Lecture 1

Set theory, probability law

Bibliography

1. Dimitri P. Bertsekas and John N. Tsitsiklis «**Introduction to Probability»**
2. Seymour Lipschutz, Marc Lipson **«Probability»**

Set theory

Probability makes extensive use of set operations**.**

A **set** is a collection of objects, which are the elements of the set. If S is a set and x is an element of S, we write $x \epsilon S$. If x is not an element of S, we write $x \notin S$.

A set can have no elements, in which case it is called the **empty set** ∅.
$\Omega - \boldsymbol{universal\ set}$

The complement of a set S, with respect to the universe Ω, is the set $S^{c} = \{x \in \Omega | x \notin S\}$.

The union of two sets S and T is the set of all elements that belong to S or T (or both), and is denoted by $S \cup T = \{x | x \in S \text{ или } x \in T\}$.

The intersection of two sets S and T is the set of all elements that belong to both S and T, and is denoted by $S \cap T = \{x | x \in S \text{ и } x \in T\}$.

The difference of two sets S and T is a set consisting of elements that belong to the set S and do not belong to the set T: $S \backslash T = \{x | x \in S \text{ и } x \notin T\}$

The algebra of sets

$$S \cup T = T \cup S.$$
$$S \cup (T \cup U) = (S \cup T) \cup U,$$
$$S \cap (T \cup U) = (S \cap T) \cup (S \cap U).$$
$$S \cup (T \cap U) = (S \cup T) \cap (S \cup U).$$
$$(S^c)^c = S.$$
$$S \cap S^c = \emptyset.$$
$$S \cup \Omega = \Omega.$$
$$S \cap \Omega = S.$$

De Morgan's laws:

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c, \qquad \left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c.$$

Examples

1. On the Venn diagram (let all possible intersections be present), select the sets:

-$A \cap (B \cup C)$; -$A^C \cap (B \cup C)$; -$(A \cup C) \cap (B \cup C)$

2. Prove the law of distributivity $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (formally by parsing boolean expressions and with Venn diagram)

3. There are 30 students on list A and 35 students on list B. 20 students are on both list A and list B.

Define:

