therefore it cannot be argued that the terms $|c_k x^k|$ of the functional series are uniformly estimated by terms of a convergent numerical series. However, the required estimate can be obtained by reducing the initial interval.

Assume that $|x| \le |x_0| - \delta$. In this case, the expression $|c_k x^k|$ can be estimated from above as follows:

$$|c_k x^k| = |c_k x_0^k| \cdot \left|\frac{x}{x_0}\right|^k \le M\left|\frac{x_0 - \delta}{x_0}\right|^k.$$

Denote $q_0 = \left|\frac{x_0-\delta}{x_0}\right|$. This value does not depend on x and belongs to the interval $(0,1)$. Thus, the numerical series $\sum_{k=0}^{\infty} M q_0^k$ does not depend on x and converges, therefore, due to the Weierstrass convergence criterion, it follows from the obtained estimate $|c_k x^k| \le M q_0^k$ that the functional series $\sum_{k=0}^{\infty} |c_k x^k|$ converges uniformly for $|x| \le |x_0| - \delta$. $\square$

Formulation of the second Abel theorem

3.14B/16:54 (04:55)

Theorem (the second Abel theorem on the convergence of a power series).

For any power series $\sum_{k=0}^{\infty} c_k x^k$, there exists a value $R \in [0, +\infty) \cup \{+\infty\}$ such that the series converges absolutely on the interval $(-R, R)$ and diverges outside the segment $[-R, R]$.

Remarks.

1. The number R is called the *radius of convergence* of the power series $\sum_{k=0}^{\infty} c_k x^k$, the interval $(-R, R)$ is called its *interval of convergence*. In particular, if $R = 0$, then the series converges only for $x = 0$ and the interval of convergence reduces to a single point 0. If $R = +\infty$, then the series converges for any values of $x \in \mathbb{R}$ and the interval of convergence coincides with the entire real axis.

2. By virtue of the first Abel theorem, uniform convergence of a power series takes place on any segment $[a, b]$ embedded in the interval of convergence of this power series.

First stage of the proof 3.14B/21:49 (14:27)

We introduce the set of non-negative x such that the series $\sum_{k=0}^{\infty} c_k x^k$ converges at these points: $K = \left\{x \ge 0 : \sum_{k=0}^{\infty} c_k x^k \text{ converges}\right\}$. This set is nonempty, since it includes point 0.

If $K = \{0\}$, then the statement of the theorem holds for $R = 0$.

If $K \ne \{0\}$, then two cases are possible.

1. The set K is bounded from above:

$$\exists\, M > 0 \quad \forall\, x \in K \quad x \le M.$$

Then we take the least upper boundary of the set K as R: $R = \sup K > 0$.

Let $x \in (0, R)$. By definition of the least upper bound, we get

$$\forall\, \varepsilon > 0 \quad \exists\, x_0 \in K \quad x_0 > R - \varepsilon.$$

Let $\varepsilon = \frac{R-x}{2}$. For this value of ε, we get that there exists $x_0 \in K$ such that $x_0 > R - \frac{R-x}{2} = \frac{R+x}{2} > x$. Since $x_0 \in K$, the series converges at the point x_0 and we obtain, according to the first Abel theorem, that the series converges absolutely on the interval $(-x_0, x_0)$ and therefore at the point x because $0 < x < x_0$. Since the choice of $x \in (0, R)$ is arbitrary, we conclude that the series converges absolutely on the interval $(-R, R)$.

Now let $x > R$. If we assume that the series converges at the point x, then this will contradict the definition of the least upper boundary R, since, for any point $x' > 0$ at which the series converges, the estimate $x' \le R$ should be true. Therefore, the series diverges at any point $x > R$.

If we consider $x < -R$ and assume that the series converges at x, then, by the first Abel theorem, the series will converge absolutely at any point x' between x and $-R$ and therefore at any point $x'' = |x'| \in (R, |x|)$. But we have already established that the series cannot converge at the point $x'' > R$. This means that the series diverges at any point $x < -R$.

So, we have proved the theorem for the case when the set K is bounded from above.

Second stage of the proof

3.14B/36:16 (04:58)

2. It remains to consider the case when the set K is not bounded from above:

$$\forall M > 0 \quad \exists x_0 \in K \quad x_0 > M. \tag{4}$$

Let $x > 0$. Then, for this value of x, by virtue of (4), there exists a point $x_0 \in K$ such that the estimate $x_0 > x$ holds. Since $x_0 \in K$, the series converges at the point x_0. Therefore, by the first Abel theorem, the series converges absolutely at the points x' when $|x'| < x_0$ and therefore at the chosen point x, as well as at the point $-x$. Since the choice of the point x is arbitrary, we conclude that the series converges at all points of the real axis; therefore, we can take $+\infty$ as R. $\square$

Remark.

The second Abel theorem can be obviously generalized to the case of power series of the form $\sum_{k=0}^{n} c_k(x - x_0)^k$, where $x_0 \neq 0$. In this case, there also exists a radius of convergence R, and the interval of convergence is $(x_0 - R, x_0 + R)$. This interval of convergence reduces to a single point x_0 when $R = 0$ and coincides with the entire real axis when $R = +\infty$.

After the existence of the radius of convergence R of a power series is established, we can ask a natural question about how to find this radius. It turns out that there exists a formula that allows us to determine the value of the radius of convergence from the coefficients c_k of a given power series. However, in order to give this formula, we need to introduce additional concepts related to the sequence limit. The following section is devoted to the description of these concepts.

Limit inferior and limit superior of a sequence

Partial limits of a sequence

3.14B/41:14 (05:05)

From the previously considered properties of subsequences (the Bolzano–Weierstrass theorem and its corollary given in [1, Ch. 7]), it follows that any sequence contains a subsequence having a finite or infinite limit. Moreover, if the initial sequence converges, then all its subsequences converge to the same limit. If the sequence is not convergent but it is bounded, then a convergent subsequence can be extracted from it. Finally, if a sequence is unbounded, then a subsequence having an infinite limit can be extracted from it.

So, for any sequence, the set of its subsequences having a finite or infinite limit is nonempty. Let us define the set $K(\{x_n\})$ of limits of such subsequences for the sequence $\{x_n\}$:

$$K(\{x_n\}) = \Big\{x \in \mathbb{R} \cup \{+\infty, -\infty\} : \exists\ \{x_{n_k}\}\ \lim_{k\to\infty} x_{n_k} = x\Big\}.$$

For any sequence $\{x_n\}$, we have $K(\{x_n\}) \neq \varnothing$.

Elements of the set $K(\{x_n\})$ are called the *partial limits* of the sequence $\{x_n\}$.

If the sequence $\{x_n\}$ converges to the limit A, then the set $K(\{x_n\})$ consists of a single element A. For example, $K(\{\frac{1}{n}\}) = \{0\}$.

When studying sequences, we gave examples of sequences that have no limit: $a_n = (-1)^n$ and $b_n = n^{(-1)^n}$. The elements of the first sequence take alternating values -1 and 1, and the elements of the second sequence take values 1, 2, 1/3, 4, 1/5, 6, …, i. e., elements with odd indices approach 0 and elements with even indices approach $+\infty$. Thus, $K(\{a_n\}) = \{-1, 1\}$, $K(\{b_n\}) = \{0, +\infty\}$.

Note that one can construct examples of sequences $\{x_n\}$ such that the set $K(\{x_n\})$ contains an infinite number of elements.

Limit superior and limit inferior of a sequence

3.15A/00:00 (08:30)

Definition.

The *limit superior* of the sequence $\{x_n\}$ (the notation $\overline{\lim}_{n\to\infty} x_n$) is $\sup K(\{x_n\})$ if the set $K(\{x_n\})$ is bounded from above. If the set $K(\{x_n\})$ is not bounded from above (in particular, if it contains the element $+\infty$), then the limit superior is equal to $+\infty$.

The *limit inferior* of the sequence $\{x_n\}$ (the notation $\underline{\lim}_{n\to\infty} x_n$) is called $\inf K(\{x_n\})$ if the set $K(\{x_n\})$ is bounded from below. If the set $K(\{x_n\})$ is not bounded from below (in particular, if it contains the element $-\infty$), then the limit inferior is equal to $-\infty$.

Limit superior and limit inferior exist for all sequences and coincide if and only if the sequence has a finite limit or an infinite limit $+\infty$ or $-\infty$; in this case, the usual limit, limit superior, and limit inferior are the same.

Cauchy–Hadamard formula for the radius of convergence of a power series

Formulation of the Cauchy–Hadamard theorem

3.15A/08:30 (04:04)

Theorem (Cauchy–Hadamard theorem on the radius of convergence of a power series).

The radius R of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ can be found by the formula

$$R = \frac{1}{\alpha}, \quad \alpha = \varlimsup_{n\to\infty} \sqrt[n]{|c_n|}. \tag{5}$$

It is assumed that $R = 0$ if $\alpha = +\infty$ and $R = +\infty$ if $\alpha = 0$. Formula (5) is called the *Cauchy–Hadamard formula.*

Remark.

In formula (5), the coefficients c_n are considered starting from $n = 1$, since a root of degree zero is not defined.

Proof of the Cauchy–Hadamard theorem

3.15A/12:34 (13:25)

We give a proof for the special case when the sequence $\{|c_n|\}$ has a usual limit (finite or infinite). The proof for the general case is given, for example, in [4, Ch. 11, Sec. 380].

So, suppose the sequence $\{|c_n|\}$ has a limit α:

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = \alpha \in \mathbb{R} \cup \{+\infty\}. \tag{6}$$

Let us consider three cases.

1. $0 < \alpha < +\infty$. In this case, formula (5) does not require special interpretations.

We want to prove two facts: if $|x| < R$, then the initial series $\sum_{n=0}^{\infty} c_n x^n$ converges, if $|x| > R$, then the initial series diverges.

We choose some point x_0 and consider the numerical series $\sum_{n=1}^{\infty} |c_n x_0^n|$ denoting its common term by a_n: $a_n = |c_n x_0^n|$.

To study the convergence of this series, we use the limit Cauchy test. Recall its formulation: if there exists a limit $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ for the series $\sum_{n=1}^{\infty} a_n$, then the series converges if $q < 1$ and diverges if $q > 1$.

Let us find the value of q, provided that $a_n = |c_n x_0^n|$ and the limit relation (6) holds:

$$q = \lim_{n\to\infty} \sqrt[n]{|c_n x_0^n|} = \lim_{n\to\infty} |x_0| \sqrt[n]{|c_n|} = |x_0| \lim_{n\to\infty} \sqrt[n]{|c_n|} = |x_0|\,\alpha.$$

Thus, $q = |x_0|\,\alpha$. Therefore, if $|x_0| < R$, i. e., $|x_0| < \frac{1}{\alpha}$, then $q = |x_0|\,\alpha < 1$ and, by the limit Cauchy test, the series $\sum_{n=1}^{\infty} |c_n x_0^n|$ converges. If $|x_0| > R$, then $q = |x_0|\,\alpha > 1$ and, by the same test, the series $\sum_{n=1}^{\infty} |c_n x_0^n|$ diverges.

Thus, the value $R = \frac{1}{\alpha}$ is the radius of convergence of the series $\sum_{n=1}^{\infty} |c_n x_0^n|$, and the same result is true for the initial series $\sum_{n=0}^{\infty} c_n x_0^n$, since, according to the second Abel theorem, the power series converges absolutely on the convergence interval.

2. $\alpha = 0$. We show that in this case $R = +\infty$, i. e., the initial series converges at any point $x \in \mathbb{R}$.

We choose some point $x_0 \in \mathbb{R}$, consider the numerical series $\sum_{n=1}^{\infty} |c_n x_0^n|$, and find the value q for it:

$$q = \lim_{n\to\infty} \sqrt[n]{|c_n x_0^n|} = \lim_{n\to\infty} |x_0| \sqrt[n]{|c_n|} = |x_0| \lim_{n\to\infty} \sqrt[n]{|c_n|} = 0.$$

Thus, $q = 0 < 1$ and, by the limit Cauchy test, the series $\sum_{n=1}^{\infty} |c_n x_0^n|$ converges for any point $x_0 \in \mathbb{R}$. So, the initial series $\sum_{n=0}^{\infty} c_n x_0^n$ converges absolutely for any point $x_0 \in \mathbb{R}$ and therefore $R = +\infty$.

3. $\alpha = +\infty$. We show that in this case $R = 0$, i. e., the initial series diverges at any point $x \neq 0$.

We choose some point $x_0 \neq 0$, consider the numerical series $\sum_{n=1}^{\infty} |c_n x_0^n|$, and find the value q for it:

$$q = \lim_{n\to\infty} \sqrt[n]{|c_n x_0^n|} = \lim_{n\to\infty} |x_0| \sqrt[n]{|c_n|} = |x_0| \lim_{n\to\infty} \sqrt[n]{|c_n|} = +\infty.$$

Taking into account the remark on the limit Cauchy test in the case $q = +\infty$, we obtain that the series $\sum_{n=1}^{\infty} |c_n x_0^n|$ diverges for any point $x_0 \neq 0$. So, the initial series $\sum_{n=0}^{\infty} c_n x_0^n$ also diverges for any point $x_0 \neq 0$ and therefore $R = 0$. □

Examples of application of the Cauchy–Hadamard formula

3.15A/25:59 (09:37)

1. Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$. In this case, $c_n = \frac{1}{n}$. Since $\lim_{n\to n} \sqrt[n]{\frac{1}{n}} = 1$ (see the convergence theorem for the sequence $\{\sqrt[n]{n}\}$ in [1, Ch. 5]), we obtain that $\alpha = 1$, $R = 1$. Therefore, this series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

Let us analyze the convergence of this series for $|x| = 1$. In the case of $x = 1$, we get the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent. In the case of $x = -1$, we get a convergent alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Thus, the domain of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is the half-interval $[-1, 1)$.

2. Consider the power series $\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}$. In this case, the sequence $\{c_n\}$ is not convergent. Indeed, let us write out the initial terms of this series:

$$\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}=\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots$$

We obtain that the coefficient c_1 (i. e., the coefficient of the first power of x) is 0, the coefficient c_2 is $\frac{1}{2}$, the coefficient c_3 is 0, and so on. The formula for the coefficients c_n takes the form

$$c_n=\begin{cases}\dfrac{1}{n}, & n=2k,\\ 0, & n=2k-1, \quad k=1,2,\dots\end{cases}$$

Therefore, the sequence $\left\{\sqrt[n]{c_n}\right\}$ contains the following elements:

$$\{\sqrt[n]{c_n}\}=\left\{0,\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt[4]{4}},0,\frac{1}{\sqrt[6]{6}},\dots\right\}.$$

This sequence has no limit, since there exist an infinite number of elements of this sequence in any neighborhood of points 0 and 1. We can also say that the sequence $\left\{\sqrt[n]{c_n}\right\}$ has two partial limits: 0 and 1.

However, according to the general Cauchy–Hadamard formula, we can determine the radius of convergence of a power series if we find the limit superior of the sequence $\left\{\sqrt[n]{c_n}\right\}$, which always exists. In our case, the limit superior is 1; it is the limit of a subsequence containing elements with even indices:

$$\overline{\lim_{n\to\infty}}\sqrt[n]{c_n}=\lim_{k\to\infty}\frac{1}{\sqrt[2k]{2k}}=1.$$

Therefore, by the Cauchy–Hadamard theorem, the radius of convergence of the series $\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}$ is equal to 1.

