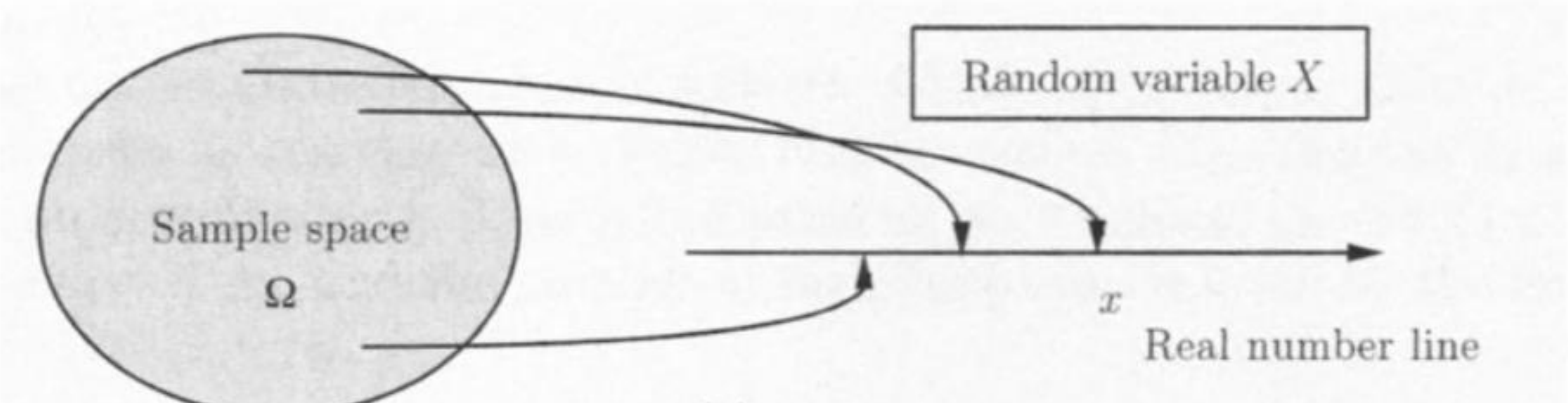
Lecture 7

Discrete random variables

Random variable

Given an experiment and the corresponding set of possible outcomes, a **random variable** associates a particular number with each outcome. We refer to this number as the numerical value or simply the value of the random variable. Mathematically, a random variable is a real-valued function of the experimental outcome.



(a)

Probability mass function

Calculation of the PMF of a Random Variable X

For each possible value x of X:

1. Collect all the possible outcomes that give rise to the event $\{X = x\}$.
2. Add their probabilities to obtain $p_X(x)$.

Example. A four-sided dice is tossed 2 times. Let $a, b -$ number of points on the first and second tosses. Find distribution for random variables $\mathrm{X} = \max\{a, b\}, Y = a + b$.

The Bernoulli random variable

Consider the toss of a coin, which comes up a head with probability p, and a tail with probability 1 - p. The Bernoulli random variable takes the two values 1 and 0, depending on whether the outcome is a head or a tail:

$$X = \begin{cases} 1, if\ a\ head \\ 0, if\ a\ tail \end{cases}$$

$$p_X(k) = \begin{cases} p, if\ k = 1 \\ 1 - p, if\ k = 0 \end{cases}$$

The binomial random variable

A coin is tossed n times. At each toss, the coin comes up a head with probability p, and a tail with probability 1 - p, independent of prior tosses. Let X be the number of heads in the n-toss sequence. We refer to X as a binomial random variable with parameters n and p. The PMF of X consists of the binomial probabilities:

$$p_X(k) = P(X = k) = C_n^k p^k q^{n-k}$$

Example. There are 10% non-standard details in the lot. Four details were selected at random. Write a binomial distribution for a discrete random variable X—the number of non-standard details.

The geometric random variable

Suppose that we repeatedly and independently toss a coin with probability of a head equal to p, where 0 < p < 1. The geometric random variable is the number X of tosses needed for a head to come up for the first time. Its PMF is given by:

$$p_X(k) = q^{k-1}p$$

Naturally, the use of coin tosses here is just to provide insight. More generally, we can interpret the geometric random variable in terms of repeated independent trials until the first "success.«

Example. The shooter fires several shots at the target until the first hit, with only 4 rounds. The probability of hitting with one shot is 0.6. Find the distribution of X.