-number of students only in list A

-number of students only in list B

-number of students in List A or List B

-number of students contained in exactly one list

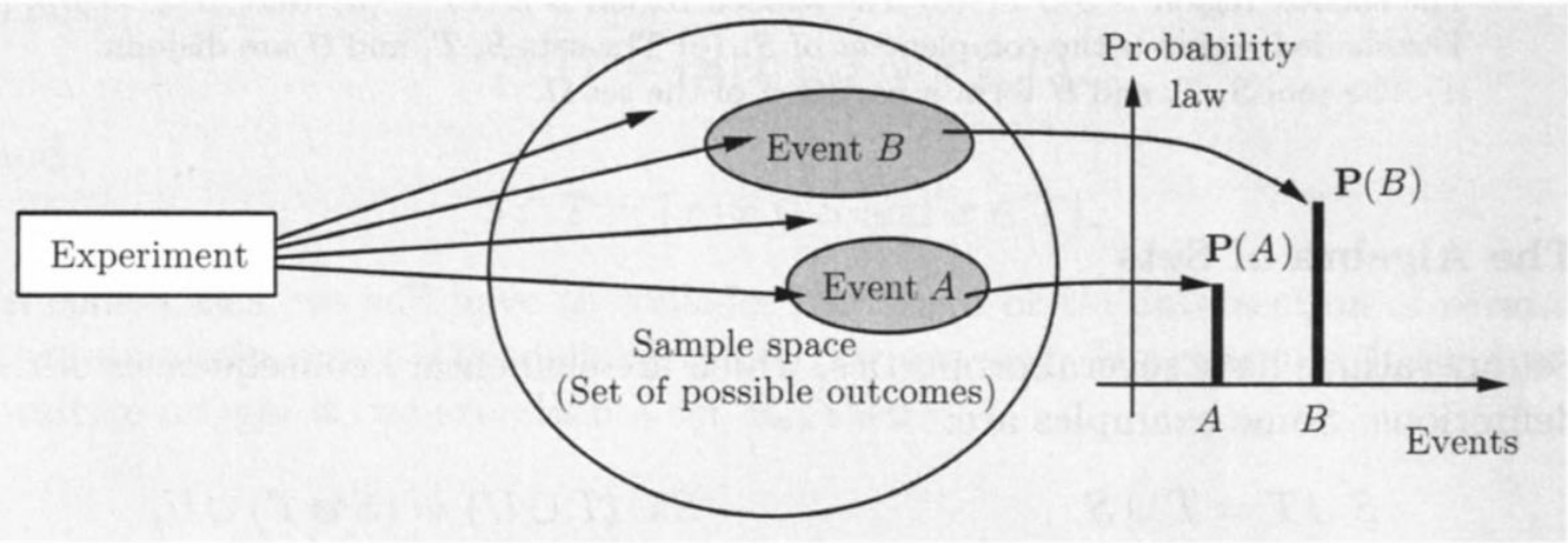
Probabilistic models

A probabilistic model is a mathematical description of an uncertain situation. It must be in accordance with a fundamental framework. Its two main ingredients are listed below and are visualized

Elements of a Probabilistic Model

- The **sample space** Ω, which is the set of all possible **outcomes** of an experiment.
- The **probability law**, which assigns to a set A of possible outcomes (also called an **event**) a nonnegative number $\mathbf{P}(A)$ (called the **probability** of A) that encodes our knowledge or belief about the collective "likelihood" of the elements of A. The probability law must satisfy certain properties to be introduced shortly.

Probabilistic models



The main ingredients of probabilistic model

Axioms of probability

Let S be a sample space, let Ω be the class of all events, and let P be a real-valued function defined on Ω. Then P is called a *probability function*, and $P(A)$ is called the *probability* of the event A, when the following axioms hold:

- [**P**1] For any event A, we have $P(A) \geq 0$.
- [**P**2] For the certain event S, we have $P(S) = 1$.
- [**P**3] For any two disjoint events A and B, we have $P(A \cup B) = P(A) + P(B)$
- [**P**3] For any infinite sequence of mutually disjoint events $A1$, $A2$, $A3$, . . ., we have

$$P(A1 \cup A2 \cup A3 \cup \cdots) = P(A1) + P(A2) + P(A3) + \cdots$$

Furthermore, when P does satisfy the above axioms, the sample space S will be called a *probability space*.

Theorems of probability spaces

Theorem 1: The impossible event or, in other words, the empty set has probability zero, that is, $P(\emptyset) = 0$

Proof: For any event A, we have $A \cup \emptyset = A$ where A and $\emptyset$ are disjoint. By [**P**3], $\mathrm{P(A)} = \mathrm{P}(A \cup \emptyset) = P(A) + P(\emptyset)$. Adding $P(A)$ to both sides gives $P(\emptyset) = 0$.

Theorem 2 (Complement Rule): For any event A, we have $P(A^c) = 1 - P(A)$

Proof: $S = A \cup A^c$ where A and A^c are disjoint. By [**P**2], $P(S) = 1$. Thus, by [**P**3],

$$1 = P(S) = \mathrm{P}(A \cup A^c) = P(A) + \mathrm{P}(A^c)$$

Adding -$P(A)$ to both sides gives $P(A^c) = 1 - P(A)$

Theorem 3: For any event A, we have $0 \leq P(A) \leq 1$.

Proof: By [**P**1], $P(A) \geq 0$. Hence we need only show that $P(A) \leq 1$. Since $S = A \cup A^c$ where A and A^c are disjoint, we get

$$1 = P(S) = P(A \cup A^c) = P(A + P(A^c)$$

Adding $-P(A^c)$ to both sides gives us $P(A) = 1 - P(A^c)$. Since $P(A^c) \geq 0$, we get $P(A) \leq 1$, as required.

Theorems of probability spaces

Theorem 4: If $A \subseteq B$, then $P(A) \leq P(B)$.

Proof: If $A \subseteq B$, then $B = A \cup (B \backslash A)$ where A and B A are disjoint. Hence $P(B) = P(A) + P(B \backslash A)$. By [**P**1], we have $P(B \backslash A) \geq 0$; hence $P(A) \leq P(B)$.

