Lecture 3

Conditional probability, probability multiplication theorem, independent events

Conditional probability

The conditional probability is the probability that one event will occur given that another event has already occurred.

It is known that event B has occurred. Knowing this, calculate the probability of event A. The conditional probability of event A relative to event B is calculated by the formula:

$$\boldsymbol{P(A|B) = \frac{P(AB)}{P(B)}, P(B) \neq 0}$$

Example. 2 fair dice are tossed in turn. What is the probability that the sum of the points is 10 if it is known that the number 5 fell on one die.

$$\boldsymbol{B = \{a_1 = 5\}, A = \{a_1 + a_2 = 10\}}$$

$$\boldsymbol{P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\{a_1=5, a_2=5\})}{P(\{a_1=5\})} = \frac{1}{36} : \frac{1}{6} = \frac{1}{6}}$$

Probability multiplication theorem

Theorem. The probability of the joint occurrence of two events A and B is equal to the product of the probability of one of them by the conditional probability of the other, calculated under the condition that the first event has already occurred:

$$\boldsymbol{P(AB) = P(A) \cdot P(B|A)}$$

Proof: by definition of conditional probability $P(B|A) = \frac{P(AB)}{P(A)}$, so $P(AB) = P(A)P(A|B)$

Corollary. Consider an event A that occurs if and only if each of the events $A_1, \dots, A_n$ happened, that is $A = A_1 A_2 \dots A_n$, so

$$\boldsymbol{P(A_1 A_2 \dots A_n) = P(A_1) P(A_2|A_1) \dots P(A_n|A_1 A_2 \dots A_{n-1})}$$

Probability multiplication theorem

Example. There is a deck of 36 cards. 3 cards are drawn consecutively. What is the probability that there are no spades among these cards?

Consider events : $A_1 = \{the\ first\ card\ is\ not\ a\ spade\}$;

$A_2 = \{the\ second\ card\ is\ not\ a\ spade\}$; $A_3 = \{the\ third\ card\ is\ not\ a\ spade\}$. Then, according to the condition of the problem, one should find $P(A_1A_2A_3)$. According to the probability multiplication theorem, we have: $P(A_1A_2A_3) = \mathrm{P}(A_1)P(A_2|A_1)P(A_3|A_1A_2)$.

$\mathrm{P}(A_1) = \frac{27}{36}$; $P(A_2|A_1) = \frac{26}{35}$; $P(A_3|A_1A_2) = \frac{25}{34}$

$$P(A_1A_2A_3) \approx 0{,}41$$

Independent events

Corollary. The probability of the joint occurrence of several events that are independent in the aggregate is equal to the product of the probabilities of these events:

$$\boldsymbol{P(A_1A_2 \ldots A_n) = P(A_1)P(A_2) \ldots P(A_n)}$$

Proof: Consider 3 events A, B, C. $P(ABC) = P(AB \cdot C)$. The independence of the events A,B,C implies the independence of AB and C, as well as A and B. We use the probability multiplication theorem for two independent events:

$$P(AB \cdot C) = P(AB)P(C) \text{ и } P(AB) = P(A)P(B)$$

Finally we get: $P(ABC) = P(A)P(B)P(C)$

Example. There are 3 boxes containing 10 details each. The first box contains 8, the second box 7 and the third box 9 standard details. One item is drawn at random from each box. Find the probability that all the details taken out are standard.

Examples

Consider an example of the joint application of the addition and multiplication theorems.

Example. Probabilities of occurrence of each of three independent events A_1, A_2, A_3 respectively equal p_1, p_2, p_3. Find the probability of occurrence of only one of these events.

Example. The printing house has 4 flatbed printing presses. For each machine, the probability that it is running at a given time is 0.9. Find the probability that at least one machine is running at the moment.

Theorem of addition of joint event probabilities

Two events are called **joint** , if the appearance of one of them does not exclude the appearance of the other in the same experience.

Example. The event A is the appearance of four points when throwing a die, B is the appearance of an even number of points.

Theorem. The probability of the occurrence of at least one of the two joint events is equal to the sum of the probabilities of these events without the probability of their joint occurrence:

$$P(A+B) = P(A) + P(B) - P(AB)$$

Proof: event A+B will occur if one of the events occurs $A\bar{B}, \bar{A}B, AB$. According to the theorem of addition of probabilities of incompatible events: $P(A+B) = P(A\bar{B}) + P(\bar{A}B) + P(AB)$.(*)

$$P(A) = P(A\bar{B}) + P(AB) \Rightarrow P(A\bar{B}) = P(A) - P(AB)$$
$$P(B) = P(\bar{A}B) + P(AB) \Rightarrow P(\bar{A}B) = P(B) - P(AB)$$

Substitute these expressions into (*):

$$P(A+B) = P(A) - P(AB) + P(B) - P(AB) + P(AB) = P(A) + P(B) - P(AB)$$

Tasks

Tasks

1. A four-sided dice is tossed twice. Let X,Y be the results of the first and second throws, respectively. Determine P(A|B), where A={max(X,Y)=4}, B={min(X,Y)=2}.

2. To signal an accident, two independently operating signaling devices are installed. The probability that the signaling device will work in case of an accident is 0.95 for the first signaling device and 0.9 for the second one. Find the probability that only one signaling device will work in case of an accident.

3. There are 15 textbooks randomly arranged on a library shelf, 5 of them bound. The librarian takes 3 textbooks at random. Find the probability that at least one of the selected textbooks will be bound.

4. Two shooters shoot at a target. The probability of hitting the target with one shot for the first shooter is 0.7, and for the second - 0.8. Find the probability that in one shot only one of the shooters hits the target.

Tasks

5. The device consists of three elements that work independently. The probabilities of no-failure operation (during time t) of the first, second and third elements, respectively, are equal to 0.6; 0.7; 0.8. Find the probabilities that during time t the following will work without fail: a) only one element; b) only two elements; c) all three elements.

Homework

1. There are 10 balls in a box, 4 of which are colored. The boy took 3 balls at random. Find the probability that at least one of the balls taken is colored.
2. The probability of one hit on the target with one volley of two guns is 0.38. Find the probability of hitting the target with one shot by the first of the guns, if it is known that for the second gun this probability is equal to 0.8.
3. A six-sided dice is tossed twice. Let X,Y be the results of the first and second throws, respectively. Determine P(A|B) and P(B|A), where A={max(X,Y)=6}, B={min(X,Y)=3}.

Homework

4. Two cards are drawn one after the other from a deck of 36 cards. Find the probability that the second card will be a heart if before:

a) a heart was drawn;
b) a card of a different suit was drawn.

5. Five computers were purchased for a computer laboratory, and the probability of defect for one computer is 0.1. What is the probability that it will be necessary to replace:

a) none of the computers;
b) at least one computer;
c) only one computer
d) more than two computers