Lecture 4

Total probability theorem, Bayes' rule, Bernoulli experiments, the local Moivre-Laplace theorem, integral Laplace theorem

Total probability theorem

Theorem. Probability of event *B*, which can occur only together with one of the events $A_1, \dots, A_n$, forming a complete group:

$$\boldsymbol{P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

Example. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1). 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

Let $A_i = \{playing\ with\ the\ opponent\ of\ type\ \boldsymbol{i}\}$, we have:

$$\mathrm{P}(A_1) = 0{,}5; \mathrm{P}(A_2) = 0{,}25; \mathrm{P}(A_3) = 0{,}25$$

Let $B = \{winning\}$, we have $P(B|A_1) = 0{,}3;\ P(B|A_2) = 0{,}4;\ P(B|A_3) = 0{,}5;$

Thus, by the total probability theorem, the probability of winning is:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) = 0{,}375$$

Bayes' rule

Let event B occur as a result of the implementation of one of the events $A_1, \dots, A_n$. We determine the probability that one or another hypothesis took place.

The probability of event B is determined by the total probability theorem:

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Suppose that a test was performed, as a result of which an event appeared *B.* Define the conditional probability $P(A_1|B)$. By the multiplication theorem, we get :

$$P(BA_1) = P(B)P(A_1|B) = P(A_1)P(B|A_1)$$

So $P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(B)}$, write $P(B)$ according to the total probability formula :

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{\sum_{i=1}^{n} P(B|A_1)P(A_1)}$$

Bayes' rule

Let event B occur as a result of the implementation of one of the hypotheses $A_1, \dots, A_n$. We determine the probability that one or another hypothesis took place:

$$\boldsymbol{P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}}$$

Example. Let's consider the first example. Let the player win. What is the probability that he played a game with a player of type 1?

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)} = \frac{0{,}3 \cdot 0{,}5}{0{,}375} = 0{,}4$$

Bernoulli experiments

A **Bernoulli trial** (or **binomial trial**) is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

1. The experiment is repeated a fixed number of times (n times).
2. Each trial has only two possible outcomes, “success” and “failure”. The possible outcomes are exactly the same for each trial.
3. The probability of success remains the same for each trial. We use p for the probability of success (on each trial) and $q = 1 - p$ for the probability of failure.
4. The trials are independent (the outcome of previous trials has no influence on the outcome of the next trial).

Bernoulli experiments

Let n independent trials be performed, in each of which event A can either appear with probability p or not appear with probability $q = 1 - p$.Let us determine the probability that in a series of n independent trials the event A will repeat exactly k times.

$p^k q^{n-k}$ – the probability that in n trials event A occurs k times and does not occur n−k times; such events can be C_n^k

According to the theorem of addition of probabilities of incompatible events

$$\boldsymbol{P_n(k) = C_n^k p^k q^{n-k}}$$

Bernoulli experiments

Example. Flip a coin 12 times, count the number of heads.

Here $n = 12$. Each flip is a trial. It is reasonable to assume the trials are independent.

Each trial has two outcomes heads (success) and tails (failure).

The probability of success on each trial is $p = 1/2$ and the probability of failure is $q = 1 - 1/2 = 1/2$.

This is an example of a Bernoulli Experiment with 12 trials.

Example. A shooter makes 6 shots at a target. The probability of hitting with one shot is 0.3. Find the probability that he hit 4 times.

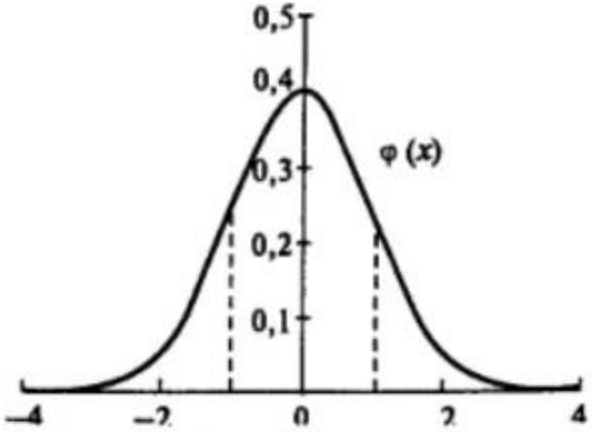
The local Moivre-Laplace theorem

Let n be big enough.