Note that this series diverges at both endpoints of the convergence interval $(-1,1)$, since, for the value $x=1$ and for the value $x=-1$, we get the same series $\sum_{n=1}^{\infty}\frac{1}{2n}$, which differs from the harmonic series only by the factor $\frac{1}{2}$.

Remark.

The Cauchy–Hadamard formula can also be used to find the radius of convergence R of power series of the form $\sum_{k=0}^{n}c_k(x-x_0)^k$, where $x_0\neq 0$, since the radius of convergence is determined by the coefficients c_k only and does not depend on the center x_0.

Properties of power series

Continuity of the sum of a power series

3.15A/35:36 (08:26)

In all the theorems of this section, we consider the power series

$$\sum_{k=0}^{n} c_k x^k. \tag{7}$$

We assume that series (7) has a radius of convergence $R > 0$ (the case $R = +\infty$ is also allowed). We denote the sum of this series by $S(x)$. This sum is defined for all $x \in (-R, R)$.

REMARK.

All the results obtained in this section remain valid for power series of the form $\sum_{k=0}^{n} c_k (x - x_0)^k$, where $x_0 \neq 0$. Recall that the convergence interval has the form $(x_0 - R, x_0 + R)$ for such series, where the radius of convergence R can be found by the Cauchy–Hadamard formula.

THEOREM 1 (ON THE CONTINUITY OF THE SUM OF A POWER SERIES).

The function $S(x)$ is continuous on the convergence interval $(-R, R)$.

PROOF.

Let $x_0 \in (-R, R)$. Choose a segment $[-R + \varepsilon, R - \varepsilon]$ containing the point x_0 and nested in the convergence interval $(-R, R)$:

$$x_0 \in [-R + \varepsilon, R - \varepsilon] \subset (-R, R).$$

As ε, we can take $\frac{R - |x_0|}{2}$. In the case $R = +\infty$, we can consider an arbitrary segment of the real axis containing the point x_0.

Using the first and second Abel theorems, we obtain that series (7) converges uniformly on the segment $[-R + \varepsilon, R - \varepsilon]$. In addition, the general term of the series (7) has the form $c_k x^k$ and therefore is a continuous function. These two facts imply, by virtue of the theorem on the continuity of the uniform limit, that the limit function $S(x)$ is continuous on the segment $[-R + \varepsilon, R - \varepsilon]$; therefore, it is continuous at the point x_0 belonging to this segment.

Since the point $x_0 \in (-R, R)$ was chosen arbitrarily, we obtain that the function $S(x)$ is continuous on the entire convergence interval $(-R, R)$. $\square$

Integration of a power series

3.15B/00:00 (13:22)

THEOREM 2 (ON THE INTEGRATION OF A POWER SERIES).

Series (7) can be term-by-term integrated on any segment nested in the convergence interval $(-R, R)$, i. e., to find the integral of the sum $S(x)$ of series (7), it suffices to find the sum of the series whose terms are the integrals of the terms of the initial series. Moreover, the radius of convergence of the integrated series coincides with the radius of convergence of the initial series.

PROOF.

First, we prove that the radius of convergence R' of the integrated series is equal to the radius of convergence R of the initial series (7).

Let $|x| < R$. For definiteness, we will perform integration from 0 to x. Consider the following series obtained by term-by-term integration of series (7):

$$\sum_{k=0}^{\infty} \int_0^x c_k t^k \, dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1} x^{k+1} = c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \dots$$

Denote by d_k the coefficient of x^k in the resulting series: $d_k = \frac{c_{k-1}}{k}$.

To find the radius of convergence R' of the integrated power series, we use the Cauchy–Hadamard formula:

$$\frac{1}{R'} = \varlimsup_{n\to\infty} \sqrt[n]{|d_n|} = \varlimsup_{n\to\infty} \sqrt[n]{\frac{|c_{n-1}|}{n}} = \frac{\varlimsup_{n\to\infty} \sqrt[n]{|c_{n-1}|}}{\lim_{n\to\infty} \sqrt[n]{n}}.$$

In the denominator, we indicated the usual limit, since the sequence $\{\sqrt[n]{n}\}$ is convergent. The limit of this sequence is 1. We transform the numerator as follows:

$$\varlimsup_{n\to\infty} \sqrt[n]{|c_{n-1}|} = \varlimsup_{n\to\infty} |c_{n-1}|^{\frac{1}{n-1}\frac{n-1}{n}} = \varlimsup_{n\to\infty} \left(\sqrt[n-1]{|c_{n-1}|} \right)^{\frac{n-1}{n}}.$$

By the Cauchy–Hadamard formula, we get $\varlimsup_{n\to\infty} \sqrt[n-1]{|c_{n-1}|} = \frac{1}{R}$, the limit of the exponent $\frac{n-1}{n}$, as $n \to \infty$, is 1. Thus, the limit of the numerator is $\frac{1}{R}$ and finally we get

$$\frac{1}{R'} = \varlimsup_{n\to\infty} \sqrt[n]{|d_n|} = \frac{1}{R}.$$

So, we have proved that the radii of convergence R' and R coincide.

To complete the proof of the theorem, it remains for us to prove that the sign of the integral can be moved under the sign of an infinite sum:

$$\int_0^x S(t)\, dt = \int_0^x \Big(\sum_{k=0}^{\infty} c_k t^k \Big)\, dt = \sum_{k=0}^{\infty} \int_0^x c_k t^k \, dt.$$

This formula is valid by virtue of the corollary on the integration of a uniformly converging functional series, since all conditions of this corollary are fulfilled: the initial series (7) converges uniformly on the segment $[0, x]$ nested in the convergence interval $(-R, R)$ and, in addition, all terms of the initial series (7) are continuous functions on this segment. $\square$

Differentiation of a power series

3.15B/13:22 (12:46)

Now we turn to the differentiation operation and consider the series obtained by term-by-term differentiation of the initial series (7):

$$\sum_{k=0}^{\infty}(c_k x^k)' = \sum_{k=1}^{\infty} k c_k x^{k-1}. \tag{8}$$

In this case, the summation starts with $k = 1$, since when differentiating the first term c_0, which does not depend on x, we get 0. If d_k is the coefficient for the degree x^k of the differentiated series, then it follows from formula (8) that $d_{k-1} = kc_k$, whence $d_k = (k+1)c_{k+1}$. Let us find the radius of convergence R' of the power series (8) by the Cauchy–Hadamard formula:

$$\begin{aligned}\frac{1}{R'} &= \overline{\lim_{n\to\infty}} \sqrt[n]{|d_n|} = \overline{\lim_{n\to\infty}} \bigl((n+1)|c_{n+1}|\bigr)^{\frac{1}{n}} = \\ &= \overline{\lim_{n\to\infty}} \Bigl(\sqrt[n+1]{n+1}\Bigr)^{\frac{n+1}{n}} \cdot \overline{\lim_{n\to\infty}} \Bigl(\sqrt[n+1]{|c_{n+1}|}\Bigr)^{\frac{n+1}{n}}.\end{aligned} \tag{9}$$

The first limit on the right-hand side of (9) is 1, since both the limit of the base $\sqrt[n+1]{n+1}$ and the limit of the exponent $\frac{n+1}{n}$ are 1 as $n \to \infty$. For the second limit, we obtain that the exponent $\frac{n+1}{n}$ approaches 1 and $\overline{\lim}_{n\to\infty} \sqrt[n+1]{|c_{n+1}|} = \frac{1}{R}$ by the Cauchy–Hadamard formula, where R is the radius of convergence of the initial series (7). Thus, the right-hand side of equality (9) approaches $\frac{1}{R}$.

So, we have proved that $\frac{1}{R'} = \frac{1}{R}$. This means that series (8) obtained by formal differentiation of the terms of the initial series (7) has the same radius of convergence as the initial series.

Theorem 3 (on the differentiation of a power series).

Series (7) can be term-by-term differentiated at any point in the convergence interval $(-R, R)$, i. e., to find the derivative of the sum $S(x)$ of the series (7), it suffices to find the sum of the series whose members are the derivatives of the terms of the initial series. Moreover, the radius of convergence of the differentiated series coincides with the radius of convergence of the initial series.

Proof.

We have already proved the statement about the coincidence of the convergence radii.

It remains to prove that the sign of differentiation can be moved under the sign of an infinite sum:

$$S'(x) = \Big(\sum_{k=0}^{\infty} c_k x^k\Big)' = \sum_{k=0}^{\infty} \big(c_k x^k\big)' = \sum_{k=1}^{\infty} k c_k x^{k-1} \ .$$

This statement is true by virtue of the corollary on the differentiation of the functional series, since all conditions of this corollary are fulfilled: the formally differentiated series (8) uniformly converges on any segment nested in the convergence interval $(-R, R)$, all the terms of the series (8) are continuous functions, and the initial series (7) also converges uniformly on this segment (note that, in the indicated corollary, it was only required that the initial series converge at one point). □

After differentiating the series (8) composed of differentiated members of the initial series (7) we obtain a power series with the same radius of convergence. The sum of this series will be the second derivative $S''(x)$ of the sum of the initial series. Such a process can be continued infinitely. Therefore, the following statement holds.

Corollary (on the infinite differentiability of a power series).

The sum $S(x)$ of power series (7) is an infinitely differentiable function on the convergence interval; to find its derivative of order m, it is enough to find the sum of the series obtained from the initial series by differentiating its terms the required number of times:

$$S^{(m)}(x) = \Big(\sum_{k=0}^{\infty} c_k x^k\Big)^{(m)} = \sum_{k=m}^{\infty} \big(c_k x^k\big)^{(m)}.$$

18. Taylor series

Real analytic functions and their expansions into Taylor series

3.15B/26:08 (13:22)

Definition.

A function f is called a *real analytic function* on the interval (x_0-R, x_0+R) if it can be expanded on this interval into a convergent power series centered at x_0:

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k. \tag{1}$$

From representation (1), using the theorem on differentiation of a power series and the corollary on infinite differentiability of a power series, we obtain that the analytic function is infinitely differentiable on the interval $(x_0 - R, x_0 + R)$.

Let us express the coefficients c_k of the series (1) in terms of the values of the function f and its derivatives.

If we substitute the value $x = x_0$ in relation (1), then in this case all terms vanish except the term for $k = 0$, therefore relation (1) with $x = x_0$ will take the form

$$f(x_0) = c_0.$$

So, the coefficient c_0 is equal to the value of the function f at the point x_0.

From the theorem on differentiation of a power series it follows that the derivative of the function $f(x)$ at any point $x \in (x_0 - R, x_0 + R)$ can be found by means of the term-by-term differentiation of the power series (1):

$$f'(x) = \Big(\sum_{k=0}^{\infty} c_k (x - x_0)^k\Big)' = \sum_{k=1}^{\infty} k c_k (x - x_0)^{k-1} =$$

$$= c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \ldots \tag{2}$$

Substituting the value $x = x_0$ into relation (2), we obtain the following equality:

$$f'(x_0) = c_1.$$

Thus, the coefficient c_1 is equal to the value of the first derivative of the function f at the point x_0.

Differentiating equality (2), we obtain the relation defining the second derivative of the function f in the form of a power series:

$$f''(x) = \sum_{k=2}^{\infty} k(k-1)c_k(x-x_0)^{k-2} =$$

$$= 2c_2 + 3 \cdot 2c_3(x-x_0) + 4 \cdot 3c_4(x-x_0)^2 + \ldots \qquad (3)$$

Let us substitute the value $x = x_0$ in relation (3):

$$f''(x_0) = 2c_2.$$

Thus, $c_2 = \frac{f''(x_0)}{2}$.

Now we obtain a representation of the third derivative of the function f in the form of a power series:

$$f'''(x) = \sum_{k=3}^{\infty} k(k-1)(k-2)c_k(x-x_0)^{k-3} =$$

$$= 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-x_0) + 5 \cdot 4 \cdot 3c_5(x-x_0)^2 + \ldots \qquad (4)$$

Substitute the value $x = x_0$ in relation (4):

$$f'''(x_0) = 3 \cdot 2c_3.$$

In this case, the factorial appears in the denominator of the coefficient c_3 representation: $c_3 = \frac{f'''(x_0)}{3!}$.

It is easy to verify that a formula of the form $\frac{f^{(k)}(x_0)}{k!}$ remains valid for any coefficient c_k. Moreover, it will also be true for the initial coefficient of the series, since the function f is considered to be the derivative $f^{(0)}$ and the value 0! is considered equal to 1.

Let us formulate the obtained results as a theorem.

Theorem (on the properties of a real analytic function).

If the function f is a real analytic function on the interval $(x_0 - R, x_0 + R)$, then it is infinitely differentiable on this interval and is expanded on this interval into the power series with coefficients that are determined by the values of the function f and its derivatives at the point x_0 as follows:

$$c_k = \frac{f^{(k)}(x_0)}{k!}, \quad k = 0, 1, 2, \ldots$$

Substituting the found values of the coefficients c_k into relation (1), we get the equality that holds for all $x \in (x_0 - R, x_0 + R)$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{5}$$

The series on the right-hand side of equality (5) is called the *Taylor series* of the function f.

In contrast to the expansion of functions considered earlier by Taylor's formula (see [1, Ch. 22]), the sum in (5) includes an infinite number of terms but there is no remainder term.

A special case of the Taylor series is a series centered at the point $x_0 = 0$ and converging on the interval $(-R, R)$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Real analytic functions and the property of infinite differentiability

An example of an infinitely differentiable function that does not expand into a Taylor series

3.16A/00:00 (15:20)

If a function is expanded into a Taylor series on a certain interval, then, by virtue of the properties of power series, it is infinitely differentiable on this interval. The converse is not true.

Consider the function $f(x) = e^{-\frac{1}{x^2}}$ defined on the set $\mathbb{R} \setminus \{0\}$ and define it as 0 at zero:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function is infinitely differentiable at any point $x \neq 0$ as a superposition of infinitely differentiable elementary functions. Let us show that it has the same property at zero. First of all, we prove its continuity at this point. To do this, we find the limit of the function $f(x)$ as $x \to 0$:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-\frac{1}{x^2}} = \lim_{t\to\infty} e^{-t^2} = 0.$$

When calculating the limit, we made the variable change $t = \frac{1}{x}$. Note that the notation $t \to \infty$ for real numbers t means that the parameter t can take values that infinitely approach both $-\infty$ and $+\infty$. However, in any case, the value $-t^2$ approaches $-\infty$ and so we obtain a limit equal to 0. Thus, the limit of the function at zero coincides with its value at this point; this means that the function f is continuous at the point 0.

Now we show that the function f is differentiable at the point 0 and its derivative f' is continuous at the given point.

To find the derivative of the function f at the point $x \neq 0$, it suffices to use the theorem on the derivative of the superposition and formulas for the derivatives of the corresponding elementary functions:

$$f'(x) = \left(e^{-\frac{1}{x^2}}\right)' = e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}, \quad x \neq 0. \tag{6}$$

To find $f'(0)$, we need to use the definition of the derivative:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{t\to\infty} \frac{e^{-t^2}}{\frac{1}{t}} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = 0.$$

The limit is 0, since the exponential function e^{t^2} grows at infinity faster than any power function including the linear function t.

