Algorithms and Data Structures
Module 4. NP-hard problems

Lecture 13

# Algorithms for NP-hard problems. Travelling Salesman Problem.

# Time complexity

Let's recall time complexities of algorithms we studied in this course.

| Algorithm | Time complexity | Majorant |
|---|---|---|
| Binary search | $O(\log n)$ | $O(n)$ |
| Bubble/Insertion/Selection sort | $O(n^2)$ | $O(n^2)$ |
| Merge sort | $O(n \log n)$ | $O(n^2)$ |
| Graph connectivity  components detection | $O(m)$ | $O(n^2)$ |
| Kruskal's (with Union-Find Set data structure) | $O(m \log m) = O(n^2 \log n)$ | $O(n^3)$ |
| Prim's (with binary heap as priority queue) | $O(m \ \log n) = O(n^2 \log n)$ | $O(n^3)$ |
| Karatsuba's integer multiplication | $\Theta\left(n^{\log_2 3}\right)$ | $O(n^2)$ |
| Strassen's matrix multiplication | $O\left(n^{\log_2 7}\right)$ | $O(n^3)$ |
| Fast exponentiation | $O(\log n)$ | $O(n)$ |
| | | |

*(to be continued on the next slide…)*

# Time complexity

| Algorithm | Time complexity | Majorant |
|---|---|---|
| | | *(...continuation)* |
| Dijkstra's algorithm for general case | $O(nm)$ | $O(n^3)$ |
| Floyd-Warshall's | $O(n^3)$ | $O(n^3)$ |
| Needleman-Wunsch (Levenshtein's edit distance) | $O(nm)$ | $O(n^2)$ |
| Longest common subsequence | $O(nm)$ | $O(n^2)$ |

We see that for all the above algorithms there is a constant *c* such that the algorithm's time complexity is $O(n^c)$.

Such algorithms are called ***polynomial time*** algorithms.

# Time complexity

For the problem of calculating Fibonacci numbers we discussed two algorithms:

- A dynamic programming algorithm with polynomial time complexity $O(n)$.
- A recursive algorithm with time complexity $O(\varphi^n)$ for $\varphi = \frac{1+\sqrt{5}}{2}$.

The recursive algorithm is not polynomial time, it is an exponential time algorithm…

# Time complexity

Let’s consider two algorithms for a problem with time complexities $O(n)$ and $O(2^n)$.

| n | O(n) | $O(2^n)$ |
|---|---|---|
| 50 | 1.00 sec | 1 sec |
| 51 | 1.02 sec | 2 sec |
| 52 | 1.04 sec | 4 sec |
| 60 | 1.20 sec | 17 min |
| 70 | 1.40 sec | 12 days |
| 80 | 1.60 sec | 34 years |
| 90 | 1.70 sec | ~ 35 000 years |

# Time complexity

That is why polynomial time algorithms are called ***efficient***, whereas exponential time algorithms are considered *inefficient*.

For many problems no efficient algorithms are known… ☹

Moreover, for most of these problems it was proved that if a polynomial time algorithm would be designed for one of these problems, this immediately imply polynomial time algorithms for all such problems.

Such problems are called ***NP-hard***.

# Time complexity

There are thousands of NP-hard problems…

One of the most famous NP-hard problems is the Travelling Salesman Problem (TSP).

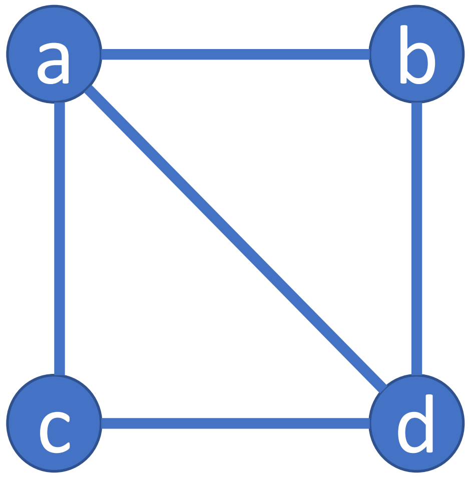
# TSP: definitions

Let $G(V, E)$ be a connected graph, $w: E \longrightarrow R_+$ be a weights function.