Theorem. If the probability p of the occurrence of the event A in each trial is constant and different from zero and one, then the probability $P_n(k)$ that the event A in n trials will occur k times is approximately equal to the value of the function $y = \frac{1}{\sqrt{npq}}\varphi(x)$ where $x = \frac{k-np}{\sqrt{npq}}$, and $\varphi(x)$ – Gaussian function.

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Properties:
1. $\varphi(x) > 0$
2. $\varphi(-x) = \varphi(x)$



That is, the probability $P_n(k)$ that the event A in n trials will occur k times is approximately equal to $P_n(k) \approx \frac{1}{\sqrt{npq}}\varphi(x)$

where $x = \frac{k-np}{\sqrt{npq}}$

The local Moivre-Laplace theorem

Example. A die is tossed 500 times. What is the probability that the number 1 will appear 50 times?

Tasks

Tasks

1. Let 3 boxes X, Y, Z be given. Box X contains 10 bulbs, 4 of which are defective. Box Y contains 6 bulbs, 1 of which is defective. Box Z contains 8 bulbs, 3 of which are defective. Find the probability that a light bulb chosen at random is good. If a good bulb is chosen, what is the probability that it is from box Z?

2. The factory uses 3 machines X, Y, Z. X produces 50% of the parts, with 3% of them defective. Y produces 30% of the details, and 4% of them are defective. Z produces 20% of the details, and 5% of them are defective. What is the probability that a randomly selected detail has a defect?

3. Consider the factory from Problem 8. Let a defective detail be produced. Find the probability that it was released by machine X.

Tasks

4. The number of trucks driving along a highway on which there is a gas station is related to the number of cars driving along the same highway as 3:2. The probability that a truck will refuel is 0.1; for a car this probability is 0.2. A car drove up to a gas station to refuel. Find the probability that it is a truck.

Tasks

5. Two equal chess players play chess. Which is more likely: to win two games out of four or three games out of six (draws are not taken into account)?

6. The coin is tossed five times. Find the probability that the “head" will fall: a) less than two times; b) at least twice.

7. There are five children in the family. Find the probability that among these children: a) two boys; b) no more than two boys; c) more than two boys; d) not less than two and not more than three boys. The probability of having a boy is assumed to be 0.51.

8. The test consists of six problems, and the test is considered passed if the student has solved at least four of them. The student can solve each problem with a probability of 0.6. What is the probability that he will pass the test?

Homework

1. The box contains 12 details made in factory 1, 20 details — in factory 2 and 18 details — in factory 3. The probability that a detail manufactured at factory 1, has an excellent quality, equal to 0.9; for details manufactured at factories 2 and 3, these probabilities are respectively 0.6 and 0.9. Find the probability that a randomly selected detail will be excellent quality.
2. The car insurance company divides drivers into three classes: class A (low risk), class B (medium risk), class C (high risk). The company assumes that of all drivers insured by it, 30% are class A, 50% are class B, 20% are class C. The probability that a class A driver will have at least one car accident during the year is 0, 01; for a class B driver this probability is 0.03, and for a class C driver it is 0.1. Mr. Jones insures his car with this company and gets into a car accident within a year. What is the probability that it belongs to class A?

Homework

3. Research by psychologists has established that men and women react differently to some life circumstances. The results of the research showed that 70% of women react positively to these situations, while 40% of men react negatively to them. 15 women and 5 men filled out a questionnaire in which they reflected their attitude to the proposed situations.

a) determine the probability that a randomly selected questionnaire contains a positive reaction.
b) a randomly extracted questionnaire contains a negative reaction. What is the probability that it was filled out by a man?

Homework

4. Two equal chess players play chess. Which is more likely: to win 4 games out of 8 or 3 games out of 6 (draws are not taken into account)?

5. The coin is tossed 4 times. Find the probability that the “head" will fall:

a) less than 3 times;

b) at least twice.

6. There are five children in the family. Find the probability that among these children:
a) two boys;

b) no more than two boys;

c) more than two boys;

d) not less than two and not more than three boys.

The probability of having a boy is assumed to be 0.51.