We proved that the function f is differentiable at the point 0 and its derivative at this point is 0. It remains to show that the function f' is continuous at the point 0. To do this, we find the limit $f'(x)$ as $x \to 0$ using the previously obtained formula (6):

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2e^{-\frac{1}{x^2}}}{x^3} = \lim_{t\to\infty} \frac{2t^3}{e^{t^2}} = 0.$$

In this case, we again used the fact that the exponential function grows at infinity faster than any power function.

So, we have proved that the function is continuously differentiable at the point 0 and $f'(0) = 0$.

To study the second derivative of the function f, we need to investigate the first derivative of the function f', for which we already have the following representation:

$$f'(x) = \begin{cases} e^{-\frac{1}{x^2}} \cdot \dfrac{2}{x^3}, & x \neq 0, \\ \quad 0, & x = 0. \end{cases}$$

At all points except the point 0, the second derivative can be found by differentiation formulas:

$$f''(x) = \left(e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}\right)' = e^{-\frac{1}{x^2}} \cdot \left(\frac{2}{x^3}\right)^2 + e^{-\frac{1}{x^2}} \cdot \left(-\frac{6}{x^4}\right) = e^{-\frac{1}{x^2}} \left(\frac{4}{x^6} - \frac{6}{x^4}\right).$$

Thus, the derivative $f''(x)$ can be represented as $e^{-\frac{1}{x^2}} P_6\left(\frac{1}{x}\right)$, where $P_6(t)$ is a polynomial of degree 6. It follows from this representation that the limit $f''(x)$ as $x \to 0$ is 0:

$$\lim_{x\to 0} f''(x) = \lim_{x\to 0} e^{-\frac{1}{x^2}} P_6\left(\frac{1}{x}\right) = \lim_{t\to\infty} \frac{P_6(t)}{e^{t^2}} = 0.$$

At the point 0, the second derivative also exists and is equal to zero:

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x\to 0} \frac{\frac{2e^{-\frac{1}{x^2}}}{x^3}}{x} = \lim_{x\to 0} \frac{2e^{-\frac{1}{x^2}}}{x^4} = \lim_{t\to\infty} \frac{2t^4}{e^{t^2}} = 0.$$

Using the method of mathematical induction, we can prove for the function f that its derivative of any order k is representable for $x \neq 0$ in the form $f^{(k)}(x) = P(\frac{1}{x})e^{-\frac{1}{x^2}}$, where $P(t)$ is some polynomial. Using this representation and finding limits similar to those given above, we can prove that, at the point 0, there exist derivatives of f of any order that are equal to zero.

So, the function f is infinitely differentiable on the set $\mathbb{R}$, and all its derivatives at the point 0 are equal to zero. If the function f were expanded into a Taylor series in a certain interval $(-R, R)$ with a center at the point $x_0 = 0$, then this would mean that the function f is identically equal to zero on this interval:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0.$$

But from the definition of the function f, it follows that it is nonzero at any point $x \neq 0$. The obtained contradiction means that the function f does not expand into a Taylor series in a neighborhood of zero, i. e., it is not a real analytic function in this neighborhood. Note that the function f expands into a Taylor series in a neighborhood of any point $x \neq 0$.

We have shown that *there exist infinitely differentiable functions that cannot be expanded into a Taylor series.*

Additional remarks on the properties of the considered function

3.16A/15:20 (03:39)

The reason for the “bad” behavior of the function $e^{-\frac{1}{x^2}}$ in a neighborhood of the point 0 becomes more clear if we consider this function not on the real axis but on a complex plane consisting of numbers of the form $z = x + iy$, where $x, y \in \mathbb{R}$ and i is the *imaginary unit* for which the relation $i^2 = -1$ holds. We do not even need to specify how the function e^z behaves for complex numbers. It is enough to note that if we consider the numbers $z = iy$, where $y \in \mathbb{R}$, and find the limit as $y \to 0$, then, for the function $f(z)$, we get

$$\lim_{y\to 0} e^{-\frac{1}{(iy)^2}} = \lim_{y\to 0} e^{\frac{1}{y^2}} = \lim_{t\to\infty} e^{t^2} = +\infty.$$

Therefore, the point 0 on the complex plane is a singular point for the function $e^{-\frac{1}{z^2}}$. This circumstance is the reason that in a neighborhood of zero this function does not expand into a Taylor series neither on the complex plane, nor on the real axis.

Sufficient condition for the existence of a Taylor series. Expansions of exponent, sine, and cosine into a Taylor series

Relationship between the existence of a Taylor series and the behavior of the remainder term of Taylor's formula

3.16A/18:59 (06:28)

Suppose that the function f is infinitely differentiable in a neighborhood of the point x_0. Then, for the function f, Taylor's formula holds for any $n \in \mathbb{N}$ (see [1, Ch. 22]):

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + r_n(x_0, x).$$

Here $r_n(x_0, x)$ is the remainder term of Taylor's formula. Recall the representation of the remainder term in the Lagrange form. In this representation, the point ξ appears; this is some point located between x_0 and x:

$$r_n(x_0, x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}. \tag{7}$$

We rewrite Taylor's formula in the following form:

$$r_n(x_0, x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

If the remainder term vanishes as $n \to \infty$ for all x from some neighborhood of the point x_0, then the limit of the right-hand side of the equality also exists and is equal to 0, i. e., there exists a limit of partial sums $\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ as $n \to \infty$ and this limit is $f(x)$.

But the limit of partial sums, by definition, is the sum of the power series $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$. Thus, in this case, the function f expands into a Taylor series in some neighborhood of the point x_0:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Therefore, if we succeed in formulating the condition under which the remainder term of Taylor's formula vanishes as $n \to \infty$ on some interval, then this condition will be sufficient for the function to expand into a Taylor series on this interval.

Sufficient condition for the existence of a Taylor series

3.16A/25:27 (08:54)

Theorem (on a sufficient condition for the existence of a Taylor series).

Let the function f be infinitely differentiable on the interval (x_0-R, x_0+R) and let the condition for uniform boundedness of the derivatives of the function f of all orders on this interval be fulfilled:

$$\exists\, M>0 \quad \forall\, k\in\mathbb{N} \quad \forall\, x\in(x_0-R, x_0+R) \quad |f^{(k)}(x)|\le M. \tag{8}$$

Then the function f expands into a Taylor series on the interval (x_0-R, x_0+R).

Remark.

It follows from this theorem that the function $e^{-\frac{1}{x^2}}$ considered in the previous example does not satisfy condition (8) in a neighborhood of the point 0, since if it satisfies this condition, then it would expand into a Taylor series in this neighborhood, but we proved that this is not true. Therefore, it can be stated that, for any interval $(-R, R)$, the set of all derivatives of a given function is not bounded, i. e., for any value $M>0$, there exists a point $x\in(-R,R)$ and an order of derivative k such that the value $|f^{(k)}(x)|$ is greater than M.

Proof.

We show that, under condition (8), the remainder term of the Taylor formula $r_n(x_0,x)$ vanishes as $n\to\infty$ for all $x\in(x_0-R, x_0+R)$. As noted above, this ensures the existence of a Taylor series for the function f on a given interval.

If $x\in(x_0-R, x_0+R)$, then using the representation of the remainder term in the Lagrange form (7) and condition (8), we obtain the following estimate for $r_n(x_0,x)$:

$$|r_n(x_0,x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!}|x-x_0|^{n+1} < \frac{MR^{n+1}}{(n+1)!}. \tag{9}$$

In this estimate, we took into account that the point ξ is located between x_0 and x and therefore also belongs to the interval (x_0-R, x_0+R).

The right-hand side of estimate (9) does not depend on x and vanishes as $n\to\infty$, since the factorial $(n+1)!$ grows faster than any exponential function R^{n+1} (see the theorem on the convergence of the sequence $\left\{\frac{q^n}{n!}\right\}$ in [1, Ch. 5]). Thus, we have proved that $\lim_{n\to\infty} r_n(x_0,x)=0$ for any $x\in(x_0-R, x_0+R)$; therefore, the function f expands into a Taylor series on the interval (x_0-R, x_0+R). $\square$

Taylor series expansion of exponent, sine, and cosine

3.16A/34:21 (09:28)

1. Consider the function e^x in some neighborhood of the point 0: $x \in (-R, R)$, $R > 0$. This is an infinitely differentiable function, and its derivatives of any order coincide with the initial function:

$$(e^x)^{(k)} = e^x, \quad k \in \mathbb{N}.$$

Therefore, all derivatives are uniformly estimated by the value e^R on the interval $(-R, R)$:

$$|(e^x)^{(k)}| = |e^x| < e^R.$$

Thus, all the conditions of the previous theorem are satisfied; therefore, the function e^x expands into a Taylor series on the interval $(-R, R)$. Given Taylor's formula centered at point 0 for the function e^x, we obtain the following expansion of this function into a Taylor series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{10}$$

Since the value $R > 0$ can be chosen arbitrarily, the expansion (10) is valid for any point $x \in \mathbb{R}$. Consequently, the radius of convergence of this power series is $+\infty$ (note that this result can also be obtained by the Cauchy–Hadamard formula).

2. Consider the functions $\sin x$ and $\cos x$ in some neighborhood of the point 0: $x \in (-R, R)$, $R > 0$. These are infinitely differentiable functions, and the following formulas are valid for their derivatives of any order:

$$(\sin x)^{(k)} = \sin\Big(x + \frac{k\pi}{2}\Big), \quad (\cos x)^{(k)} = \cos\Big(x + \frac{k\pi}{2}\Big), \quad k \in \mathbb{N}.$$

All derivatives of these functions are estimated by 1 for any $x \in \mathbb{R}$:

$$|(\sin x)^{(k)}| = \Big|\sin\Big(x + \frac{k\pi}{2}\Big)\Big| \le 1, \quad |(\cos x)^{(k)}| = \Big|\cos\Big(x + \frac{k\pi}{2}\Big)\Big| \le 1.$$

By the previous theorem, the functions $\sin x$ and $\cos x$ expand in the Taylor series on the interval $(-R, R)$ for any $R > 0$. Given Taylor's formulas centered at point 0 for the functions $\sin x$ and $\cos x$, we obtain the following expansions:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

These expansions are valid for any point $x \in \mathbb{R}$, the radius of convergence of the obtained power series is $+\infty$.

Taylor series expansion of a power function

3.16B/00:00 (04:25)

When considering Taylor's formula for a power function of the form $(1+x)^\alpha$, $\alpha \in \mathbb{R}$, we noted that Taylor's formula for this function is defined only for $|x| < 1$. It is natural to expect that a similar restriction will occur for the expansion of this function into a Taylor series. Terms of the series can be obtained from the corresponding Taylor's formula:

$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots =$$
$$= 1 + \sum_{k=1}^{\infty} \frac{\alpha\dots(\alpha-k+1)}{k!}x^k. \tag{11}$$

THEOREM (ON THE EXPANSION OF A POWER FUNCTION INTO A TAYLOR SERIES).

The power function $(1+x)^\alpha$ expands into the Taylor series (11) on the interval $(-1, 1)$:

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha\dots(\alpha-k+1)}{k!}x^k. \tag{12}$$

PROOF[5].

First, we investigate the convergence of series (11) for a fixed value of x. To do this, we use the limit D'Alembert test. Denote the common term of the series by $a_k = \frac{\alpha\dots(\alpha-k+1)}{k!}x^k$ and find the limit $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ under the assumption that $a_k \ne 0$ for all $k \in \mathbb{N}$ (note that the situation $a_k = 0$ is possible only for $\alpha \in \mathbb{N}$; in this case, the Taylor series turns into a polynomial of finite degree α):

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\alpha\dots(\alpha-k)x^{k+1}\cdot k!}{(k+1)!\cdot\alpha\dots(\alpha-k+1)x^k}\right| =$$
$$= \lim_{k\to\infty}\frac{|\alpha-k|}{k+1}\cdot|x| = |x|.$$

Thus, according to the limit d'Alembert test, series (11) converges for $|x| < 1$ and diverges for $|x| > 1$.

To prove that the series (11) converges to the function $(1+x)^\alpha$ for $|x| < 1$, it suffices to show that, for these values of x, we have the limit relation $\lim_{n\to\infty} r_n(x_0, x) = 0$, where $r_n(x_0, x)$ is the remainder term in the corresponding Taylor's formula with n terms.

[5]There is no proof of this theorem in video lectures.

We use the representation of the remainder term in the Cauchy form:

$$r_n(x_0, x) = \frac{f^{(n+1)}\big(x_0 + \theta(x - x_0)\big)(1-\theta)^n}{n!}(x - x_0)^{n+1}.$$

Here θ is some value lying in the range from 0 to 1.

For the function $f(x) = (1+x)^\alpha$, in the case $x_0 = 0$, we get

$$r_n(0, x) = \frac{\alpha \dots (\alpha - n)(1+\theta x)^{\alpha-n-1}(1-\theta)^n}{n!} x^{n+1} =$$

$$= \Big(\frac{(\alpha-1)\dots\big(\alpha-1-(n-1)\big)}{n!}x^n\Big)\times$$

$$\times\Big(\alpha x(1+\theta x)^{\alpha-1}\Big)\cdot\frac{(1-\theta)^n}{(1+\theta x)^n}. \qquad (13)$$

The expression $b_n(x) = \frac{(\alpha-1)\dots(\alpha-1-(n-1))}{n!}x^n$ is a common term of the series (11) corresponding to the exponent $(\alpha-1)$. We have already proved that this series converges for $|x| < 1$, therefore, due to the necessary convergence condition, $b_n(x) \to 0$ as $n \to \infty$ for all $|x| < 1$.

Thus, in order to prove that $r_n(0, x)$ vanishes as $n \to \infty$, it suffices to establish the boundedness of the two remaining factors on the right-hand side of equality (13).

The first of these factors is $\alpha x(1+\theta x)^{\alpha-1}$. It does not depend on n and is bounded for any $|x| < 1$.

The second of the factors is $\frac{(1-\theta)^n}{(1+\theta x)^n}$. Note that the following estimate holds for any $|x| < 1$:

$$1 + \theta x \geq 1 - \theta|x| > 1 - \theta.$$

Using this estimate, we obtain

$$\Big|\frac{(1-\theta)^n}{(1+\theta x)^n}\Big| < \Big|\frac{1-\theta}{1-\theta}\Big|^n = 1.$$

So, the first factor on the right-hand side of (13) vanishes, the other two are bounded, therefore $r_n(0, x_0)$ also vanishes as $n \to \infty$ for any $|x| < 1$. This means that the series (11) converges to the function $(1+x)^\alpha$ for $|x| < 1$. $\square$

Taylor series expansions of the logarithm and arcsine

Taylor series expansion of the logarithm

3.16B/04:25 (10:55)

Having the expansion of a power function into a Taylor series (12), we can easily obtain expansions for other elementary functions using theorems on the integration of a power series.

THEOREM (ON THE EXPANSION OF THE LOGARITHM INTO A TAYLOR SERIES).

The function $\ln(1+x)$ expands in the following Taylor series on the interval $(-1,1)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}. \tag{14}$$

PROOF.

