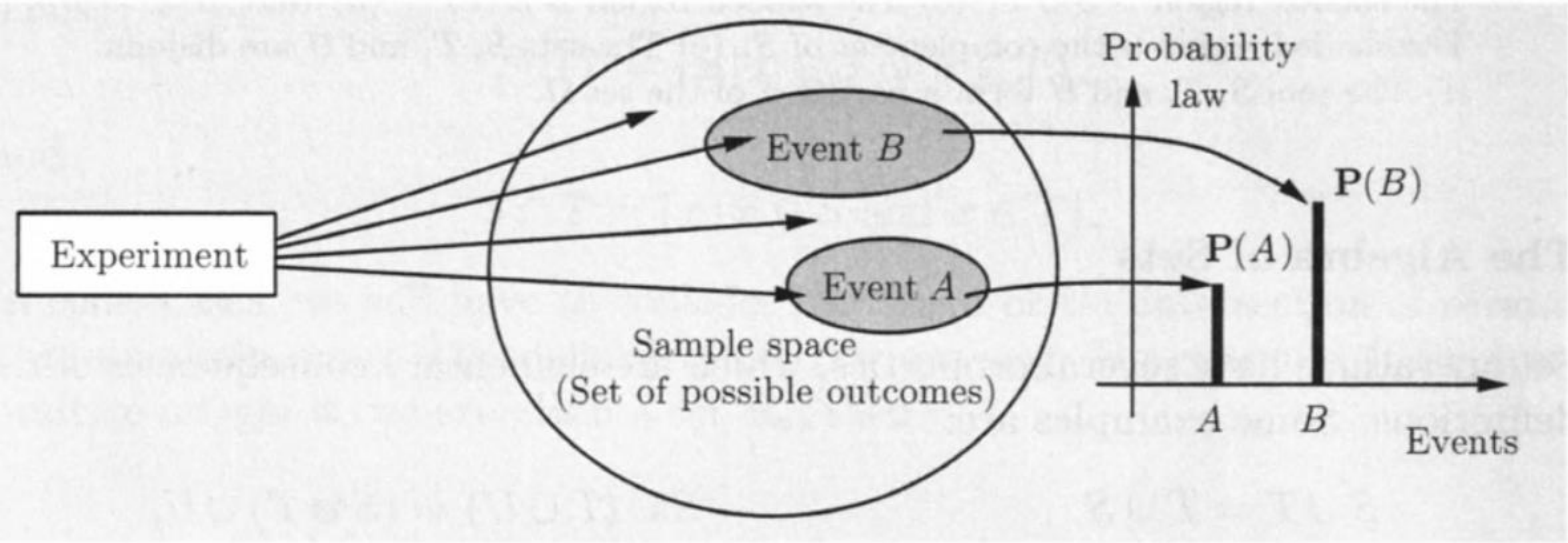
Lecture 2

Probabilistic models, enumerative combinatorics, conditional probability, probability multiplication theorem, independent events, theorem of addition of joint event probabilities

Probabilistic models



The main ingredients of probabilistic model

Discrete uniform probability law

Discrete Uniform Probability Law

If the sample space consists of n possible outcomes which are equally likely (i.e., all single-element events have the same probability), then the probability of any event A is given by

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{n}.$$

Examples

1. A two-digit number is conceived. Find the probability that the conceived number will be:

a) a randomly named two-digit number;

b) a randomly named two-digit number, the digits of which are different.

2. A cube, all sides of which are painted, is sawn into a thousand cubes of the same size, which are then thoroughly mixed. Find the probability that a randomly drawn cube has colored faces:

a) one;

b) two;

c) three.

Examples

3. The box contains 15 details, 10 of which are painted. The assembler draws three details at random. Find the probability that the extracted details will be painted.
4. The shop employs six men and four women. Seven people were randomly selected by personnel numbers. Find the probability that there will be three women among the selected persons.

Enumerative combinatorics

- **Enumerative combinatorics** is an area of combinatorics that deals with the number of ways that certain patterns can be formed. Two examples of this type of problem are **counting combinations** and **counting permutations**.
- **Sum rule.** If two actions A and B are mutually exclusive, and action A can be performed in m ways, and B in n ways, then any one of these actions (either A or B) can be performed in n + m ways.