The Poisson random variable

To get a feel for the Poisson random variable think of a binomial random variable with very small p and very large n. For example, let X be the number of typos in a book with a total of n words. Then X is binomial, but since the probability p that anyone word is misspelled is very small X can also be well-modeled with a Poisson PMF :

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \lambda = np$$

Example. The textbook was published in 100 000 copies. The probability that a textbook is bound incorrectly is 0.0001. Find the probability that the circulation contains exactly five defective books.

Expectation

Mathematical expectation is the average expected value when testing is repeated many times.
Let random variable X takes values $x_1, \dots, x_n$ with probabilities $p_1, \dots, p_n$ respectively. Then the mathematical expectation $\mathrm{E}(X)$ of a given random variable is equal to the sum of the products of all its values and the corresponding probabilities.

$$\mathrm{E}(X) = x_1 p_1 + \cdots + x_n p_n = \sum_{i=1}^{n} x_i p_i$$

Example. Consider two independent coin tosses, each with a 3/4 probability of a head, and let X be the number of heads obtained. Find the mathematical expectation of X.

Properties of expectation:

1. $\mathrm{E}(C) = C$
2. $\mathrm{E}(CX) = CE(X)$
3. $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$, if X and Y are independent
4. $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$

Expectation

1. Expectation of binomial random variable:

$$\mathbf{E}(\boldsymbol{X}) = \boldsymbol{np}$$

Proof: let $X_i, i = 1, \dots, n -$ the number of occurrences of the event in the i-th trial, then $X = X_1 + X_2 + \cdots + X_n$. $\mathrm{E}(X) = E(X_1) + E(X_2) + \cdots + E(X_n)$, $E(X_1) = E(X_2) = \cdots = E(X_n) = p \Rightarrow \mathbf{E}(\boldsymbol{X}) = \boldsymbol{np}$

2. Expectation of Poisson random variable:

$$\mathbf{E}(\boldsymbol{X}) = \boldsymbol{\lambda}$$

3. Expectation of geometric random variable :

$$\mathbf{E}(\boldsymbol{X}) = \frac{\mathbf{1}}{\boldsymbol{p}}$$

Variance

Variance allows you to estimate how far the values of a random variable are scattered relative to the expectation.

$$var(X) = E[(X - E(X))^2]$$

Standard deviation $\sigma = \sqrt{var(X)}$.

If we deviate from the mathematical expectation to the right and to the left by the standard deviation, then we get the interval on which the most probable values of the random variable will be concentrated.

Theorem. $\boldsymbol{var(X) = E\left(X^2\right) - (E(X))^2}$

Properties:

1. $\boldsymbol{var(\mathrm{C}) = 0}$
2. $\boldsymbol{var(\mathrm{C}X) = C^2 var(X)}$
3. $\boldsymbol{var(X + Y) = var(X) + var(Y)}$, **if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent**

Variance

1. Variance of binomial random variable:

$$\boldsymbol{var(X) = npq}$$

2. Variance of Poisson random variable:

$$\boldsymbol{var(X) = \lambda}$$

3. Variance of geometric random variable :

$$\boldsymbol{var(X) = \frac{q}{p^2}}$$

Example

1. There are three non-standard details in a lot of 10 details. Two details were selected at random. Find the expectation, variance and standard deviation of a discrete random variable X - the number of non-standard details among two selected.
2. There are 10% non-standard details in the lot. Four details were selected at random. Write a binomial distribution for a discrete random variable X—the number of non-standard details. Find the expectation, variance and standard deviation of a discrete random variable X.

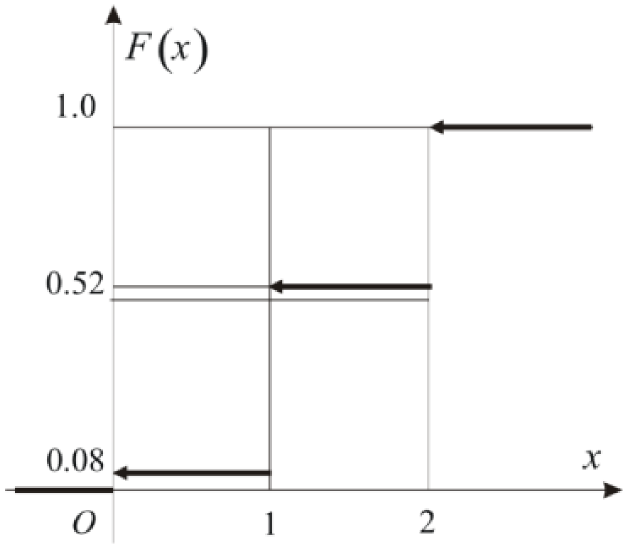
Cumulative distribution function

The cumulative distribution function, or CDF can describe all kinds of random variables with a single mathematical concept:

$$\boldsymbol{F(x) = P(X < x)}$$

Example (discrete case):