Find the derivative of the function $\ln(1+x)$:

$$\bigl(\ln(1+x)\bigr)' = \frac{1}{1+x}.$$

The derivative is a power function of the form $(1+x)^\alpha$ for $\alpha = -1$. This function expands into the Taylor series (12) on the interval $(-1,1)$. Note that the required expansion can be obtained in a simpler way using the formula of the sum of the terms of an infinite geometric progression with the ratio $(-x)$:

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k.$$

We integrate both sides of this equality from 1 to x assuming that $|x| < 1$:

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^k\,dt.$$

The left-hand side of the resulting equality is

$$\int_0^x \frac{1}{1+t}\,dt = \ln|1+t|\Big|_0^x = \ln|1+x| = \ln(1+x). \tag{15}$$

On the right-hand side, we can use the theorem on the integration of a power series and move the sign of the integral under the sign of the infinite sum:

$$\int_0^x \sum_{k=0}^{\infty} (-1)^k t^k\,dt = \sum_{k=0}^{\infty} \int_0^x (-1)^k t^k\,dt = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{k+1}\Big|_0^x =$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}. \tag{16}$$

By virtue of the theorem on the integration of a power series, the radius of convergence of the series obtained on the right-hand side of equality (16) coincides with the radius of convergence of the initial series.

Equating the expressions in the right-hand sides of equalities (15) and (16), we obtain the proved equality (14). $\square$

Taylor series expansion of the arcsine 3.16B/15:20 (13:03)

Now consider the function $\arcsin x$. This function is differentiable on the interval $(-1, 1)$. Find its derivative:

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}.$$

This derivative can be represented as $(1-t)^\alpha$, where $t = x^2$ and $\alpha = -\frac{1}{2}$.

Given the expansion of the power function in the Taylor series (12), we obtain

$$\begin{aligned}
(1+(-t))^{-\frac{1}{2}} &= 1 + \left(-\frac{1}{2}\right)(-t) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}(-t)^2 + \\
&\quad + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{3!}(-t)^3 + \dots = \\
&= 1 + \frac{t}{2} + \frac{1\cdot 3}{2^2 \cdot 2!}t^2 + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}t^3 + \dots = \\
&= 1 + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdots(2k-1)}{2^k \cdot k!} t^k.
\end{aligned}$$

The resulting expression can be simplified by using the *double factorial* function, which is denoted by $n!!$ and is equal to the product of all numbers from 1 to n of the same parity as the number n: $n!! = n(n-2)(n-4)\dots$ For example, $5!! = 5\cdot 3\cdot 1 = 15$, $6!! = 6\cdot 4\cdot 2 = 48$.

In our case, $1\cdot 3\cdots(2k-1) = (2k-1)!!$ and, in addition, $2^k \cdot k! = (2\cdot 1)(2\cdot 2)\cdots(2\cdot k) = (2k)!!$. Using double factorials and taking into account that $t = x^2$, we obtain the Taylor series for the function $\frac{1}{\sqrt{1-x^2}}$ for $|x| < 1$:

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} x^{2k}. \tag{17}$$

Now we integrate both sides of equality (17) from 0 to x for $|x| < 1$:

$$\int_0^x (1-t^2)^{-\frac{1}{2}}\,dt = x + \int_0^x \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} t^{2k}\,dt.$$

On the left-hand side, we get $\arcsin x$, on the right-hand side, we apply the theorem on the integration of a power series and move the sign of the integral under the sign of the infinite sum:

$$x + \int_0^x \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} t^{2k}\,dt = x + \sum_{k=1}^{\infty} \int_0^x \frac{(2k-1)!!}{(2k)!!} t^{2k}\,dt =$$

$$= x + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{x^{2k+1}}{2k+1}.$$

The resulting power series converges for $|x| < 1$ and is equal to the function $\arcsin x$. Thus, we have proved the following theorem.

Theorem (on the expansion of the arcsine into a Taylor series).

The function $\arcsin x$ expands into a Taylor series on the interval $(-1, 1)$:

$$\arcsin x = x + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{x^{2k+1}}{2k+1}.$$

Additional remarks

3.16B/28:23 (06:25)

1. Acting in a similar way, we can obtain the Taylor series expansion for the function $\arctan x$ on the interval $(-1, 1)$. To do this, it is sufficient to find the derivative $(\arctan x)' = \frac{1}{1+x^2} = (1+x^2)^{-1}$, expand the power function $(1+x^2)^{-1}$ into a Taylor series, and then integrate both sides of the resulting equality.

2. If we have some Taylor series and it is required to determine a function that expands into this series, then sometimes it is possible to solve this problem by performing term-by-term integration of the initial series. If, as a result of integration, a series arises corresponding to some known function $f(x)$, then this means, by virtue of the theorem on the differentiation of a power series, that the initial series corresponds to the function $f'(x)$. Another way to solve this problem is the term-by-term differentiation of the initial series and finding the function $g(x)$ corresponding to the differentiated series. In this case, the initial series will correspond to the function obtained by integrating the function $g(x)$. All considered series will have the same interval of convergence.

3. Having the expansion of the function into a Taylor series, we can find approximate values of this function by calculating some initial number of terms of this series. To estimate the error of the found approximation, we can use, for example, the estimate of the remainder term of the corresponding Taylor's formula. For various classes of functions, there exist other types of their expansion into series that converge in one sense or another. In the case of periodic functions, their expansions into the Fourier series are widely used. This kind of functional series is studied in the two final chapters of this book.

19. Fourier series in Euclidean space

Real Euclidean space and its properties

Definition of a real Euclidean space

3.17A/00:00 (07:51)

Definition.

A *real Euclidean space* E is a linear space equipped with the scalar product operation.

Recall the definition of a *linear space*. This is a nonempty set of elements (*vectors*) equipped with the operation $f+g$ of addition of vectors f and g and the operation αf of multiplication of a vector f by a real number α. The operations introduced satisfy the following axioms (called *axioms of a linear space*):

L1) $\forall\, f, g \in E \quad f+g = g+f$,

L2) $\forall\, f, g, h \in E \quad (f+g)+h = f+(g+h)$,

L3) $\exists\, \mathbf{0} \in E \quad \forall\, f \in E \quad f+\mathbf{0} = \mathbf{0}+f = f$,

L4) $\forall\, f \in E \quad \exists\,(-f) \in E \quad f+(-f) = \mathbf{0}$,

L5) $\forall\, \alpha, \beta \in \mathbb{R} \quad \forall\, f \in E \quad \alpha(\beta f) = (\alpha\beta)f$,

L6) $\forall\, f \in E \quad 1 \cdot f = f$,

L7) $\forall\, \alpha, \beta \in \mathbb{R} \quad \forall\, f \in E \quad (\alpha+\beta)f = \alpha f + \beta f$,

L8) $\forall\, \alpha \in \mathbb{R} \quad \forall\, f, g \in E \quad \alpha(f+g) = \alpha f + \alpha g$.

The element $\mathbf{0} \in E$ from the axiom L3 is called the *zero vector* of the space E; the element $-f$ from the axiom L4 is called the *opposite vector* to the vector f. To distinguish between the number 0 and the zero vector $\mathbf{0}$, we will use a bold face for the zero vector. The difference $f-g$ of the vectors is understood as the sum of the vector f and the vector opposite to the vector g: $f-g \overset{\text{def}}{=} f+(-g)$.

The *scalar product* operation (f,g) associates the vectors $f, g \in E$ with a real number and satisfies the following axioms (called *axioms of the scalar product*):

S1a) $\forall\, f \in E \quad (f,f) \ge 0$,

S1b) $\big((f,f) = 0\big) \Leftrightarrow (f = \mathbf{0})$, S2) $\forall f, g \in E \quad (f, g) = (g, f)$,

S3) $\forall f, g, h \in E \quad \alpha, \beta \in \mathbb{R} \quad (\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$.

REMARK.

An example of a real Euclidean space is the set of vectors on the plane. For them, the scalar product is defined by the formula $(\overline{a}, \overline{b}) = |\overline{a}| \cdot |\overline{b}| \cos \varphi$, where φ denotes the angle between the vectors $\overline{a}$ and $\overline{b}$.

In what follows, we will usually omit the word “real” in the name of a real Euclidean space.

Norm of a vector and its properties

3.17A/07:51 (10:05)

Taking into account the axiom S1a of the scalar product, we can introduce such a characteristic of the vector f from the Euclidean space E as the *norm* (notation $\|f\|$). The norm of the vector $f \in E$ is defined as follows:

$$\|f\| \stackrel{\text{def}}{=} \sqrt{(f, f)} = (f, f)^{\frac{1}{2}}.$$

It follows from the axiom S1a that the norm of a vector always exists and is non-negative. From the axiom S1b, it follows that the norm $\|f\|$ is 0 if and only if the vector f coincides with the zero vector $\mathbf{0}$.

REMARK.

In the case of the Euclidean space of vectors on the plane, the norm is the usual length of a vector.

THEOREM (CAUCHY–BUNYAKOVSKY INEQUALITY FOR THE NORM OF A VECTOR).

For the norm of vectors of Euclidean space, the following inequality holds (this inequality is called the *Cauchy–Bunyakovsky inequality*):

$$\forall f, g \in E \quad |(f, g)| \le \|f\| \cdot \|g\|. \tag{1}$$

PROOF.

We choose an arbitrary real number λ and consider the following scalar square: $(\lambda f + g, \lambda f + g)$. By virtue of the axiom S1a, this scalar square is non-negative:

$$(\lambda f + g, \lambda f + g) \ge 0. \tag{2}$$

Let us transform this scalar product using the axioms S2 and S3:

$$(\lambda f + g, \lambda f + g) = \lambda^2 (f, f) + \lambda(g, f) + \lambda(f, g) + (g, g) =$$
$$= \lambda^2 (f, f) + 2\lambda(f, g) + (g, g).$$

We have obtained a quadratic equation for the number λ. It follows from inequality (2) that this equation has at most one real root. This means that the discriminant D of the equation is non-positive:

$$D = (f, g)^2 - (f, f) \cdot (g, g) \leq 0.$$

Rewriting the last inequality in the form $(f, g)^2 \leq (f, f) \cdot (g, g)$, extracting the square root from both sides, and taking into account the definition of the norm of a vector, we obtain inequality (1). □

Using inequality (1), we can prove that the triangle inequality holds for the norm of a vector.

Theorem (triangle inequality for the norm of a vector).

For the norm of a vector in Euclidean space, the *triangle inequality* holds:

$$\forall\, f, g \in E \quad \|f + g\| \leq \|f\| + \|g\|. \tag{3}$$

Proof.

By definition, we have

$$\|f + g\|^2 = (f + g, f + g).$$

We transform the right-hand side of the equality using the axioms S2 and S3:

$$(f + g, f + g) = |(f + g, f + g)| = |(f, f) + 2(f, g) + (g, g)| \leq$$
$$\leq (f, f) + 2|(f, g)| + (g, g) = \|f\|^2 + 2|(f, g)| + \|g\|^2.$$

Now we use estimate (1):

$$\|f\|^2 + 2|(f, g)| + \|g\|^2 \leq \|f\|^2 + 2\|f\| \cdot \|g\| + \|g\|^2 = \big(\|f\| + \|g\|\big)^2.$$

Thus, we have proved the following estimate:

$$\|f + g\|^2 \leq \big(\|f\| + \|g\|\big)^2.$$

Taking the square root of both sides of the last inequality, we obtain estimate (3). □

Metric and definition of the limit of a sequence in Euclidean space

3.17A/17:56 (06:48)

Using the norm of a vector, we can define yet another characteristic of vectors in Euclidean space called the *metric*: $\rho(f, g) \overset{\text{def}}{=} \|f - g\|$. It is easy to prove that a metric so defined satisfies the following properties (called *axioms of a metric space*):

M1a) $\forall\, f, g \in E \quad \rho(f, g) \geq 0$,

M1b) $\big(\rho(f, g) = 0\big) \Leftrightarrow (f = g)$,

M2) $\forall\, f, g \in E \quad \rho(f, g) = \rho(g, f)$,

M3) $\forall\, f, g, h \in E \quad \rho(f, h) \leq \rho(f, g) + \rho(g, h)$.

In particular, the axiom M3 immediately follows from the triangle inequality:

$$\rho(f, h) = \|f - h\| = \|f - g + g - h\| = \|(f - g) + (g - h)\| \leq$$
$$\leq \|f - g\| + \|g - h\| = \rho(f, g) + \rho(g, h).$$

Thus, any Euclidean space is a metric space. The value $\rho(f, g)$ determines the measure of proximity of the vectors f and g (in other words, this is the distance between the vectors in a given space).

Using the concept of metric, we can define the limit of a sequence of elements of a metric (in our case, Euclidean) space.

DEFINITION.

Let $\{f_n\}$ be a sequence of vectors of the Euclidean space E. It is said that the *limit* of a given sequence as $n \to \infty$ is equal to the vector $f \in E$ if $\lim_{n\to\infty} \rho(f_n, f) = 0$. We denote this fact as follows: $\lim_{n\to\infty} f_n = f$ or $f_n \to f$ as $n \to \infty$.

We define the limit for the sequence of vectors of the space E through the limit of the numerical sequence $\{\rho(f_n, f)\}$. This makes it easy to prove many properties of the limit of a sequence of vectors using the already proven properties of the limit of a numerical sequence.

Given the definition of a metric, we obtain that $f_n \to f$ as $n \to \infty$ if

$$\lim_{n\to\infty} \|f_n - f\| = 0. \tag{4}$$

Using the definition of the norm, we can reformulate the definition of the limit of a sequence of vectors of Euclidean space in terms of the limit of a numerical sequence defined by the scalar product: $f_n \to f$ as $n \to \infty$ if

$$\lim_{n\to\infty} (f_n - f, f_n - f) = 0.$$

Of all the above versions of the definition of the limit in Euclidean space, the definition (4) based on the norm is the most suitable for what follows. If the limit relation (4) holds, then they say that the sequence $\{f_n\}$ *converges to the vector f in the norm of the Euclidean space E.*

Fourier series with respect to an orthonormal sequence of vectors in Euclidean space

Definition of an orthonormal sequence and Fourier series

3.17A/24:44 (06:35)

Hereinafter, we will consider sequences with elements indexed from zero. The set of non-negative integer indices will be denoted by $\mathbb{N}_0$. Thus, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

DEFINITION.

Consider the sequence of vectors $\{\psi_k\}$, $k \in \mathbb{N}_0$, in the Euclidean space E. A sequence $\{\psi_k\}$ is called an *orthonormal sequence of vectors*, or an *orthonormal system of vectors*, if the equality holds for any indices $i, j \in \mathbb{N}_0$:

$$(\psi_i, \psi_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We assume that the orthonormal sequence has already been chosen; therefore, we will not explicitly indicate the dependence on this sequence in the notation introduced below.

REMARK.

Orthonormal sequences exist only in the *infinite-dimensional* Euclidean spaces. In finite-dimensional spaces, the number of orthonormal vectors cannot exceed the dimension of the space; therefore, any systems of orthonormal vectors are finite (for example, in the space of vectors on a plane, any orthonormal system consists of no more than two vectors).

Let $f \in E$. We define the following numerical sequence: $\{f_k\}$, where $f_k = (f, \psi_k)$, $k \in \mathbb{N}_0$. This set of numbers is called the *Fourier coefficients* of the vector f with respect to the orthonormal system $\{\psi_k\}$.

The formal series $\sum_{k=0}^{\infty} f_k\psi_k$ is called the *Fourier series* corresponding to the vector f:

$$f \sim \sum_{k=0}^{\infty} f_k\psi_k. \tag{5}$$

The Fourier series is formal, since we do not yet discuss the question of its convergence (and we do not even determine how to understand its convergence). That is why the symbol $\sim$ is used in relation (5) instead of the equal sign.

