Lecture 9

Continuous random variables

Continuous random variable, PDF

Expectation: $E(X) = \int_{-\infty}^{+\infty} xf(x)dx$

Variance: $var(X) = E(X^2) - (E(X))^2$, where $E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx$

Standard deviation: $\sigma = \sqrt{var(X)}$

$$\boldsymbol{F(x) = \int_{-\infty}^{x} f(t)dt}$$

$$\boldsymbol{f(x) = F'(x)}$$

Example

Example. The PDF of the random variable X has the form:

$$f(x) = \begin{cases} \frac{C}{x^4}, & x \geq 2 \\ 0, & x < 2 \end{cases}$$

Find C; CDF; E(X); var(X); $\sigma\ (X)$; $P(2 < X < 3)$, plot graphs of CDF and PDF.

Example

Example. The CDF of the random variable T, the service life of the device, is known

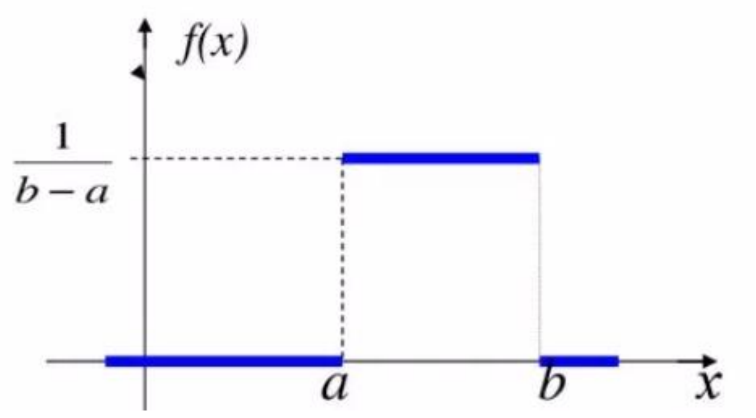
$$F(t)=\begin{cases}0, & t<0\\ kt^2, & 0\le t\le \frac{3}{4}\\ 1, & t>\frac{3}{4}\end{cases}$$

Find k, average device life, and variance.

Uniform distribution

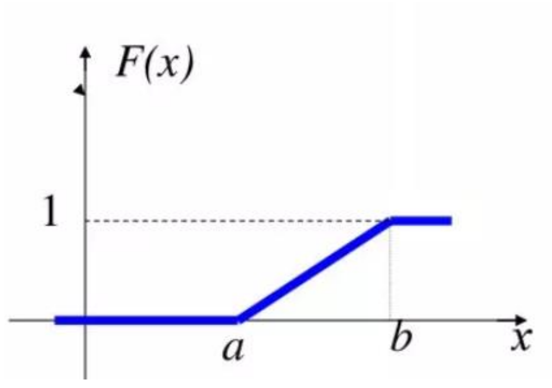
A probability distribution is called uniform if, on the interval to which all possible values of the random variable belong, the probability density function remains constant.

$$f(x) = \begin{cases} 0, & x < a \\ \dfrac{1}{b-a}, & a \le x \le b \\ 0, & x > b \end{cases}$$