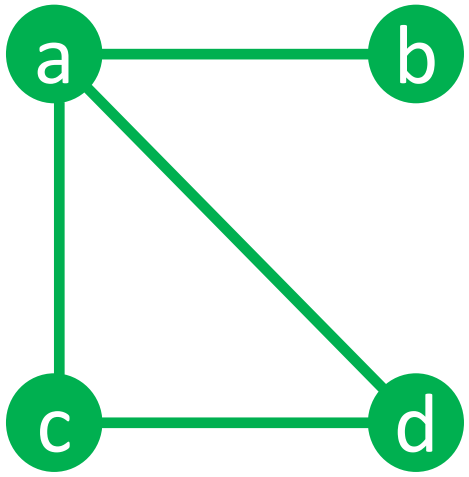
**Definitions**

- Cycle $Z$ (path $P$) is called a ***Hamiltonian cycle (Hamiltonian path)*** on $G$ iff $Z$ ($P$) contains each vertex of $G$ exactly once.
- $G(V, E)$ is called a *Hamiltonian* (*semi-Hamiltonian*) graph iff there is a Hamiltonian cycle (path) on $G$.
- The weight of $Z$ (or $P$) is defined as $w(Z) = \sum_{e \in Z} w(e)$ .

# TSP: definitions

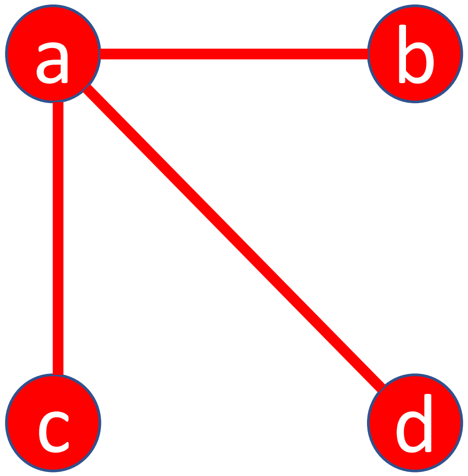



Hamiltonian graph




Semi-hamiltonian graph

Nonhamiltonian graph



# TSP: definitions

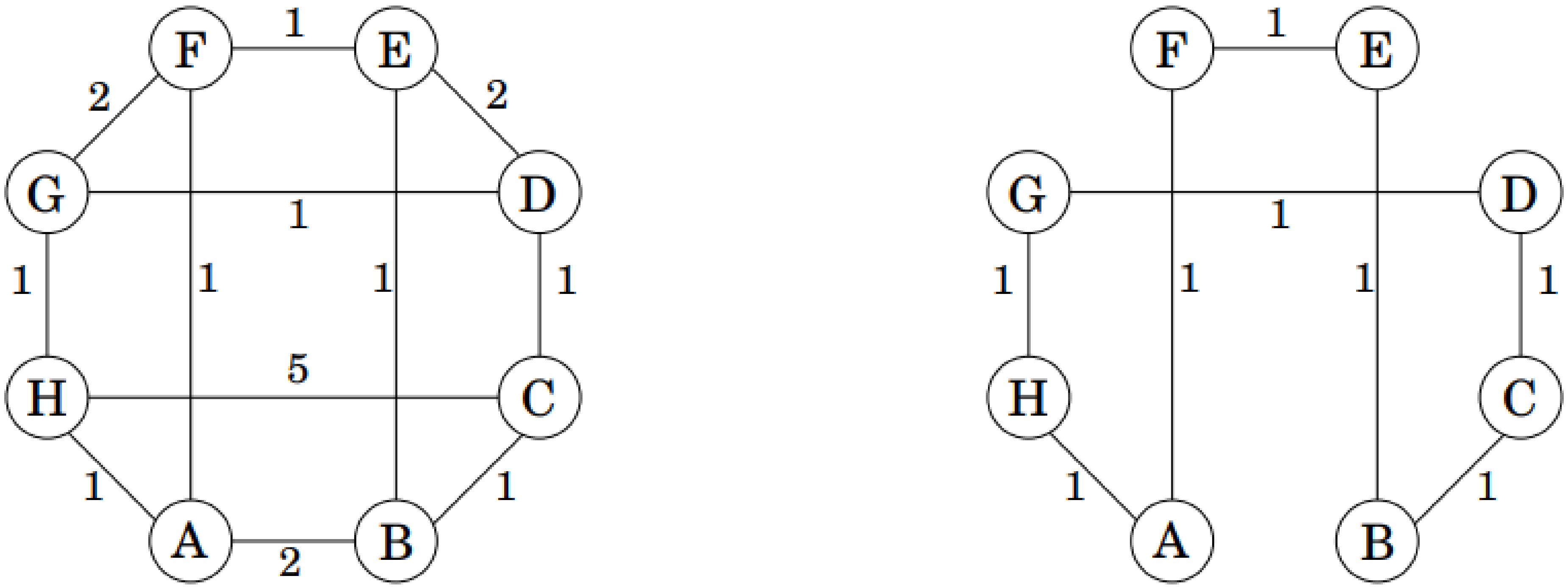
- Decision problem: is the given graph $G(V, E)$ Hamiltonian?

