Examination program
Numerical Methods of Linear Algebra for Sparse Matrices
Fall 2025

1. Basic concepts of linear algebra: basic matrix operations, transposition, inversion, determinant and its properties, eigenvalues and eigenvectors. Types of square matrices: symmetric, Hermitian, normal, unitary, orthogonal. Structures of square matrices with respect to the location of zero entries: diagonal, triangular, banded, Hessenberg, block.
2. Inner product, Eucledian inner product. Vector norms: Eucledian norm, Holder norms. Matrix norms: max column, max row, spectral, Frobenius, l1-norm, M-norm. Operator norms. Consistency of vector and matrix norms.
3. Subspace, linear independence, basis, range and kernel. Orthogonal vectors and subspaces, Gram-Schmidt orthogonalization (classic and modified). QR-factorization. Thin and full QR.
4. Eigenvalues and their multiplicities. Algebraic multiplicity, geometric multiplicity. Main definitions: simple, multiple, semisimple, defective eigenvalue. Simple, semisimple and deragotory matrix. Canonical forms of matrices: reduction to diagonal form.
5. Canonical forms of matrices: diagonal form (Eigenvalue and Singular Value Decomposition), Jordan form, Schur decomposition (Schur and quasi-Schur form). Relation between Schur and SVD, LU and Cholessky decomposition. Existence and uniqueness of each factorization.
6. Positive definite matrices. Powers of matrices. Properties of normal and Hermitian matrices.
7. Existence of a solution. Perturbation analysis, absolute and relative error. Condition number and its properties. Types of numerical errors. Well-conditioned and ill-conditioned problem. Stable and unstable algorithm. Computational costs.
8. Discretization of partial differential equations. Discretization by Finite difference method on an example of 1D and 2D Poisson's equation. Properties of the system matrix. Node numbering in 2D.
9. Graph representation of a matrix, matrix and its adjacency graph. Matrix pattern, types of patterns. Permutations and Reordering. Symmetric permutation. Fill-ins in direct solution methods. Examples of reordering algorithms.
10. Storage schemes for sparse matrices: compressed sparse row (CSR), compressed sparse column (CSC), modified CSR and CSC, diagonal format, Ellpack-Itpack format. Algorithms for basic matrix operations for sparse formats.

11. Discretization of partial differential equations. Discretization by Finite element method. Assembly process. Mesh refinement.
12. Comparison of direct and iterative methods. Overview of direct solution methods. Direct sparse methods: Gaussian elimination without and with partial pivoting. Process of computing PLU-factorization. Sparse method of Gaussian elimination with partial pivoting.
13. Classic iterative methods for linear systems. Convergence of iterative methods: necessary and sufficient conditions, convergence speed. Jacobi and Gauss-Seidel methods. Properties of diagonally dominant matrices, location of matrix eigenvalues. Convergence of Jacobi and Gauss-Seidel for diagonally dominant matrices
14. Classic iterative methods for linear systems. Convergence of iterative methods: necessary and sufficient conditions, convergence speed. Successive Over Relaxation (SOR), Symmetric Successive Over Relaxation (SSOR). Convergence of SOR and SSOR for positive definite matrices.
15. General formulation of a projection method. Search subspace and subspace of constraints. Classification of projection methods. Matrix representation of a projection method. Properties of W^HAV matrix. Example of 1D projection methods.
16. Definition of a Krylov subspace. Choice of the subspace of constraints. Arnoldi orthogonalization process (exact and modified). Properties of Arnoldi process.
17. Definition of a Krylov subspace. Derivation of Full Orthogonalization Method (FOM) for linear systems. Calculation of residual in FOM. Modifications of FOM.
18. Definition of a Krylov subspace. Derivation of Generalized Minimal Residual method (GMRES). Givens rotations. Calculation of residual in GMRES. Modifications of GMRES.
19. Definition of a Krylov subspace. Comparison of FOM and GMRES. Krylov subspace methods and polynomials. Residual polynomial, roots of residual polynomial.
20. Krylov subspace methods for symmetric linear systems. Lanczos orthogonalization process for symmetric systems. Classic Lanczos method for symmetric systems. Calculation of residual in Classic Lanczos method.
21. Krylov subspace methods for symmetric linear systems. Lanczos orthogonalization for symmetric systems. Direct Lanczos method. Properties of residuals and search directions in Direct Lanczos.
22. Krylov subspace methods for symmetric linear systems. Lanczos orthogonalization for symmetric systems. Conjugate Gradient method (CG). Properties of residuals and search directions in CG.

23. Krylov subspace methods for symmetric linear systems. Lanczos orthogonalization for symmetric systems. Generalization of CG for systems with Hermitian and nonsymmetric matrices: Conjugate Residual (CR), Generalized Conjugate Residual (GCR). Properties of residuals and search directions in CR and GCR.
24. Lanczos biorthogonalization for nonsymmetric systems. Lanczos biorthogonalization process. Properties of Lanczos biorthogonalization algorithm. Classic Lanczos method for nonsymmetric systems. Calculation of residual in Classic Lanczos method.
25. Lanczos biorthogonalization for nonsymmetric systems. Biconjugate Gradient method (BiCG). Properties of residuals and search directions in BiCG.
26. Overview of Krylov subspace methods. Definition of Krylov subspace. Classification of Krylov subspace methods. Optimal and efficient Krylov subpace methods.
27. Preconditioning technique. Types of preconditioning (left, right, split). Basic preconditioners (Jacobi, Gauss-Seidel, SOR, SSOR, incomplete LU).
28. Preconditioning technique. Types of preconditioning (left, right, split). Examples of preconditioned iterative algorithms: Preconditioned Conjugate Gradient method (PCG), Split Preconditioned Conjugate Gradient method (Split PCG), algorithms of GRMES with left and right preconditioning (LP GMRES and RP GMRES).

References

1) Lecture notes for the course
2) Yousef Saad. Iterative Methods for Sparse Linear Systems, 2nd edition. SIAM, 2003. 528 p. (available in electronic form on the course page https://edu.mmcs.sfedu.ru/mod/resource/view.php?id=22482

Additional reading

3) Gene H. Golub, Charles F. Van Loan. Matrix Computations. The Johns Hopkins University Press; 3rd edition, 1996. 728 p.
4) James W. Demmel. Applied Numerical Linear Algebra. SIAM, 1997. 184 p.
5) Ole Osterby, Zahari Zlatev. Direct Methods for Sparse Matrices. Springer-Verlag, 1983.
6) Sergio Pissanetzky. Sparse Matrix Technology. Academic Press, 1984. 312 p.