If we consider the partial sum of the Fourier series $S_n(f) = \sum_{k=0}^{n} f_k\psi_k$ for any $n \in \mathbb{N}_0$, then this sum will be some vector of the space E. Thus, the sequence of partial sums of the Fourier series $\{S_n(f)\}$ is a sequence of vectors of the Euclidean space E. The partial sums of the Fourier series will be called *partial Fourier sums*, or simply *Fourier sums*.

Formulation of the extremal property of Fourier sums

3.17A/31:19 (03:28)

THEOREM (ON THE EXTREMAL PROPERTY OF FOURIER SUMS).

Let f be some vector of the Euclidean space E, $\{\psi_k\}$ be an orthonormal sequence in E, $\{a_k\}$ be arbitrary numerical sequence. Then, for any $n \in \mathbb{N}_0$, the following relation holds:

$$\min_{a_k, k=0,\ldots,n} \Big\| f - \sum_{k=0}^{n} a_k \psi_k \Big\| = \Big\| f - \sum_{k=0}^{n} f_k \psi_k \Big\|.$$

Thus, the Fourier sum $S_n(f) = \sum_{k=0}^{n} f_k \psi_k$ has the following *extremal property*: it is the best approximation for the vector f (in the norm of the space E) among all linear combinations $\sum_{k=0}^{n} a_k \psi_k$ of the system of orthonormal vectors ψ_k, $k = 0, \ldots, n$.

Proof of the extremal property of Fourier sums

3.17A/34:47 (11:23)

We perform the following transformations using the definition of the norm and axioms of the scalar product:

$$\begin{aligned} \Big\| f - \sum_{k=0}^{n} a_k \psi_k \Big\|^2 &= \Big(f - \sum_{k=0}^{n} a_k \psi_k, f - \sum_{m=0}^{n} a_m \psi_m \Big) = (f, f) - \\ &- \Big(f, \sum_{m=0}^{n} a_m \psi_m \Big) - \Big(\sum_{k=0}^{n} a_k \psi_k, f \Big) + \Big(\sum_{k=0}^{n} a_k \psi_k, \sum_{m=0}^{n} a_m \psi_m \Big) = \\ &= \|f\|^2 - \sum_{m=0}^{n} a_m (f, \psi_m) - \sum_{k=0}^{n} a_k (\psi_k, f) + \sum_{k=0}^{n} \sum_{m=0}^{n} a_k a_m (\psi_k, \psi_m). \end{aligned} \tag{6}$$

The scalar product does not depend on the order of the factors; therefore, we can combine the first and second sums on the right-hand side of equality (6). In addition, we take into account that the scalar product (f, ψ_k) is the Fourier coefficient f_k of the vector f. By virtue of the orthonormality of the system of vectors $\{\psi_k\}$, only terms with $k = m$ remain in the last sum of the right-hand side of (6); moreover, $(\psi_k, \psi_k) = 1$. As a result, the right-hand side of (6) takes the form $\|f\|^2 - 2\sum_{k=0}^{n} a_k f_k + \sum_{k=0}^{n} a_k^2$.

We continue the transformation by adding two zero-sum terms to the resulting expression:

$$\|f\|^2 - 2\sum_{k=0}^{n} a_k f_k + \sum_{k=0}^{n} a_k^2 + \sum_{k=0}^{n} f_k^2 - \sum_{k=0}^{n} f_k^2 =$$

$$= \|f\|^2 + \sum_{k=0}^{n} (a_k^2 - 2a_k f_k + f_k^2) - \sum_{k=0}^{n} f_k^2 =$$

$$= \|f\|^2 + \sum_{k=0}^{n} (a_k - f_k)^2 - \sum_{k=0}^{n} f_k^2. \tag{7}$$

Since $\sum_{k=0}^{n}(a_k - f_k)^2 \ge 0$, the expression on the right-hand side of equality (7) will have a minimum value when this sum is equal to zero; this is equivalent to the equalities $a_k = f_k$, $k = 0, \ldots, n$.

Thus, the minimum value of the expression $\left\|f - \sum_{k=0}^{n} a_k\psi_k\right\|^2$ is achieved when $a_k = f_k$, $k = 0, \ldots, n$. Therefore, the same result is true for the initial expression $\left\|f - \sum_{k=0}^{n} a_k\psi_k\right\|$. $\square$

Bessel's inequality 3.17B/00:00 (06:08)

In the process of proving the extremal property of Fourier sums, we got the following relation:

$$\left\|f - \sum_{k=0}^{n} f_k\psi_k\right\|^2 = \|f\|^2 - \sum_{k=0}^{n} f_k^2. \tag{8}$$

Since the left-hand side of equality (8) is non-negative, the same is true for the right-hand side:

$$\|f\|^2 - \sum_{k=0}^{n} f_k^2 \ge 0.$$

Therefore, sums $\sum_{k=0}^{n} f_k^2$ can be estimated from above for any $n \in \mathbb{N}_0$:

$$\sum_{k=0}^{n} f_k^2 \le \|f\|^2.$$

We have proved that the partial sums of the series $\sum_{k=0}^{\infty} f_k^2$ with non-negative terms are uniformly bounded from above. Therefore, by the criterion for the convergence of numerical series with non-negative terms, the series $\sum_{k=0}^{\infty} f_k^2$ converges and the same estimate holds for its sum:

$$\sum_{k=0}^{\infty} f_k^2 \le \|f\|^2. \tag{9}$$

Estimate (9) is called *Bessel's inequality*.

So, although we have not yet considered the convergence of the formal Fourier series $\sum_{k=0}^{\infty} f_k\psi_k$ in the space E, we have established that, for any vector $f \in E$, the numerical series, whose terms are the squares of the Fourier coefficients f_k, is convergent and its sum is estimated from above by $\|f\|^2$.

In addition, the convergence of the series $\sum_{k=0}^{\infty} f_k^2$ implies, by the necessary condition of convergence, that $f_k^2 \to 0$ as $k \to \infty$. Therefore, the same limit relation holds for the Fourier coefficients f_k:

$$\forall\, f \in E \quad \lim_{k\to\infty} f_k = 0.$$

Fourier series over a complete orthonormal sequence of vectors

Complete sequence in Euclidean space

3.17B/06:08 (04:59)

DEFINITION.

A sequence $\{\psi_k\}$ of vectors of the Euclidean space E is called a *complete sequence*, or a *complete system*, if any vector of a given space can be approximated arbitrarily closely by a linear combination of a finite number of elements of this sequence:

$$\forall\, f \in E \quad \forall\, \varepsilon > 0 \quad \exists\, n \in \mathbb{N} \quad \exists\, a_k, k = 0, \ldots, n,$$

$$\left\| f - \sum_{k=0}^{n} a_k\psi_k \right\| < \varepsilon.$$

REMARK.

In the case of finite-dimensional space, complete systems consist of a finite number of vectors, any complete system forms the *basis* of this space, and any vector can be represented as a linear combination of basis vectors. The number of vectors in the complete system is equal to the *dimension* of space. For example, in the space of vectors on the plane, any two non-collinear vectors form a basis.

Parseval's identity for Fourier coefficients with respect to a complete orthonormal sequence

3.17B/11:07 (13:40)

THEOREM (ON FOURIER COEFFICIENTS WITH RESPECT TO A COMPLETE ORTHONORMAL SEQUENCE).

If the orthonormal sequence $\{\psi_k\}$ of vectors of the Euclidean space E is complete, then, for any vector $f \in E$, the following equality holds (this equality is called *Parseval's identity*):

$$\sum_{k=0}^{\infty} f_k^2 = \|f\|^2. \tag{10}$$

Proof.

Since we have already established that Bessel's inequality (9) holds for any orthonormal sequence, it remains for us to prove that the opposite inequality holds for a complete orthonormal sequence:

$$\|f\|^2 \le \sum_{k=0}^{\infty} f_k^2. \tag{11}$$

We choose an arbitrary vector $f \in E$ and some value $\varepsilon > 0$. Since the sequence $\{\psi_k\}$ is complete, there exists a number $n \in \mathbb{N}$ and coefficients a_k, $k = 0, \ldots, n$, such that the estimate holds:

$$\Big\| f - \sum_{k=0}^{n} a_k \psi_k \Big\| < \varepsilon. \tag{12}$$

In this estimate, the value n and the coefficients a_k, generally speaking, depend on ε.

By virtue of the theorem on the extremal property of Fourier sums, the following estimate holds for any coefficients a_k, $k = 0, \ldots, n$:

$$\Big\| f - \sum_{k=0}^{n} f_k \psi_k \Big\| \le \Big\| f - \sum_{k=0}^{n} a_k \psi_k \Big\|.$$

From this estimate and (12), the following estimate can be obtained:

$$\Big\| f - \sum_{k=0}^{n} f_k \psi_k \Big\| < \varepsilon. \tag{13}$$

In estimate (13), only n depends on ε.

Now consider the norm of the vector f and transform it using the triangle inequality as follows:

$$\|f\| = \Big\| f - \sum_{k=0}^{n} f_k \psi_k + \sum_{k=0}^{n} f_k \psi_k \Big\| \le \Big\| f - \sum_{k=0}^{n} f_k \psi_k \Big\| + \Big\| \sum_{k=0}^{n} f_k \psi_k \Big\|.$$

The first term on the right-hand side of the last inequality is estimated from above by the value ε. Let us transform the second term:

$$\Big\| \sum_{k=0}^{n} f_k \psi_k \Big\| = \Big(\sum_{k=0}^{n} f_k \psi_k, \sum_{m=0}^{n} f_m \psi_m \Big)^{\frac{1}{2}} =$$

$$= \Big(\sum_{k=0}^{n} \sum_{m=0}^{n} f_k f_m (\psi_k, \psi_m) \Big)^{\frac{1}{2}} = \Big(\sum_{k=0}^{n} f_k^2 \Big)^{\frac{1}{2}}.$$

Given that the series $\sum_{k=0}^{\infty} f_k^2$ converges and consists of non-negative terms, the expression $\left(\sum_{k=0}^{n} f_k^2\right)^{\frac{1}{2}}$ can be estimated from above by the value $\left(\sum_{k=0}^{\infty} f_k^2\right)^{\frac{1}{2}}$. So, we got the estimate

$$\|f\| < \varepsilon + \left(\sum_{k=0}^{\infty} f_k^2\right)^{\frac{1}{2}}. \tag{14}$$

Since estimate (14) is valid for any $\varepsilon > 0$, we can pass to the limit as $\varepsilon \to 0$. As a result, estimate (14) is transformed into the following non-strict inequality:

$$\|f\| \leq \left(\sum_{k=0}^{\infty} f_k^2\right)^{\frac{1}{2}}.$$

Having squared both terms of this inequality, we obtain inequality (11). The required estimate (10) follows from Bessel's inequality (9) and inequality (11). □

Interpretation of Parseval's identity as a generalization of the Pythagorean theorem

3.17B/24:47 (02:38)

Parseval's identity (10) can be considered as a generalization of the *Pythagorean theorem* to the case of infinite-dimensional Euclidean spaces. Indeed, consider the vector $\overline{a}$ on the plane and find its expansion in two orthonormal vectors $\overline{i}$ and $\overline{j}$:

$$\overline{a} = \alpha_1 \overline{i} + \alpha_2 \overline{j}. \tag{15}$$

The vector $\overline{a}$ forms the hypotenuse of a right triangle with the legs $\alpha_1 \overline{i}$ and $\alpha_2 \overline{j}$. Then, by virtue of the Pythagorean theorem, we have

$$|\overline{a}|^2 = \alpha_1^2 + \alpha_2^2. \tag{16}$$

In addition, performing scalar multiplication of both terms of the equality (15) by $\overline{i}$ and $\overline{j}$ and using the orthonormality of the vectors $\overline{i}$ and $\overline{j}$, we get $\alpha_1 = (\overline{a}, \overline{i})$, $\alpha_2 = (\overline{a}, \overline{j})$. Thus, equality (16), which follows from the Pythagorean theorem, can be rewritten in the form

$$|\overline{a}|^2 = (\overline{a}, \overline{i})^2 + (\overline{a}, \overline{j})^2.$$

The resulting equality is a finite-dimensional analogue of Parseval's identity.

Theorem on the convergence of a Fourier series with respect to a complete orthonormal sequence 3.17B/27:25 (08:11)

Once again, we turn to equality (8):

$$\Big\| f - \sum_{k=0}^{n} f_k\psi_k \Big\|^2 = \|f\|^2 - \sum_{k=0}^{n} f_k^2.$$

If we assume that the orthonormal sequence $\{\psi_k\}$ is complete and pass to the limit as n approaches infinity, then, by virtue of Parseval's identity, the right-hand side of this equality approaches zero. Then the left-hand side of the equality also approaches zero. Thus, for the Fourier series by a complete orthonormal sequence, the following limit relation holds:

$$\lim_{n\to\infty} \Big\| f - \sum_{k=0}^{n} f_k\psi_k \Big\| = 0. \qquad (17)$$

The limit relation (17) means, by definition, that the sequence of partial sums $S_n(f) = \sum_{k=0}^{n} f_k\psi_k$ of the Fourier series converges to the vector f in the norm of the space E:

$$\lim_{n\to\infty} S_n(f) = f.$$

According to the definition of a convergent series, this means that the Fourier series can be considered not as formal series, but as a converging series in the space E, and its sum is the vector f:

$$f = \sum_{k=0}^{\infty} f_k\psi_k.$$

In this case, we can already replace the symbol $\sim$ with an equal sign $=$. So, we have proved the following theorem.

Theorem (on the convergence of a Fourier series with respect to a complete orthonormal sequence in Euclidean space).

If the orthonormal sequence $\{\psi_k\}$ of vectors in the Euclidean space E is complete, then, for any vector $f \in E$, its Fourier series $\sum_{k=0}^{\infty} f_k\psi_k$ with respect to this sequence converges to the vector f in the norm of the space E, i. e., the limit relation (17) is satisfied.

Thus, we have established a number of facts related to Fourier series for abstract real Euclidean space. If we now consider specific realizations of a real Euclidean space defining the set of vectors and the required operations on this set, in particular, the scalar product, then we can immediately extend all the facts established earlier to this space including the fact of the normwise convergence of the Fourier series (provided that a complete orthonormal sequence of vectors is selected in this space).

We are interested in the *space of integrable functions* with standard operations of addition and multiplication by a number and with the scalar product defined through the integration operation. The properties of such space and the associated Fourier series will be discussed in the next, final chapter.

20. Fourier series in the space of integrable functions

Euclidean space of integrable functions

Definition of the Euclidean space of integrable functions

3.18A/00:00 (07:21)

DEFINITION.

We introduce the *space of functions* $\mathcal{R}([a,b])$ *defined and Riemann integrable on the segment* $[a,b]$. This is a linear space with operations of adding functions and multiplying a function by a number. These operations result in integrable functions and satisfy all axioms of linear space.

The *scalar product* in the space $\mathcal{R}([a,b])$ is defined as follows:

$$\forall\, f,g \in \mathcal{R}([a,b]) \quad (f,g) \overset{\text{def}}{=} \int_a^b f(x)g(x)\,dx. \tag{1}$$

In the definition of a scalar product, we use the operation of multiplication of functions. The product of integrable functions is an integral function; therefore, the scalar product is defined for all elements of the space $\mathcal{R}([a,b])$.