- Search problem: build a Hamiltonian cycle on the given graph $G(V, E)$ (return 'NULL' if $G(V, E)$ is not Hamiltonian).

- Optimization problem (=TSP): build a shortest Hamiltonian cycle on the given graph $G(V, E)$ (return 'NULL' if $G(V, E)$ is not Hamiltonian).

# TSP: definitions

A graph and its optimal Hamiltonian cycle:




http://algorithmics.lsi.upc.edu/docs/Dasgupta-Papadimitriou-Vazirani.pdf

# TSP: solving

**Theorem 1**: TSP is NP-hard.

# TSP: solving

Possible options for solving any NP-hard problem (e.g. TSP):

- Exactly but inefficiently:
  - ✓exhaustive search (brute-force, backtracking)
  - ✓smart search (branch-and-bound)
- Exactly, efficiently, but not universally:
  - ✓efficiently solvable special cases.
- Efficiently but inexactly:
  - ✓approximate algorithms,
  - ✓heuristics

# TSP: solving

**Definition**: TSP is called ***metric*** (MTSP) iff the weight function $w: E \longrightarrow R_+$ is metric.

MTSP is an important special case of TSP.

An important special case of MTSP is Euclidean TSP (ETSP): vertices are points in $R^n$ and $w$ is Euclidean distance.

# TSP: solving

**Theorem 2**: MTSP is also NP-hard.

**Theorem 3**: Even ETSP is NP-hard.

# TSP: brute force

Brute-force (exhaustive search) approach:

- Exact
- Universal
- Easily adaptable
- Very time-consuming; prohibitive time complexity even for small ($n \sim 100$) instances.

Principal idea:

1) Generate all feasible solutions.
2) For each feasible solutions calculate its cost (weight).
3) Select the best (minimum/maximum weight) feasible solution.

# TSP: brute force

For TSP, feasible solutions are Hamiltonian cycles (paths).

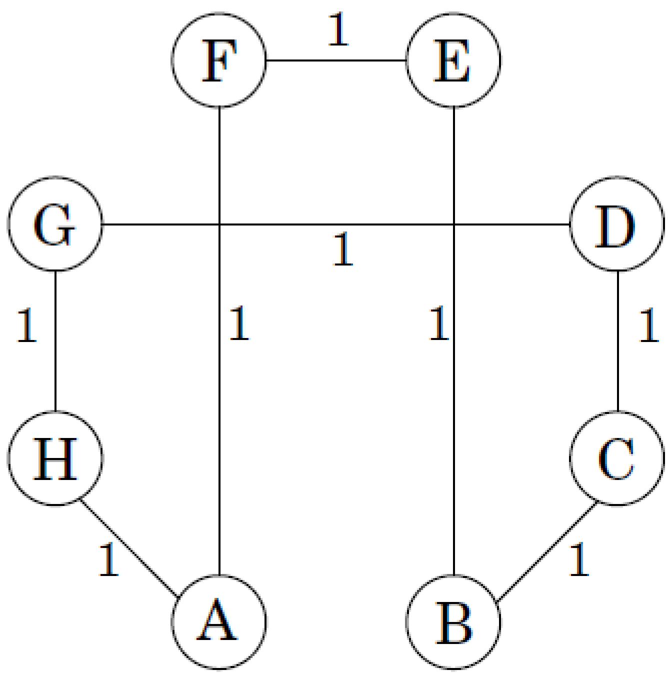
Possible representations of a Hamiltonian cycle (path):

- Vertex permutation: list the vertices in the order the cycle/path passes them.
- Edge sequence: list the edges in the order the cycle/path passes them.