$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ 1, & x > b \end{cases}$$

$$E(X) = \frac{a+b}{2}, var(X) = \frac{(b-a)^2}{12}$$



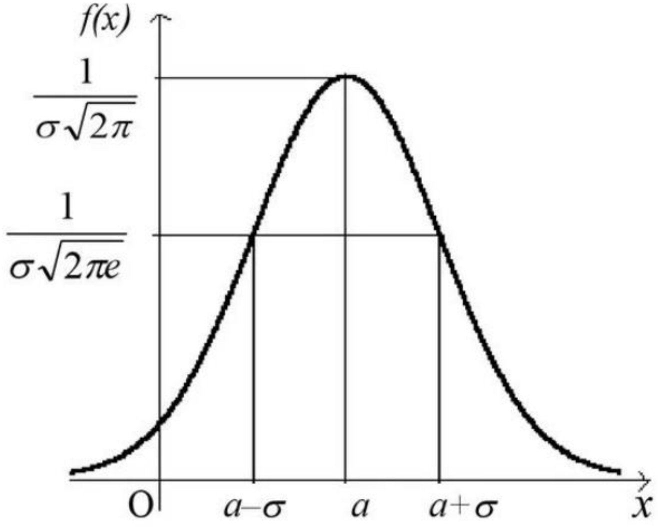
Uniform distribution

Example. Buses of a certain route run strictly according to the schedule. Movement interval 5 min. Find the probability that a passenger arriving at a stop will wait for the next bus for less than 3 minutes.

Normal distribution

A probability distribution is called normal if its PDF has the form:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$



$E(X) = a, var(X) = \sigma^2$

Central limit theorem : if X is the sum of a very large number of mutually independent random variables, the influence of each of which on the whole sum is negligible, then X has a distribution close to normal.

Normal distribution

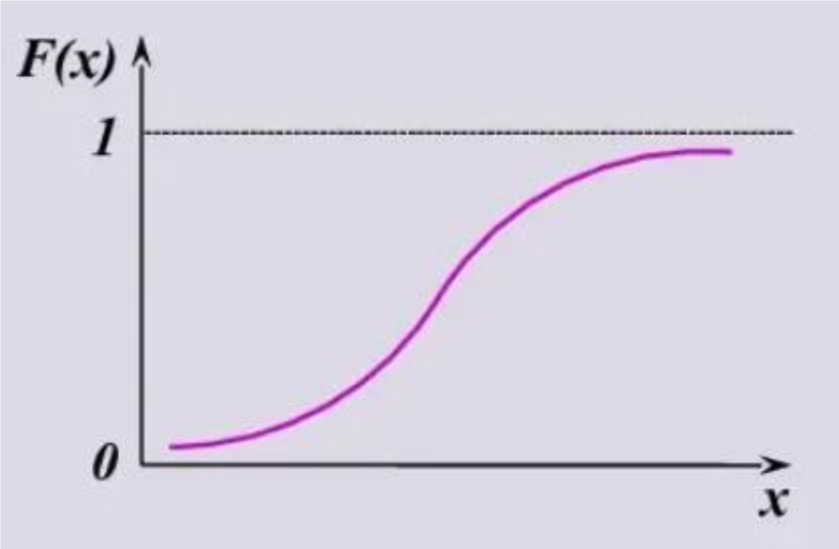
CDF:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

where

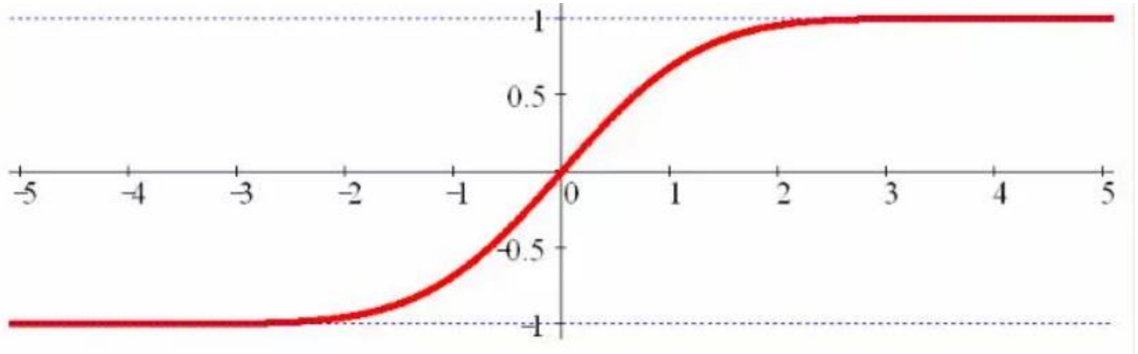
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt.$$