Example. There are 16 boys and 10 girls in the class. In how many ways can one attendant be appointed?

Enumerative combinatorics

- **Fundamental counting principle.** Let it be required to perform successively k actions. If the first action can be done in n_1 ways, the second action in n_2 ways and so on up to the k -th action, that can be done in n_k ways, then all k actions together can be done:

$$N = n_1 \cdot n_2 \cdot \cdots \cdot n_k$$

Example. There are 16 boys and 10 girls in the class. In how many ways can two attendants be appointed?

Enumerative combinatorics

- **Permutation without repetition -** combinatorial compounds, which can differ from each other only in the order of their elements. Total number of permutations without repetitions:

$$\boldsymbol{P(n) = n!}$$

- **Permutation with repetition** – is an ordered selection with repetitions in which element a_1 is repeated n_1 times, element a_2 is repeated n_2 times, and so on, until the last element a_s, is repeated n_s times. Total number of permutations with repetitions:

$$\boldsymbol{P(n, n_1, \ldots, n_s) = \frac{n!}{n_1!\, n_2! \ldots n_s!}}$$

Enumerative combinatorics

Example 1.

a) Find the number of words that can be obtained by permutation the letters of the word “HORSE”.

$$P(n) = n!; P(5) = 5! = 120$$

б) Find the number of words that can be obtained by permutation the letters of the word “FINANCE”.

$$P(n, n_1, \dots, n_s) = \frac{n!}{n_1! n_2! \dots n_s!}; P(7,2) = \frac{7!}{2!} = 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 2520$$

Enumerative combinatorics

- **Selection with return.** The number of distinct samples returning from n elements by k is $\boldsymbol{n^k}$.
- **Arrangement without repetitions (order is important).** The number of different samples without returning from n elements by k is $\boldsymbol{A_n^k} = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{k})!}$.
- **Combination without repetitions (order is not important).** The number of combinations of n elements by k is equal to $\mathrm{C}_{\boldsymbol{n}}^{\boldsymbol{k}} = \frac{\boldsymbol{n}!}{\boldsymbol{k}!(\boldsymbol{n}-\boldsymbol{k})!}$

Enumerative combinatorics

Example 2. From a deck of 52 cards, 3 cards are drawn at random. Find the number of ways this can be done if the cards are drawn

a) with a return to the deck

b) without returning to the deck

a) The number of distinct samples from n elements by k with return is n^k. $52^3 = 140608$

b) The number of distinct samples from n elements by k without return is $A_n^k = \frac{n!}{(n-k)!}$.

$$A_{52}^3 = \frac{52!}{(52-3)!} = \frac{52!}{49!} = 52 \cdot 51 \cdot 50 = 132600$$

Example 3. Determine the number of groups of 3 people that can be formed from 8 people

$$C_n^k = \frac{n!}{k!(n-k)!}; \ C_8^3 = \frac{8!}{3!(8-3)!} = 56$$

Conditional probability

The conditional probability is the probability that one event will occur given that another event has already occurred.

It is known that event B has occurred. Knowing this, calculate the probability of event A. The conditional probability of event A relative to event B is calculated by the formula:

$$\boldsymbol{P(A|B) = \frac{P(AB)}{P(B)}, P(B) \neq 0}$$

Example. 2 fair dice are tossed in turn. What is the probability that the sum of the points is 10 if it is known that the number 5 fell on one die.

$$\boldsymbol{B = \{a_1 = 5\}, A = \{a_1 + a_2 = 10\}}$$

$$\boldsymbol{P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\{a_1=5, a_2=5\})}{P(\{a_1=5\})} = \frac{1}{36} : \frac{1}{6} = \frac{1}{6}}$$

Probability multiplication theorem

Theorem. The probability of the joint occurrence of two events A and B is equal to the product of the probability of one of them by the conditional probability of the other, calculated under the condition that the first event has already occurred:

$$\boldsymbol{P(AB) = P(A) \cdot P(B|A)}$$

Proof: by definition of conditional probability $P(B|A) = \frac{P(AB)}{P(A)}$, so $P(AB) = P(A)P(A|B)$

Corollary. Consider an event A that occurs if and only if each of the events $A_1, \dots, A_n$ happened, that is $A = A_1 A_2 \dots A_n$, so

$$\boldsymbol{P(A_1 A_2 \dots A_n) = P(A_1) P(A_2|A_1) \dots P(A_n|A_1 A_2 \dots A_{n-1})}$$

Probability multiplication theorem

Example. There is a deck of 36 cards. 3 cards are drawn consecutively. What is the probability that there are no spades among these cards?