Let us verify that axioms S1–S3 are satisfied for the scalar product introduced in this way. Axiom S3 immediately follows from the linearity of the integral with respect to integrands:

$$\forall\, f,g,h \in \mathcal{R}([a,b]) \quad \forall\, \alpha,\beta \in \mathbb{R}$$

$$(\alpha f + \beta g, h) = \int_a^b \big(\alpha f(x) + \beta g(x)\big)h(x)\,dx =$$

$$= \alpha \int_a^b f(x)h(x)\,dx + \beta \int_a^b g(x)h(x)\,dx = \alpha(f,h) + \beta(g,h).$$

Axiom S2 also holds:

$$\forall\, f,g \in \mathcal{R}([a,b])$$

$$(f,g) = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = (g,f).$$

Axiom S1a holds by virtue of the theorem on the non-negativeness of the integral of the non-negative function:

$$\forall\, f \in \mathcal{R}([a,b]) \quad (f,f) = \int_a^b f^2(x)\,dx \geq 0.$$

However, the axiom S1b may not hold: if the function f is integrable on the segment $[a,b]$, then the condition $(f,f)=0$ is not equivalent to the fact that $f(x)$ vanishes on this segment. We can consider a function that vanishes at all points of the segment $[a,b]$, except for a finite number of points at which it is equal to 1. This function is integrable, but the integral of $f^2(x)$ on the segment $[a,b]$ will be zero.

Ways to satisfy the axiom associated with the zero scalar product

3.18A/07:21 (07:46)

This problem can be solved in two ways.

Method 1.

We can narrow the class of functions under consideration and restrict ourselves, for example, to the class $C([a,b])$ of all functions continuous on the segment $[a,b]$ with the same scalar product (1). In this case, the axiom S1b holds by virtue of the theorem on the integral of a positive continuous function: if the function is continuous and non-negative on the segment $[a,b]$ and takes a positive value at least at one point, then the integral of this function on the segment $[a,b]$ is positive. Thus, the equality $\int_a^b f^2(x)\,dx = 0$ in this case will be equivalent to the identity $f(x) \equiv 0$.

Method 2.

Instead of the integrable functions themselves, we can consider the *classes of equivalent integrable functions* as elements of the space $\mathcal{R}([a,b])$.

The class H_0 containing all functions equivalent to the function $f(x) \equiv 0$ is defined as follows:

$$H_0 \stackrel{\text{def}}{=} \left\{ f : \int_a^b f^2(x)\,dx = 0 \right\}.$$

This class is considered as the zero element of the space $\mathcal{R}([a,b])$. Functions f and g belong to the same class H (i. e., are equivalent) if $f - g \in H_0$. In particular, functions f and g will be equivalent if they differ only in a finite number of points of the segment $[a,b]$. It is easy to show that with such a definition of equivalent functions, the entire set of functions integrable on $[a,b]$ splits into pairwise disjoint equivalence classes such that any integrable function belongs to exactly one class.

If $f \in H_f$, $g \in H_g$, then the sum of the classes $H_f + H_g$, by definition, is the class containing the function $f+g$, the product of αH_f of the class H_f

by a number α is the the class containing the function αf, and the scalar product of the classes is $(H_f, H_g) \stackrel{\text{def}}{=} \int_a^b f(x)g(x)\,dx$. It can be shown that so defined operations are well posed, i. e., they do not depend on the choice of "representatives" of the source classes, and all the required axioms are fulfilled for them including the axiom S1b.

When considering the elements of such a space, we can still use the notation of some function f from the class H_f instead of this class itself and suppose that this notation means *any* function equivalent to f.

We assume that we modified the definition of the Euclidean space $\mathcal{R}([a,b])$ using method 2. Note that this modification will not affect the following definitions, but it is required to justify many important facts related to the properties of the Euclidean space (for example, the property of the uniqueness of the limit of a sequence converging in the norm of this space).

Norm in the space of integrable functions 3.18A/15:07 (01:07)

Having the scalar product, we can define the *norm* of the function $f \in \mathcal{R}([a,b])$:

$$\|f\| \stackrel{\text{def}}{=} \sqrt{(f,f)} = \Big(\int_a^b f^2(x)\,dx\Big)^{\frac{1}{2}}.$$

The Cauchy–Bunyakovsky inequality and the triangle inequality hold for such a norm, since in the previous chapter we proved the validity of these inequalities for an arbitrary Euclidean space.

Constructing an orthonormal sequence of integrable functions

A sequence of trigonometric functions and a proof of its orthogonality 3.18A/16:14 (08:26)

We choose the segment $[-\pi,\pi]$ as the integration segment and introduce the following sequence of functions: $\{\sin kx, \cos kx\}$, $k = 0, 1, \ldots$

First, we show that this system is *orthogonal*, i. e., the scalar product of any two different functions from a given system vanishes. To prove this fact, we will use the following trigonometric identities:

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big), \tag{2}$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big), \tag{3}$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x - y) + \sin(x + y)\big). \tag{4}$$

Let us prove, for example, the orthogonality of the functions $\sin k_1 x$ and $\cos k_2 x$ with different coefficients k_1 and k_2 using formula (4):

$$\int_{-\pi}^{\pi} \sin k_1 x \cos k_2 x \, dx =$$

$$= \frac{1}{2}\Big(\int_{-\pi}^{\pi} \sin(k_1 - k_2)x \, dx + \int_{-\pi}^{\pi} \sin(k_1 + k_2)x \, dx\Big) = 0.$$

The equality to zero of the obtained integrals follows from two facts: the integrands $\sin(k_1 + k_2)x$ and $\sin(k_1 - k_2)x$ are odd; the integral of an odd function over a symmetric segment vanishes. There is no need to even use the formula (4), since it can be noted that the *initial integrand* is an odd function (as a product of the odd function $\sin k_1 x$ and the even function $\cos k_2 x$).

We simultaneously proved the orthogonality of the functions $\sin kx$ and the constant 1 (which is also included in this system of functions, since $\cos kx \equiv 1$ when $k = 0$).

The orthogonality of the functions $\sin kx$ and $\cos kx$ for the same coefficient $k = 1, 2, \ldots$ is proved similarly.

Now we prove the orthogonality of the functions $\cos k_1 x$ and $\cos k_2 x$ for $k_1 \neq k_2$ using formula (3):

$$\int_{-\pi}^{\pi} \cos k_1 x \cos k_2 x \, dx =$$

$$= \frac{1}{2}\Big(\int_{-\pi}^{\pi} \cos(k_1 - k_2)x \, dx + \int_{-\pi}^{\pi} \cos(k_1 + k_2)x \, dx\Big) =$$

$$= \frac{1}{2}\Big(\frac{\sin(k_1 - k_2)x}{k_1 - k_2}\Big|_{-\pi}^{\pi} + \frac{\sin(k_1 + k_2)x}{k_1 + k_2}\Big|_{-\pi}^{\pi}\Big) = 0.$$

This result follows from the equality $\sin k\pi = 0$, $k \in \mathbb{Z}$. We simultaneously proved the orthogonality of the functions $\cos kx$, $k \neq 0$, and the constant 1.

In the same way, using formula (2), the orthogonality of the functions $\sin k_1 x$ and $\sin k_2 x$ may be proved for $k_1 \neq k_2$.

Normalization of the obtained sequence of trigonometric functions

3.18A/24:40 (07:23)

We have proved that the sequence of functions $\{\sin kx, \cos kx\}$, $k = 0, 1, \ldots$, is orthogonal in the space $\mathcal{R}([-\pi, \pi])$. It remains to transform it into an orthonormal sequence.

To do this, we use the *normalization* operation applicable for any nonzero vector f of an arbitrary Euclidean space E. If $f \neq \mathbf{0}$, then the vector $\frac{1}{\|f\|}f$ has the norm 1:

$$\left\|\frac{1}{\|f\|}f\right\| = \frac{1}{\|f\|} \cdot \|f\| = 1.$$

In our case, all functions of the sequence are nonzero, except for the function $\sin kx$ for $k = 0$, which we will not consider hereinafter.

First, we normalize the constant function $\cos 0x \equiv 1$. For this function, we have

$$\|1\| = \left(\int_{-\pi}^{\pi} dx\right)^{\frac{1}{2}} = \sqrt{2\pi}.$$

Therefore, we must take the function $\frac{1}{\sqrt{2\pi}}$ as the first element of an orthonormal sequence.

Now we normalize the function $\sin kx$, $k \in \mathbb{N}$:

$$\|\sin kx\| = \left(\int_{-\pi}^{\pi} \sin^2 kx\, dx\right)^{\frac{1}{2}}.$$

Find the integral using the formula $\sin^2 kx = \frac{1}{2}(1 - \cos 2kx)$:

$$\int_{-\pi}^{\pi} \sin^2 kx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2kx)\, dx = \frac{x}{2}\bigg|_{-\pi}^{\pi} - \frac{1}{2} \cdot \frac{\sin 2kx}{2k}\bigg|_{-\pi}^{\pi}.$$

The second term on the right-hand side of the last equality is 0, the first term is π. Thus, the functions $\frac{\sin kx}{\sqrt{\pi}}$ should be taken as normalized sine functions.

Similarly, we can normalize the functions $\cos kx$, $k \in \mathbb{N}$, using the formula $\cos^2 kx = \frac{1}{2}(1 + \cos 2kx)$ and taking into account that the integral of $\cos 2kx$ over the segment $[-\pi, \pi]$ vanishes.

Thus, we get the following *orthonormal sequence of functions*: $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}\right\}$, $k \in \mathbb{N}$. Here the parameter k does not take the value 0: we must indicate the constant function separately, since it has a different normalizing factor.

Orthonormal sequence of functions for an arbitrary segment

3.18A/32:03 (08:49)

A similar sequence of orthonormal functions can be defined on any segment $[a, b]$. To do this, we use trigonometric functions with a period equal to the length of this segment.

In the case of the space $\mathcal{R}([-l,l])$, the period must be equal to $2l$, so we need to take the sequence $\left\{\sin\frac{\pi kx}{l}, \cos\frac{\pi kx}{l}\right\}$, $k = 0, 1, \ldots$, as the initial orthogonal sequence. For example, for the function $f(x) = \sin\frac{\pi kx}{l}$, we have

$$f(x+2l) = \sin\frac{\pi k(x+2l)}{l} = \sin\Big(\frac{\pi kx}{l} + 2\pi k\Big) = \sin\frac{\pi kx}{l} = f(x).$$

Proving the orthogonality of a given sequence and normalizing its nonzero elements is performed in the same way as in the case of the sequence considered above for the space $\mathcal{R}([-\pi,\pi])$.

In the case of the segment $[a,b]$, we should consider the sequence of functions $\left\{\sin\frac{2\pi kx}{b-a}, \cos\frac{2\pi kx}{b-a}\right\}$, $k = 0, 1, \ldots$, and normalize its nonzero elements. These functions will have a period $(b-a)$.

In order to simplify the proof of orthogonality and the calculation of normalization coefficients in the space $\mathcal{R}([a,b])$, it is convenient to transform the corresponding integrals by replacing the original integration segment $[a,b]$ with the segment $\left[-\frac{b-a}{2}, \frac{b-a}{2}\right]$ which has the same length, but is symmetrical with respect to the origin. This transformation can be performed by corollary 3 on the integration of periodic functions from the theorem on the change of variables in a definite integral.

Constructing a formal Fourier series for integrable functions

Fourier coefficients and Fourier series for integrable functions

3.18B/00:00 (08:57)

Given an orthonormal sequence $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}\right\}$, $k \in \mathbb{N}$, we can find the *Fourier coefficients* for any function $f \in \mathcal{R}([-\pi,\pi])$:

$$\tilde{a}_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\,dx;$$

$$\tilde{a}_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x)\cos kx\,dx, \quad k \in \mathbb{N};$$

$$\tilde{b}_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x)\sin kx\,dx, \quad k \in \mathbb{N}.$$

Thus, the formal *Fourier series* for the function f with respect to a given orthonormal sequence has the form

$$f(x) \sim \frac{\tilde{a}_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty}\Big(\frac{\tilde{a}_k}{\sqrt{\pi}}\cos kx + \frac{\tilde{b}_k}{\sqrt{\pi}}\sin kx\Big). \tag{5}$$

Bessel's inequality for the Fourier coefficients will be as follows:

$$\tilde{a}_0^2 + \sum_{k=1}^{\infty}(\tilde{a}_k^2 + \tilde{b}_k^2) \le \int_{-\pi}^{\pi} f^2(x)\,dx. \tag{6}$$

Recall that we have the square of the norm $\|f\|$ on the right-hand side of inequality (6).

The series indicated on the left-hand side of (6) is a convergent numerical series. It follows from the convergence of this numerical series that its common term should approach zero: $(\tilde{a}_k^2+\tilde{b}_k^2) \to 0$ as $k \to \infty$. Therefore, the following limit relations hold for any integrable function f:

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0,$$

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0.$$

Another representation of the Fourier series 3.18B/08:57 (09:54)

Representation (5) of the Fourier series can be simplified by using the following set of coefficients:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx; \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \quad k \in \mathbb{N}; \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx, \quad k \in \mathbb{N}. \end{aligned} \tag{7}$$

Let us express the initial Fourier coefficients $\tilde{a}_k$, $\tilde{b}_k$ in terms of the corresponding new coefficients a_k, b_k:

$$\sqrt{2\pi}\tilde{a}_0 = \pi a_0, \quad \tilde{a}_0 = \frac{\pi a_0}{\sqrt{2\pi}} = \sqrt{\frac{\pi}{2}}a_0;$$

$$\sqrt{\pi}\tilde{a}_k = \pi a_k, \quad \tilde{a}_k = \frac{\pi a_k}{\sqrt{\pi}} = \sqrt{\pi}a_k, \quad k \in \mathbb{N};$$

$$\sqrt{\pi}\tilde{b}_k = \pi b_k, \quad \tilde{b}_k = \frac{\pi b_k}{\sqrt{\pi}} = \sqrt{\pi}b_k, \quad k \in \mathbb{N}.$$

In the formal Fourier series (5), we replace the coefficients $\tilde{a}_k$, $\tilde{b}_k$ with formulas containing the coefficients a_k, b_k:

$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}}a_0 + \sum_{k=1}^{\infty}\Big(\frac{\sqrt{\pi}a_k}{\sqrt{\pi}}\cos kx + \frac{\sqrt{\pi}b_k}{\sqrt{\pi}}\sin kx\Big).$$

In the resulting representation of the Fourier series, we can reduce all factors of the form $\sqrt{\pi}$. Finally we get

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx). \tag{8}$$

Also we can transform Bessel's inequality (6) in a similar way:

$$\frac{\pi}{2}a_0^2 + \sum_{k=1}^{\infty}(\pi a_k^2 + \pi b_k^2) \le \int_{-\pi}^{\pi} f^2(x)\,dx,$$

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx. \tag{9}$$

Formulas (7)–(9), as the simplest ones, are traditionally used when writing the Fourier formal series (and related relations) for integrable functions. It should be noted that, in all these formulas, the factor $\frac{1}{\pi}$ is indicated before the integral sign, and an additional factor $\frac{1}{2}$ appears before the terms containing the coefficient a_0.