Normal distribution

Laplace function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-z^2/2} dz$



Probability of hitting the segment $[\alpha, \beta]$:

$$P(\alpha \le X \le \beta) = \Phi\left(\frac{\beta - a}{\sigma}\right) - \Phi\left(\frac{\alpha - a}{\sigma}\right)$$

Deviation probability :

$$P(|X - a| < \delta) = 2\Phi(\delta/\sigma)$$

Three sigma rule

Deviation probability : $P(|X-a|<\delta)=2\Phi(\delta/\sigma)$

Let $\delta=\sigma t$

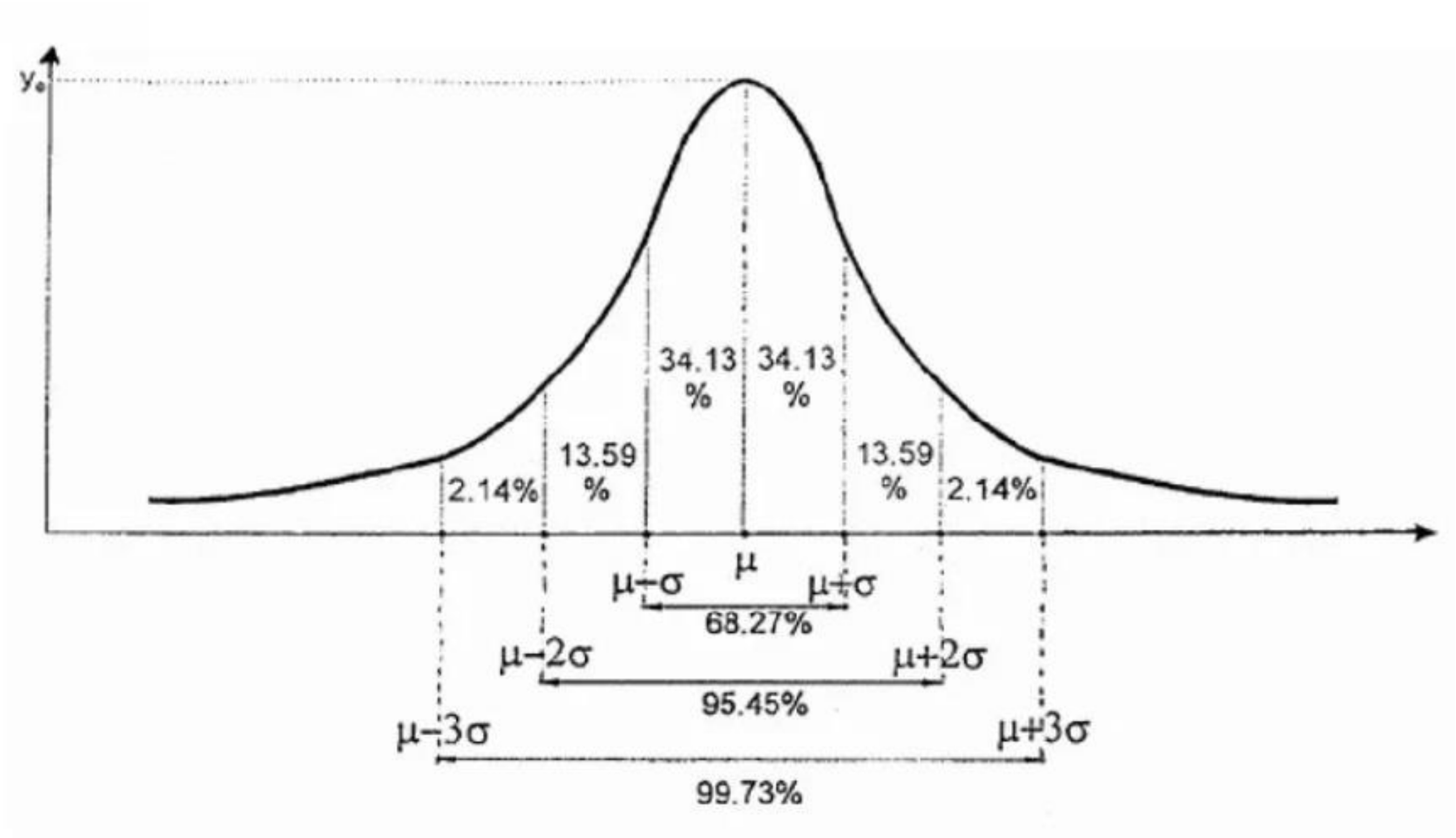
$$P(|X-a|<\sigma t)=2\Phi(t)$$

Let $t=3$

$$P(|X-a|<3\sigma)=2\Phi(3)=2\cdot 0{,}49865=0{,}9973$$

Three sigma rule: with a normal distribution, almost all values of the quantity with a probability of 0.9973 lie no further than three sigma in any direction from the mathematical expectation.

Three sigma rule



Normal distribution

Example. The machine stamps the details. The length of the detail X is controlled, which is normally distributed with the mathematical expectation (design length) equal to 50 mm. In fact, the length of the manufactured parts is not less than 32 and not more than 68 mm. Find the probability that the length of a part taken at random: a) is greater than 55 mm; b) less than 40 mm.

Tasks

Tasks

1. Find A; CDF; plot graphs of CDF and PDF; $P(3 < X < 7)$; E(X); var(X); $\sigma(X)$.

$$f(x) = \begin{cases} A(x-1), & x \in [2;4] \\ 0, & x \notin [2;4] \end{cases}$$

2. The CDF of the random variable X has the form:

$$F(x) = \begin{cases} 0, & x < 0 \\ 1-\cos x, & 0 \le x \le \frac{\pi}{2} \\ 1, & x > \frac{\pi}{2} \end{cases}.$$

Find PDF; E(X); var(X); plot graphs of CDF and PDF.

Tasks

3. All values of a uniformly distributed random variable X lie on the interval $[2;8]$. Find the probability that the random variable X falls within the interval (3;5). Find PDF, CDF(X), E(X), var(X), plot graphs of CDF and PDF.