ξ	0	1	2
p	0.08	0.44	0.48



$$F(x) = \begin{cases} 0; & x \in (-\infty, 0] \\ 0.08; & x \in (0, 1] \\ 0.52; & x \in (1, 2] \\ 1; & x \in (2, \infty] \end{cases}$$

Cumulative distribution function

Properties of CDF:

1. $0 \leq F(x) \leq 1$

2. $F(x) -$ non-decreasing function

Proof: let $x_2 > x_1$, then $F(x_2) = P(X < x_2) = P(X < x_1) + P(x_1 \leq X < x_2) \Rightarrow$

$P(x_1 \leq X < x_2) = P(X < x_2) - P(X < x_1)$, and $P(x_1 \leq X < x_2) = F(x_2) - F(x_1) \geq 0$

3. $P(a \leq X \leq b) = F(b) - F(a)$

4. $P(X = a) = 0$

5. If the possible values of X belong to the interval (a, b), that $F(x) = 0, x \leq a$;
$F(x) = 1, x \geq b$

6. $\lim\limits_{x \to -\infty} F(x) = 0$, $\lim\limits_{x \to +\infty} F(x) = 1$

Tasks

Tasks

1. There are four standard details in a batch of six details. Three details were selected at random. Find the distribution, expectation, variance and standard deviation of a discrete random variable X - the number of standard parts among the selected. Plot CDF.
2. There are three tasks in the examination paper. The probability of a correct solution by a student of the first problem is 0.9, the second - 0.6, the third - 0.1. Find the distribution, expectation, variance and standard deviation of a number of correctly solved problems. Plot CDF.
3. Two students are taking an exam in probability theory. For one of them, the probability of passing the exam is 0.9, for the other - 0.6. Construct a distribution of X - number of students who passed the exam. Find the distribution, expectation, variance and standard deviation of a number of correctly solved problems. Plot CDF.
4. The probability that the shooter will hit the target with one shot is 0.8. The shooter is given cartridges until he misses. It is required: a) to draw up a distribution law for a discrete random variable X—the number of cartridges issued to the shooter; b) find the most probable number of cartridges issued to the shooter.

Tasks

5. The distribution of the random variable X has the form:

X	-2	1	2	5	7
p	0.3	0.1	0.1	0.3	0.2

Find its mathematical expectation, variance and standard deviation. Find the CDF of the random variable X and plot it.

6. The test consists of three questions. Each question has four possible answers, one of which is correct. Find the distribution of the number of correct answers with simple guessing, find the expectation and variance.

7. Discrete random variable X takes three possible values : $x_1 = 4$ with a probability $p_1 = 0.5$; $x_2 = 6$ with a probability $p_2 = 0.3$ and x_3 with a probability p_3. Find x_3 and p_3 knowing that E(X) = 8.

8. Find the variance of a discrete random variable X—the number of occurrences of event A in five independent trials, if the probability of occurrence of events A in each trial is 0.2.

Homework

1. There are 10% non-standard details in the lot. Four details were selected at random. Write a distribution law for a discrete random variable X—the number of non-standard details among four selected ones. Find the distribution, expectation, variance and standard deviation of a discrete random variable X. Plot CDF.
2. Write a binomial distribution law for a discrete random variable X—the number of occurrences of the “heads" in two tosses of a coin. Find the distribution, expectation, variance and standard deviation of a discrete random variable X. Plot CDF.
3. A worker maintains 3 machines, the probabilities of failure of each of which within an hour, respectively, are 0.2; 0.15; 0.1. Find a distribution, expectation, variance and standard deviation of a discrete random variable X – the number of machines that do not require repair within an hour. Plot CDF.
4. The device consists of three independently operating elements. The probability of failure of each element in one experiment is 0.1. Find the distribution, expectation of number of failed elements in one experiment.

Homework

5. Find the variance of a discrete random variable X—number of occurrences of event A in two independent tests, if the probability of occurrence of an event in these tests are the same and it is known that E(X) = 1.2.

6. The distribution of the random variable X has the form:

X	-3	-2	1	3
p	1/7	3/7	2/7	1/7

Find its mathematical expectation, variance and standard deviation. Find the CDF of the random variable X and plot it.