Theorem 5: For any two events A and B, we have $P(A \backslash \mathrm{B}) = P(A) - P(A \cap B)$

Proof: $\mathrm{A} = (A \backslash \mathrm{B}) \cup (A \cap B)$ where $A \backslash \mathrm{B}$ and $A \cap B$ are disjoint. By [**P**3] we have $\mathrm{P(A)} = P(A \backslash \mathrm{B}) + P(A \cap B)$

Theorem (Addition Rule) 6: For any two events A and B,

$$\mathrm{P}(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: $A \cup B = (A \backslash \mathrm{B}) \cup B$ where $A \backslash \mathrm{B}$ and B are disjoint sets. Thus, using Theorem 5,

$$\mathrm{P}(A \cup B) = P(A \backslash \mathrm{B}) + P(B) = P(A) - P(A \cap B) + P(B) = P(A) + P(B) - P(A \cap B)$$

Discrete uniform probability law

Discrete Uniform Probability Law

If the sample space consists of n possible outcomes which are equally likely (i.e., all single-element events have the same probability), then the probability of any event A is given by

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{n}.$$

Discrete uniform probability law. Example 1

Consider the experiment of rolling a pair of 4-sided dice. We assume the dice are fair, and we interpret this assumption to mean that each of the sixteen possible outcomes has the same probability of $\frac{1}{16}$. To calculate the probability of an event, we must count the number of elements of the event and divide by 16.

Here are some event probabilities calculated in this way:

$$\mathbf{P}\big(\{\text{the sum of the rolls is even}\}\big) = 8/16 = 1/2,$$

$$\mathbf{P}\big(\{\text{the sum of the rolls is odd}\}\big) = 8/16 = 1/2,$$

$$\mathbf{P}\big(\{\text{the first roll is equal to the second}\}\big) = 4/16 = 1/4,$$

$$\mathbf{P}\big(\{\text{the first roll is larger than the second}\}\big) = 6/16 = 3/8,$$

$$\mathbf{P}\big(\{\text{at least one roll is equal to 4}\}\big) = 7/16.$$

Practical lesson

Examples

1. Prove the De Morgan's law: $(A \cup B)^c = A^c \cap B^c$ (formally by parsing boolean expressions and with Venn diagram).
2. For a six-sided die, consider the events A = {the number of points is even}, B = {the number of points is greater than 3}. Find and compare sets in both sides of the equality:

$$-(A \cup B)^c = A^c \cap B^c$$
$$-(A \cap B)^c = A^c \cup B^c$$

3. 2 coins and a dice are tossed.

a) describe a suitable set of all outcomes and find the number of elements in it

b) describe the set of outcomes of events:

- -A= {2 heads rolled and even number of points};
- -B= {score rolled 2};
- -C= {dropped 1 heads and an odd number of points};
- -A and B
- -only in B
- -B and C
- -A but not B

Examples

4. Give examples of sets for which $A \cap B = A \cap C, but\ B \neq C$; $A \cup B = A \cup C, but\ B \neq C$.

5. A two-digit number is conceived. Find the probability that the conceived number will be:

a) a randomly named two-digit number;

b) a randomly named two-digit number, the digits of which are different.

6. A cube, all sides of which are painted, is sawn into a thousand cubes of the same size, which are then thoroughly mixed. Find the probability that a randomly drawn cube has colored faces:

a) one;

b) two;

c) three.

Homework

1. Prove the De Morgan's law: $(A \cap B)^c = A^c \cup B^c$(formally by parsing boolean expressions and with Venn diagram).
2. There are 50 students on the course who study German or French or do not study either language. There are also students who do not study any language and study two languages at once. It is known that 25 students study French, 20 study German, 5 students study both languages. Determine the number of students who study:
   - French only
   - do not learn German
   - learn French or German
   - do not learn any language

3. Two dice are thrown. Find the probabilities of the following events:

a) the sum of the rolled points is equal to seven;

b) the sum of the dropped points is equal to eight, and the difference is four;

c) the sum of the dropped points is equal to eight, if it is known that their difference is equal to four;

d) the sum of the dropped points is five, and the product is four.

Homework

4. 2 coins and a dice are tossed.

a) describe a suitable set of all outcomes and find the number of elements in it
b) describe the set of outcomes of events:
   - -A= {only one head rolled and number of points more than 2};
   - -B= {only one tail rolled};
   - -C= {dropped 2 heads and an even number of points};
   - -A and B
   - -A or B
   - -only in C
   - -A and C
   - -C but not B