4. Buses on a certain route run strictly according to schedule. The interval between buses is 5 minutes. Find the probability that a passenger arriving at the stop will wait less than 3 minutes for the next bus.

5. The mathematical expectation and standard deviation of a normally distributed random variable X are equal to 10 and 2, respectively. Find the probability that, as a result of the test, X will take on a value contained in the interval (12, 14).

Homework

1. The PDF of the random variable X has the form:

$$f(x) = \begin{cases} \dfrac{C}{x^4}, & x \geq 2 \\ 0, & x < 2 \end{cases}$$

Find C; E(X); $P(2 < X < 3)$.

2. Random variable X is given by the CDF:

$$F(x) = \begin{cases} 0, x \leq 0 \\ x^2, 0 < x \leq 1 \\ 1, x > 1 \end{cases}$$

Find PDF; E(X); var(X); $P(0.5 < X < 1)$; plot graphs of CDF and PDF.

Homework

3. The CDF of the random variable X has the form:

$$F(x)=\begin{cases}0, & x<0\\ \frac{1}{8}x^3, & 0\le x\le 2\\ 1, & x>2\end{cases}.$$

Find PDF; E(X); var(X); $P(1 < X < 2)$; plot graphs of CDF and PDF.

Homework

4. A random continuous variable X is defined by its PDF

$$f(x) = \begin{cases} \frac{1}{16}, & x \in (-5;11) \\ 0, & x \notin (-5;11) \end{cases}$$

a) find the CDF
b) schematically plot the graph of the functions f (x) and F(x);
c) calculate the expectation E(X) and the variance var(X);
d) determine the probability that X will take a value from the interval (1;3).

5. All values of a uniformly distributed random variable X lie on the interval (2;8) . Find the probability of the random variable X falling into the interval (3;5).

6. The mathematical expectation and standard deviation of a normally distributed random variable X are equal to 12 and 3, respectively. Find the probability that, as a result of the test, X will take on a value contained in the interval (10, 15).