Representing a Hamiltonian cycle/path as a vertex permutation is a bit easier, since we just need to check that all neighbors in the permutation are neighbors (adjacent vertices) in the graph (plus, for cycle: the last vertex is adjacent to the first one). For edge sequence representation checking validity is more complicated.

# TSP: brute force

So, we need to generate all $n!$ possible vertex permutations.




In case of *cycle* we need to generate $(n-1)!$

AFEBCDGH

FEBCDGHA

EBCDGHAF

…

For an *undirected* graph: only $\frac{(n-1)!}{2}$:

AHGDCBEF

HGDCBEFA

…

# TSP: brute force

Generating permutations [Lectures Notes on Algorithm Analysis and Computational Complexity (Fourth Edition) - Ian Parberry: http://ianparberry.com/books/free/license.html].

Problem: given positive integer $n$, generate all possible permutations of $1, \dots, n$.

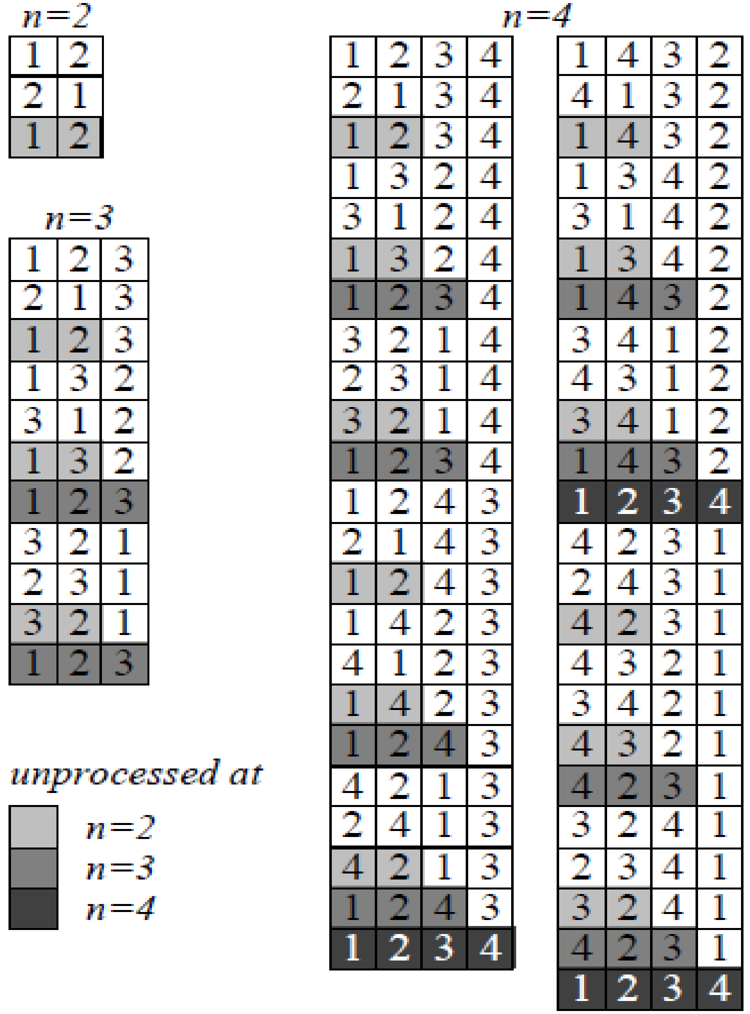
Idea of the generation algorithm:

- Create array `A[1..n]`.
- Initialization: `for each i: A[i] = i`.
- For each *k* successively swap $A[k]$ with $A[i]$ for $i = 1, \dots, k$.