Convergence of the Fourier series in mean square in the case of periodic continuous functions

Weierstrass theorems on uniform approximation

3.18B/18:51 (08:15)

The question arises when the formal Fourier series for the function f converges to this function (and in what sense this convergence should be understood).

Let us study the convergence of the Fourier series in the norm of the space of integrable functions $\mathcal{R}([-\pi,\pi])$. In this case, the following theorem will help us (we accept this theorem without proof; see its proof, for example, in [18, Ch. 14, Sec. 69.1]).

Theorem (Weierstrass theorem on the uniform approximation of continuous periodic functions by trigonometric polynomials).

Any continuous 2π-periodic function f can be uniformly approximated with arbitrary accuracy on the segment $[-\pi,\pi]$ by trigonometric polynomials $T_n(x) = \alpha_0 + \sum_{k=1}^{n}(\alpha_k\cos kx + \beta_k\sin kx)$:

$$\forall\, \varepsilon > 0 \quad \exists\, T_n(x) \qquad \sup_{x\in[-\pi,\pi]} |f(x) - T_n(x)| < \varepsilon. \tag{10}$$

Remark.

There is a similar statement about the uniform approximation of a function continuous on the segment $[a, b]$ by the usual polynomials $P_n(x) = \sum_{k=0}^{n} c_k x^k$:

$$\forall\, \varepsilon > 0 \quad \exists\, P_n(x) \qquad \sup_{x\in[a,b]} |f(x) - P_n(x)| < \varepsilon.$$

This statement is called the *Weierstrass theorem on the uniform approximation of continuous functions by polynomials* (see, for example, [18, Ch. 14, Sec. 69.2]).

Convergence of the Fourier series for continuous periodic functions 3.18B/27:06 (13:15)

If the function f can be approximated by the trigonometric polynomials $T_n(x)$ uniformly on the segment $[-\pi, \pi]$, then it can be approximated by the same polynomials in the norm of the space $\mathcal{R}([-\pi, \pi])$.

Indeed, let us write the estimate (10) in a slightly modified form:

$$\sup_{x\in[-\pi,\pi]} |f(x) - T_n(x)| < \frac{\varepsilon}{\sqrt{2\pi}}. \tag{11}$$

Then, for the norm of the difference $f - T_n$ in the space $\mathcal{R}([-\pi, \pi])$, using the corollary of the theorem on the comparison of integrals and estimate (11), we have

$$\|f - T_n\| = \Bigl(\int_{-\pi}^{\pi} \bigl(f(x) - T_n(x)\bigr)^2\, dx\Bigr)^{\frac{1}{2}} \le$$

$$\le \Bigl(\sup_{x\in[-\pi,\pi]} \bigl(f(x) - T_n(x)\bigr)^2 \cdot 2\pi\Bigr)^{\frac{1}{2}} =$$

$$= \sup_{x\in[-\pi,\pi]} |f(x) - T_n(x)| \cdot \sqrt{2\pi} < \frac{\varepsilon}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = \varepsilon.$$

Thus, we have proved that the following condition is satisfied:

$$\forall\, \varepsilon > 0 \quad \exists\, T_n(x) \qquad \|f - T_n\| < \varepsilon. \tag{12}$$

This condition means that the function f can be approximated with any accuracy by trigonometric polynomials in the norm of the space $\mathcal{R}([-\pi, \pi])$. The approximation in the norm of the space $\mathcal{R}([-\pi, \pi])$ is also called the *mean-square approximation*.

So, we have proved that any continuous 2π-periodic function can be approximated with any accuracy in the norm of the space $\mathcal{R}([-\pi, \pi])$ by a trigonometric polynomial, i. e., by some linear combination of functions from the orthonormal sequence $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}\right\}$, $k \in \mathbb{N}$.

Therefore, if we consider the subspace of the space $\mathcal{R}([-\pi,\pi])$ of all continuous 2π-periodic functions, then, for this subspace, the considered orthonormal sequence will be complete. This immediately implies, by virtue of the theorem on the convergence of the Fourier series by a complete orthonormal sequence in Euclidean space, that the Fourier series (8) of any continuous 2π-periodic function f converges to this function in the norm of the space $\mathcal{R}([-\pi,\pi])$, i. e., in mean square.

In the next section, we show that our orthonormal sequence of trigonometric functions is complete in the wider subspace of the space $\mathcal{R}([-\pi,\pi])$, namely, in the subspace of all piecewise continuous functions. Thus, we justify the convergence of the Fourier series in mean square for all piecewise continuous functions.

Convergence of the Fourier series in mean square in the case of piecewise continuous functions

Formulation of the theorem on the approximation of a piecewise continuous function

3.19A/00:00 (02:51)

Definition.

A *piecewise continuous function* on the segment $[a,b]$ is a function that is continuous at all points of a given segment except for a finite number of discontinuity points of the first kind.

It follows from the additivity property of the integral with respect to the integration segment that any piecewise continuous function on the segment $[a,b]$ is integrable on this segment.

Theorem (on mean-square approximation of a piecewise continuous function by trigonometric polynomials).

For any piecewise continuous function f defined on the segment $[-\pi,\pi]$ and for any value $\varepsilon > 0$, there exists a trigonometric polynomial $T_n(x)$ such that $\|f - T_n\| < \varepsilon$. Given the definition of a norm in the space $\mathcal{R}([-\pi,\pi])$, this inequality can be rewritten as follows:

$$\left(\int_{-\pi}^{\pi}\bigl(f(x) - T_n(x)\bigr)^2\,dx\right)^{\frac{1}{2}} < \varepsilon.$$

First stage of the proof

3.19A/02:51 (12:07)

For simplicity, we consider a function f that has one discontinuity point of the first kind $x = 0$, although the same result can be obtained for a function with any finite number of discontinuity points. In addition, we assume that the inequality $f(-\pi) \neq f(\pi)$ holds for this function.

The idea of the proof is to transform the function f into a continuous 2π-periodic function by means of a "small" change to the given function (from the point of view of the norm in $\mathcal{R}([-\pi, \pi])$) and then to apply the Weierstrass theorem on uniform approximation by trigonometric polynomials to this transformed function.

Choose some small value $\delta > 0$ and define an auxiliary function $\tilde{f}_\delta$:

$$\tilde{f}_\delta(x) = \begin{cases} f(\pi) + \dfrac{f(-\pi+\delta) - f(\pi)}{\delta}(x+\pi), & x \in [-\pi, -\pi+\delta), \\ f(\delta) - \dfrac{f(-\delta) - f(\delta)}{2\delta}(x-\delta), & x \in (-\delta, \delta), \\ f(x), & x \in [-\pi+\delta, -\delta] \cup [\delta, \pi]. \end{cases} \tag{13}$$

The function $\tilde{f}_\delta$ differs from the initial function f only on the intervals $[-\pi, -\pi+\delta)$ and $(-\delta, \delta)$; the new function is linear on these intervals. The function $\tilde{f}_\delta$ is continuous at all points of the segment $[-\pi, \pi]$ and, moreover, its values coincide at the points $-\pi$ and π (and are equal to $f(\pi)$). Thus, the function $\tilde{f}_\delta$ can be extended to the entire real axis $\mathbb{R}$ resulting in a continuous 2π-periodic function on $\mathbb{R}$.

Second stage of the proof

3.19A/14:58 (06:33)

We assume that the estimate $|f(x)| \leq M$, $x \in [-\pi, \pi]$, holds for the initial function f. Then the same estimate holds for the function $\tilde{f}_\delta$, and we obtain an estimate for the difference between the functions f and $\tilde{f}_\delta$:

$$\forall\, x \in [-\pi, \pi] \quad |f(x) - \tilde{f}_\delta(x)| \leq |f(x)| + |\tilde{f}_\delta(x)| \leq 2M. \tag{14}$$

Given (13) and (14), we can estimate the square of the norm $\|f - \tilde{f}_\delta\|$:

$$\|f - \tilde{f}_\delta\|^2 = \int_{-\pi}^{\pi} \bigl(f(x) - \tilde{f}_\delta(x)\bigr)^2\, dx =$$

$$= \int_{-\pi}^{-\pi+\delta} \bigl(f(x) - \tilde{f}_\delta(x)\bigr)^2\, dx + \int_{-\delta}^{\delta} \bigl(f(x) - \tilde{f}_\delta(x)\bigr)^2\, dx \leq$$

$$\leq (2M)^2\delta + (2M)^2 \cdot 2\delta = 12M^2\delta.$$

So, we have obtained the following estimate for the norm $\|f - \tilde{f}_\delta\|$:

$$\|f - \tilde{f}_\delta\| \le 2\sqrt{3}M\sqrt{\delta}. \tag{15}$$

Third stage of the proof

3.19A/21:31 (06:13)

Let us choose some value $\varepsilon > 0$ and put $\delta = \frac{\varepsilon^2}{48M^2}$. Then estimate (15) takes the form

$$\|f - \tilde{f}_\delta\| \le 2\sqrt{3}M \cdot \frac{\varepsilon}{4\sqrt{3}M} = \frac{\varepsilon}{2}. \tag{16}$$

Now we apply the Weierstrass theorem on the uniform approximation of a continuous periodic function by trigonometric polynomials to the function $\tilde{f}_\delta$ using the previously selected value ε:

$$\exists\, T_n(x) \qquad \sup_{x\in[-\pi,\pi]} |\tilde{f}_\delta(x) - T_n(x)| < \frac{\varepsilon}{2\sqrt{2\pi}}.$$

Using the same reasoning as in passing from (11) to (12), we get an estimate for the norm $\|\tilde{f}_\delta - T_n\|$:

$$\|\tilde{f}_\delta - T_n\| < \frac{\varepsilon}{2}. \tag{17}$$

Combining estimates (16) and (17) and using the triangle inequality for the norm, we finally obtain

$$\|f - T_n(x)\| = \|f - \tilde{f}_\delta + \tilde{f}_\delta - T_n(x)\| \le$$
$$\le \|f - \tilde{f}_\delta\| + \|\tilde{f}_\delta - T_n(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \ \square$$

Convergence of the Fourier series for piecewise continuous functions

3.19A/27:44 (07:41)

The proved theorem means that if in the space $\mathcal{R}([-\pi,\pi])$ we consider the subspace of all piecewise continuous functions, then, for this subspace, the orthonormal sequence $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}\right\}$, $k \in \mathbb{N}$, will be complete.

Thus, the previously obtained result on the convergence in mean square of the Fourier series for continuous 2π-periodic functions can be generalized to the case of piecewise continuous functions.

Theorem (on the convergence in mean square of the Fourier series for piecewise continuous functions).

The Fourier series (8) of any piecewise continuous function f converges to this function with respect to the norm of the space $\mathcal{R}([-\pi,\pi])$, i. e., in mean square:

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\Bigl(f(x)-\frac{a_0}{2}-\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)\Bigr)^2\,dx=0.$$

In addition, for any piecewise continuous functions, Bessel's inequality (9) turns into *Parseval's identity*:

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx.$$

Pointwise convergence of the Fourier series

3.19A/35:25 (04:58)

In this section, we briefly consider the *pointwise convergence* of the Fourier series (8). This type of convergence means that there exists a limit of partial sums of the Fourier series $S_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$ as $n\to\infty$ at any point $x\in[-\pi,\pi]$.

It is clear that pointwise convergence does not follow from the mean-square convergence. Therefore, we can expect that pointwise convergence will take place for a narrower class of functions (compared to the class of piecewise continuous and even continuous 2π-periodic functions). There are several classes of functions with the required properties; we will consider only one of them, which can be described in a very simple way.

Definition.

A function f is called *piecewise continuously differentiable* on the segment $[a,b]$ if it is piecewise continuous and also has a continuous derivative on each interval of continuity (moreover, the function has one-sided derivatives at the endpoints of each such interval).

We give the following theorem without proof (see, for example, [9, Ch. 8, Sec. 5.6]).

Theorem (on the pointwise convergence of the Fourier series).

Let the function f be piecewise continuously differentiable on $[-\pi,\pi]$. Then its Fourier series converges at any point of the given segment and the following equalities hold for the limit of partial sums $S_n(x)$ of the Fourier series:

$$\lim_{n\to\infty}S_n(x)=\frac{f(x+0)+f(x-0)}{2},\quad x\in(-\pi,\pi);$$

$$\lim_{n\to\infty}S_n(\pm\pi)=\frac{f(-\pi+0)+f(\pi-0)}{2}.$$

Recall that $f(x+0)$ means the right-hand limit of the function f at the point x and $f(x-0)$ means its left-hand limit. Thus, at all points of the continuity of the function f, the Fourier series converges to this function.

Uniform convergence of the Fourier series

Formulation of the theorem on the uniform convergence of the Fourier series

3.19B/00:00 (05:26)

We have two main facts so far: the mean-square convergence of the Fourier series for piecewise continuous functions and the pointwise convergence of the Fourier series for piecewise continuously differentiable functions. Now let us turn to the question for which subspace of the space of integrable functions $\mathcal{R}([-\pi,\pi])$ there exists *uniform convergence* of the Fourier series. The answer to this question is given by the following theorem.

THEOREM (ON THE UNIFORM CONVERGENCE OF THE FOURIER SERIES).

Let the function f be a 2π-periodic function that is differentiable on $[-\pi,\pi]$ and let its derivative be piecewise continuous. Then the Fourier series of the function f converges to this function uniformly on $[-\pi,\pi]$:

$$S_n(x) \overset{[-\pi,\pi]}{\rightrightarrows} f(x), \quad n\to\infty.$$

Before proving this theorem, we formulate and prove an auxiliary lemma that generalizes the integration formula by parts for a definite integral to a wider class of functions.

LEMMA (GENERALIZATION OF THE INTEGRATION FORMULA BY PARTS)[6].

Suppose that the functions u and v are differentiable on the segment $[a,b]$ and their derivatives are piecewise continuous. Then the integration formula by parts holds for them:

$$\int_a^b uv'\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'v\,dx. \tag{18}$$

PROOF.

Earlier, we proved the integration formula by parts under the assumption that the functions u and v are continuously differentiable on the entire segment $[a,b]$ (see the corresponding theorem in Chapter 7 and the remark to it). The lemma to be proved states that the integration formula by parts

[6] There is no statement and proof of this lemma in video lectures.

remains valid if the derivatives of the functions u and v have a finite number of discontinuities of the first kind.

We assume that the function $u'(x)$ has a unique discontinuity at the point $c \in (a, b)$ and the function $v'(x)$ has a unique discontinuity at the point $d \in (a, b)$, moreover, $c < d$ (the case with several discontinuities can be analyzed in a similar way).

On the segments $[a, c]$, $[c, d]$, $[d, b]$, the functions u and v are continuous and have derivatives defined on the corresponding intervals, and these derivatives can be defined by continuity at the endpoints of the corresponding segments, since one-sided limits exist at discontinuity points of the first kind. Therefore, on these segments, all the conditions of the original theorem on integration by parts are satisfied and therefore the equalities hold:

$$\int_a^c uv'\,dx = u(c)v(c) - u(a)v(a) - \int_a^c u'v\,dx,$$

$$\int_c^d uv'\,dx = u(d)v(d) - u(c)v(c) - \int_c^d u'v\,dx,$$

$$\int_d^b uv'\,dx = u(b)v(b) - u(d)v(d) - \int_d^b u'v\,dx.$$

If we summarize the obtained equalities and use the property of additivity of the integrals with respect to the integration segment (see the remark on the first theorem on the additivity of a definite integral in Chapter 6), then we get $\int_a^b uv'\,dx$ on the left-hand side and we get the expression $u(b)v(b) - u(a)v(a) - \int_a^b u'v\,dx$ on the right-hand side. Formula (18) is proved. $\square$

Now let us prove the theorem.