Consider events : $A_1 = \{the\ first\ card\ is\ not\ a\ spade\}$;

$A_2 = \{the\ second\ card\ is\ not\ a\ spade\}$; $A_3 = \{the\ third\ card\ is\ not\ a\ spade\}$. Then, according to the condition of the problem, one should find $P(A_1A_2A_3)$. According to the probability multiplication theorem, we have: $P(A_1A_2A_3) = \mathrm{P}(A_1)P(A_2|A_1)P(A_3|A_1A_2)$.

$\mathrm{P}(A_1) = \frac{27}{36}$; $P(A_2|A_1) = \frac{26}{35}$; $P(A_3|A_1A_2) = \frac{25}{34}$

$$P(A_1A_2A_3) \approx 0{,}41$$

Independent events

Two events are said to be independent if the occurrence of one of them makes it neither more nor less probable that the other occurs.

An event B is called **independent** of the event A if the occurrence of the event A does not change the probability of the event B, that is, if the conditional probability of the occurrence of the event B is equal to its unconditional probability:

$$P(B|A) = P(B)$$

If event B does not depend on A, then event A does not depend on B.

For independent events, the probability multiplication theorem has the form:

$$\boldsymbol{P(AB) = P(A)P(B)}$$

Two events are called independent if the probability of their joint occurrence is equal to the product of the probabilities of these events, otherwise the events are called dependent.

Example. Find the probability of joint hitting the target by two guns, if the probability of hitting the target by the first gun (event A) is 0.8, the probability of hitting the target by the second gun (event B) is 0.7.

Tasks

Tasks

1. There are five identical products in the box, and three of them are painted. Two items were retrieved at random. Find the probability that among the two items taken out there will be: a) one painted item; b) two painted products; c) at least one painted product.
2. There are 100 details in a box, 10 of them are defective. Four details were randomly extracted. Find the probability that among the extracted details: a) there are no defective ones; b) no good ones.

Tasks

3. Find the number of words made up of the letters of the word:

-FATHER

-MATHEMATICS

4. Two boys are taking part in a tennis tournament. The winner is the one who wins 2 games in a row or 3 games in total. In how many ways can the tournament be played? (use tree diagram)

5. There are 8 boys and 6 girls in the class. Determine the number of ways to choose:

a) 1 person from the class;

b) 1 boy and one girl;

c) 1 president and 1 vice president

Tasks

6. A farmer buys 3 cows, 2 pigs, 4 ducks from a man who has 6 cows, 5 pigs, 8 ducks. How many ways are there to make this purchase?

7. There are 7 blue and 5 red balls in a box. Determine the number of ways: a) choose 2 balls;

b) choose 2 balls of the same color

8. Mom has 2 apples and 3 pears. Every day for 5 days in a row, she gives out one piece of fruit. In how many ways can this be done?

Homework

1. There are 12 married couples at the party. Find the number of ways to choose 1 man and 1 woman so that they are not married.
2. Determine the number of ways in which 9 toys can be divided among 4 children, if all children need to be given 2 toys, and the youngest - 3.
3. There are 10 people in the class - 6 boys and 4 girls. Determine the number of ways you can choose: a) 4 people from the class; b) 2 boys and 2 girls; c) President, Vice President, Treasurer and Secretary
4. Determine the number of different signals, each of which consists of 6 vertically arranged flags, which can be made up of 4 identical red flags and 2 identical blue flags.
5. There are 10 white and 5 black balls. In how many ways can 7 balls be chosen so that 3 of them are black?
6. There are 12 students in the group, including 8 excellent students. 9 students were randomly selected from the list. Find the probability that there are five excellent students among the selected students.
7. There are 10 details in a box, four of which are painted. The man randomly took three details. Find the probability that at least one of the selected parts is colored.