# TSP: brute force

Call: `ProcessPermutations(A,k)`

```
Function ProcessPermutations(A,k)
if k = 1 then Process(A)
else
        ProcessPermutations(A, k-1);
        for i = k-1 downto 1 do
        {
          swap A[k] and A[i];
          ProcessPermutations(A, k-1);
          swap A[k] and A[i];
        }
```

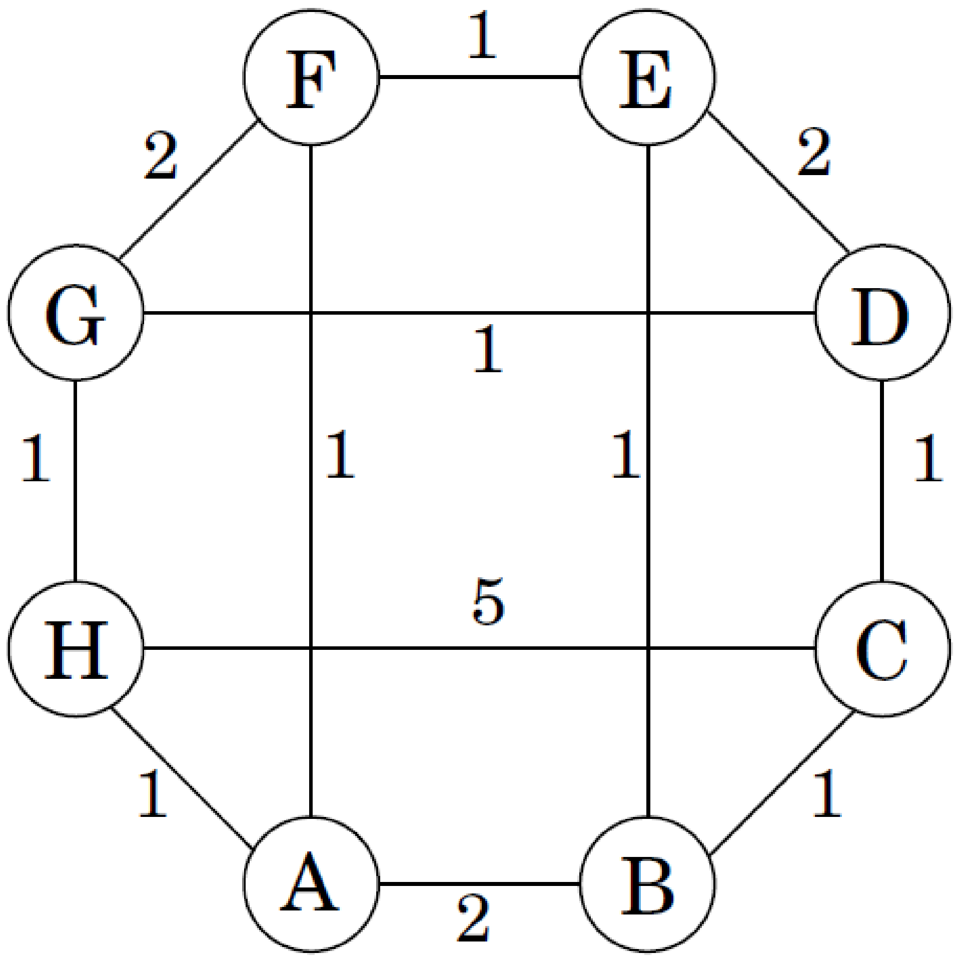


# TSP: brute force

What the procedure `Process()` is for?

- Check whether the current permutation represents a feasible solution (Hamiltonian cycle).
- If it does, yield the current feasible solution (Hamiltonian cycle), calculate its weight and compare to the current champion.

# TSP: brute force




Example:

- Generate 7! permutations, fix A as the 1st vertex.
- Permutation 'aBCDEFGH' is feasible, its weight is 11.
- Permutation 'aBCDEFHG' is not feasible because F and H are not adjacent in the graph.
- Permutation 'aFEBCHGD' is not feasible (doesn't represent a Hamiltonian cycle) because D is not adjacent to A.