First stage of the proof 3.19B/05:26 (13:35)

We begin by proving the uniform convergence of the Fourier series on the segment $[-\pi, \pi]$.

Recall the formula for partial Fourier sums:

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

By virtue of the Weierstrass uniform convergence test, it suffices to estimate the common term of the Fourier series $(a_k \cos kx + b_k \sin kx)$ by a value independent of x and being a common term of some convergent numerical series.

The following obvious estimate holds:

$$|a_k \cos kx + b_k \sin kx| \le |a_k| + |b_k|. \tag{19}$$

Therefore, to prove the uniform convergence of the Fourier series, it suffices to prove that the numerical series $\sum_{k=1}^{\infty}(|a_k| + |b_k|)$ converges.

Note that although the convergence of the series $\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$ follows from Bessel's inequality (9), this fact does not guarantee the convergence of the series $\sum_{k=1}^{\infty}(|a_k| + |b_k|)$. For example, we know that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges but the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Let us transform the coefficients a_k and b_k using the properties of the function f. We start with the coefficients a_k:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx.$$

Since the function f is differentiable and its derivative is piecewise continuous, we can apply the integration formula by parts (18) by setting $u = f(x)$, $dv = \cos kx \, dx$, whence $v = \frac{\sin kx}{k}$:

$$a_k = \frac{1}{\pi}\Big(\frac{f(x) \sin kx}{k} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{f'(x) \sin kx}{k} \, dx \Big).$$

The term $\frac{f(x)\sin kx}{k}\Big|_{-\pi}^{\pi}$ vanishes due to the fact that the function $f(x) \sin kx$ has the period 2π (in addition, we can see that $\sin(-\pi k) = \sin \pi k = 0$). Thus, we get

$$a_k = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f'(x) \sin kx}{k} \, dx = -\frac{1}{k}\Big(\frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin kx \, dx \Big).$$

The expression in parentheses indicated on the right-hand side of the obtained equality is the Fourier coefficient corresponding to the function $\sin kx$ for the function $f'(x)$. To distinguish this coefficient from the coefficient b_k of the initial function f, we denote it by b_k':

$$b_k' = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin kx \, dx.$$

So, we got the following equality:

$$|a_k| = \frac{|b_k'|}{k}. \tag{20}$$

Similarly, we can transform the coefficient b_k:

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx =$$
$$= \frac{1}{\pi}\Big(-\frac{f(x)\cos kx}{k}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{f'(x)\cos kx}{k}\,dx\Big) =$$
$$= \frac{1}{k}\Big(\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos kx\,dx\Big).$$

The expression $\frac{f(x)\cos kx}{k}\Big|_{-\pi}^{\pi}$ vanishes due to the 2π-periodicity of the function $f(x)\cos kx$. Denoting $a_k' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos kx\,dx$, we obtain

$$|b_k| = \frac{|a_k'|}{k}. \tag{21}$$

Applying equalities (20) and (21), we can rewrite estimate (19) as follows:

$$|a_k\cos kx + b_k\sin kx| \le \frac{|a_k'| + |b_k'|}{k}. \tag{22}$$

Second stage of the proof

3.19B/19:01 (04:48)

Using the relation $xy \le \frac{1}{2}(x^2+y^2)$, which is valid for any real x and y and can be obtained from the obvious inequality $(x+y)^2 \ge 0$, we estimate the right-hand side of equality (22):

$$\frac{|a_k'| + |b_k'|}{k} = \frac{|a_k'|}{k} + \frac{|b_k'|}{k} \le \frac{1}{2}\Big((a_k')^2 + \frac{1}{k^2}\Big) + \frac{1}{2}\Big((b_k')^2 + \frac{1}{k^2}\Big) =$$
$$= \frac{1}{2}\big((a_k')^2 + (b_k')^2\big) + \frac{1}{k^2}.$$

The estimate (22) takes the form

$$|a_k\cos kx + b_k\sin kx| \le \frac{1}{2}\big((a_k')^2 + (b_k')^2\big) + \frac{1}{k^2}.$$

On the right-hand side of the estimate, we got the common term of the converging number series. Indeed, we previously established the convergence of the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$; the series $\frac{1}{2}\sum_{k=1}^{\infty}\big((a_k')^2 + (b_k')^2\big)$ converges due to Bessel's inequality for the Fourier coefficients of the function $f'(x)$.

Therefore, according to the Weierstrass test, the initial Fourier series for the function f converges uniformly:

$$S_n(x) \overset{[-\pi,\pi]}{\rightrightarrows} S(x), \quad n\to\infty.$$

Moreover, the function $S(x)$ is continuous on $[-\pi,\pi]$ as the sum of a uniformly converging functional series with continuous terms.

Third stage of the proof

3.19B/23:49 (10:00)

It remains for us to prove that the sum of the Fourier series $S(x)$ coincides with the function $f(x)$ on the segment $[-\pi, \pi]$.

Recall that, by virtue of the criterion for uniform convergence of the functional series in terms of the supremum limit, the following limit relation is fulfilled:

$$\lim_{n\to\infty} \sup_{x\in[-\pi,\pi]} |S(x) - S_n(x)| = 0.$$

We write this relation in the language ε–N:

$$\forall\, \varepsilon > 0 \quad \exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \qquad \sup_{x\in[-\pi,\pi]} |S(x) - S_n(x)| < \frac{\varepsilon}{2\sqrt{2\pi}}.$$

Therefore, for all $x \in [-\pi, \pi]$, the estimate holds:

$$|S(x) - S_n(x)| < \frac{\varepsilon}{2\sqrt{2\pi}}.$$

We square both parts of this estimate, integrate from $-\pi$ to π, and then extract the square root from both parts:

$$\left(\int_{-\pi}^{\pi} |S(x) - S_n(x)|^2\, dx\right)^{\frac{1}{2}} < \left(\frac{\varepsilon^2 \cdot 2\pi}{4 \cdot 2\pi}\right)^{\frac{1}{2}} = \frac{\varepsilon}{2}.$$

The resulting relation means that the following condition is true for the norm $\|S - S_n\|$ in the space $\mathcal{R}([-\pi, \pi])$:

$$\forall\, \varepsilon > 0 \quad \exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \qquad \|S - S_n\| < \frac{\varepsilon}{2}. \tag{23}$$

On the other hand, the function f, which, by condition, is continuous on the segment $[-\pi, \pi]$, belongs to the subspace of all piecewise continuous functions defined on $[-\pi, \pi]$; therefore, by the mean-square convergence theorem of the Fourier series, the sequence of partial Fourier sums $\{S_n(x)\}$ converges to the function f in the norm of the space $\mathcal{R}([-\pi, \pi])$:

$$\lim_{n\to\infty} \|f - S_n\| = 0.$$

This fact can be written as follows:

$$\forall\, \varepsilon > 0 \quad \exists\, N_2 \in \mathbb{N} \quad \forall\, n > N_2 \qquad \|f - S_n\| < \frac{\varepsilon}{2}. \tag{24}$$

From conditions (23) and (24), we obtain that the following estimate is satisfied for all $n > \max\{N_1, N_2\}$:

$$\begin{aligned} &\|f - S\| = \|f - S_n + S_n - S\| \le \\ &\quad \le \|f - S_n\| + \|S_n - S\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The resulting estimate $\|f - S\| < \varepsilon$ is independent of n and is valid for any $\varepsilon > 0$. When $\varepsilon \to 0$, we get

$$\|f - S\| = 0.$$

Given the definition of the norm, the last equality can be written in the form

$$\Big(\int_{-\pi}^{\pi} \big(f(x) - S(x)\big)^2\,dx\Big)^{\frac{1}{2}} = 0\ .$$

The functions f and S are continuous on $[-\pi, \pi]$, so the integrand $\big(f(x) - S(x)\big)^2$ is continuous and non-negative. If it has a positive value at least at one point of the segment $[-\pi, \pi]$, then its integral must be positive by the theorem on the integral of a positive continuous function. Therefore, the integrand is identically equal to zero: $|f(x) - S(x)|^2 \equiv 0$. It follows that $S(x) = f(x)$ for all $x \in [-\pi, \pi]$. $\square$

Decreasing rate of Fourier coefficients for differentiable functions

3.19B/33:49 (06:32)

Let the function f be 2π-periodic and differentiable up to the order n, let its derivatives $f', \ldots, f^{(n-1)}$ be 2π-periodic and the derivative $f^{(n)}$ be piecewise continuous.

In the proof of the previous theorem, we have obtained relations (20) and (21) connecting the Fourier coefficients a_k, b_k of the function f with the Fourier coefficients a'_k, b'_k of its derivative f' (provided that the function f is 2π-periodic and has a piecewise continuous derivative). We can combine these relations in the form of the following equality:

$$|a_k| + |b_k| = \frac{1}{k}(|a'_k| + |b'_k|), \quad k \in \mathbb{N}. \tag{25}$$

Since in our case the derivative f' is also 2π-periodic and differentiable, we can use a similar equality for its coefficients and relate these coefficients to the coefficients a''_k, b''_k of the function f'':

$$|a'_k| + |b'_k| = \frac{1}{k}(|a''_k| + |b''_k|), \quad k \in \mathbb{N}. \tag{26}$$

Combining equalities (25) and (26), we obtain

$$|a_k| + |b_k| = \frac{1}{k^2}(|a''_k| + |b''_k|), \quad k \in \mathbb{N}.$$

Repeating these steps and expressing the coefficients $a_k^{(m-1)}$, $b_k^{(m-1)}$ of the function $f^{(m-1)}$ in terms of the coefficients $a_k^{(m)}$, $b_k^{(m)}$ of its derivative $f^{(m)}$

for $m = 3, \ldots, n-1$, we finally obtain the following equality relating the coefficients a_k, b_k of the initial function f and the coefficients $a_k^{(n)}$, $b_k^{(n)}$ of its derivative $f^{(n)}$:

$$|a_k| + |b_k| = \frac{1}{k^n}(|a_k^{(n)}| + |b_k^{(n)}|), \quad k \in \mathbb{N}.$$

Moreover, the relation $\lim_{n\to\infty}(|a_k^{(n)}| + |b_k^{(n)}|) = 0$ follows from Bessel's inequality for the Fourier series of the function $f^{(n)}$.

Thus, we have proved the following theorem.

Theorem (on the rate of decrease of Fourier coefficients for differentiable functions).

If the function f is 2π-periodic and n times differentiable, its derivatives $f', \ldots, f^{(n-1)}$ are also 2π-periodic, and the derivative $f^{(n)}$ is piecewise continuous, then the following limit relations are valid for the Fourier coefficients of the function f:

$$a_k = o\Big(\frac{1}{k^n}\Big), \quad b_k = o\Big(\frac{1}{k^n}\Big), \quad k \to \infty.$$

Thus, the better the differential properties of the function f, the faster its Fourier coefficients decrease, which allows the use of partial sums of the Fourier series with fewer terms for a good approximation of such functions.

References

1. *Abramyan M. E.* Lectures on differential calculus of functions of one variable. – Rostov-on-Don; Taganrog: SFedU Press, 2020. – 228 p.

2. *Abramyan M. E.* Lectures on differential calculus of functions of one variable. (In Russian.) – Rostov-on-Don; Taganrog: SFedU Press, 2020. – 220 p.

3. *Bartle R. G., Sherbert D. R.* Introduction to real analysis. – NY: John Wiley & Sons, Inc., 2000. – 388 p.

4. *Fichtengolts G. M.* Course of differential and integral calculus. Vol. 2. (In Russian.) – M: Fizmatlit Publ., 2001. – 810 p.

5. *Fichtengolts G. M.* Course of differential and integral calculus. Vol. 3. (In Russian.) – M: Fizmatlit Publ., 2001. – 662 p.

6. *Ilyin V. A., Poznyak E. G.* Fundamentals of Mathematical Analysis. Part 1. (In Russian.) – M.: Fizmatlit Publ., 2005. – 648 p.

7. *Ilyin V. A., Poznyak E. G.* Fundamentals of Mathematical Analysis. Part 2. (In Russian.) – M.: Fizmatlit Publ., 2002. – 464 p.

8. *Ilyin V. A., Sadovnichy V. A., Sendov B. Kh.* Mathematical analysis. Part 1. (In Russian.) – M.: Moscow University Press, 1985. – 662 p.

9. *Ilyin V. A., Sadovnichy V. A., Sendov B. Kh.* Mathematical analysis. Part 2. (In Russian.) – M.: Moscow University Press, 1987. – 358 p.

10. *Konev V. V.* Higher mathematics. Part 2. Textbook. – Tomsk: TPU Press, 2009. – 134 p. – URL: `http://portal.tpu.ru:7777/SHARED/k/KONVAL/Textbooks/Tab1/Konev-Higher_Mathematics_TextBook.pdf` (date of the access 31.01.2021).

11. *Kudryavtsev L. D.* Course of mathematical analysis. Vol. 1. (In Russian.) – M.: Drofa Publ., 2003. – 704 p.

12. *Kudryavtsev L. D.* Course of mathematical analysis. Vol. 2. (In Russian.) – M.: Drofa Publ., 2004. – 720 p.

13. *Kudryavtsev L. D.* Course of mathematical analysis. Vol. 3. (In Russian.) – M.: Drofa Publ., 2006. – 351 p.

14. *Nikolsky S. M.* The course of mathematical analysis. Vol. 1. (In Russian.) – M.: Nauka Publ., 1983. – 464 p.

15. *Nikolsky S. M.* The course of mathematical analysis. Vol. 2. (In Russian.) – M.: Nauka Publ., 1983. – 449 p.

16. *Pilidi V. S.* Mathematical analysis. (In Russian.) – Rostov-on-Don: Phoenix Publ., 2008. – 240 p.

17. *Piskunov N. S.* Differential and integral calculus. – M.: Mir Publ., 1969. – 895 p.

18. *Ter-Krikorov A. M., Shabunin M. I.* The course of mathematical analysis. (In Russian.) – M.: Nauka Publ., 1988. – 816 p.

19. *Zorich V. A.* Mathematical analysis. Part 1. (In Russian.) – M.: Nauka Publ., 1997. – 568 p.

20. *Zorich V. A.* Mathematical analysis. Part 2. (In Russian.) – M.: Nauka Publ., 1984. – 640 p.

Index

Educational edition

Abramyan Mikhail Eduardovich

Lectures on integral calculus of functions of one variable and series theory

Computer typesetting by *M. E. Abramyan*

Подписано в печать 18.06.2021 г.
Бумага офсетная. Формат 60×84 1/16. Тираж 30 экз.
Усл. печ. лист. 14,65. Уч. изд. л. 9,5. Заказ № 8058.

Издательство Южного федерального университета.

Отпечатано в отделе полиграфической, корпоративной и сувенирной продукции
Издательско-полиграфического комплекса КИБИ МЕДИА ЦЕНТРА ЮФУ.
344090, г. Ростов-на-Дону, пр. Стачки, 200/1, тел (863) 243-41-66.

9 785927 538294