Numerical Methods of Linear Algebra for Sparse Matrices

Course for Master Degree students in Southern Federal University

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Outline

- Overview of the course: description, aims and learning outcomes
- Prerequisites and study materials, workload and assessment
- Course map
- Course structure in detail:
  - Lectures
  - Individual project
  - Practical assignments

Description of the course

- **Course title:** Numerical Methods of Linear Algebra for Sparse Matrices
- **Specialty:** MSc. in Applied Mathematics and Informatics
- **Master Degree programme:** Computational Modeling in Technology and Finance, Modern Software Development
- **Language of instruction**: English
- **Status of the subject:** major subject, compulsory module
- **Period:** one semester
- **Workload: 5 ECTS**
  - 180 hours total, including 34 hours of lectures, 52 hours of practice, 58 hours of independent study, and 36 hours to prepare for exam
  - Lectures: 2 hours per week
  - Practice: 3 hours per week

Aims of the course

- Learn effective solution methods for linear sparse systems of large and extralarge dimension
- Learn different storage schemes for sparse matrices and algorithms for basic sparse matrix operations
- Study direct and **iterative solution methods** for linear systems with sparse matrices
  - Direct solution methods
  - Projection methods
  - **Krylov subspace methods**
- Understand preconditioning techniques and use different types of preconditioners

Learning outcomes: knowledge

On successful completion of the course, students are expected to expected to have the following knowledge, skills and abilities:

- Knowledge of
  - main sparse storage formats for large sparse matrices;
  - direct solution methods for large sparse linear systems;
  - classic iterative and projection solution methods for linear systems;
  - Krylov subspace solution methods for large sparse linear systems;
  - Preconditioning methods

Learning outcomes: skills

On successful completion of the course, students are expected to expected to have the following knowledge, skills and abilities:

- Skills
  - applying sparse matrix technology to investigate modern numerical problems of large size;
  - use of numerical algorithms to solve large sparse linear systems;
  - writing programs in modern mathematical software packages to work with sparse matrices.

Learning outcomes: abilities

On successful completion of the course, students are expected to expected to have the following knowledge, skills and abilities:

- Abilities
  - use technology of sparse matrices for solving discretized problems of mathematical physics;
  - apply a suitable numerical solution method for a given sparse linear system and justify its suitability both theoretically and practically;
  - employ preconditioning techniques to precondition a given sparse linear system by selecting appropriate type of preconditioner;
  - implement direct and iterative algorithms for solving sparse linear systems in the form of program code;
  - use modern mathematical software (Matlab) for programming numerical solution methods for sparse linear systems.

Prerequisites for the course

- Calculus
- Linear Algebra
- Numerical Analysis
- Ordinary Differential Equations
- Partial Differential Equations
- Scientific Computing (knowledge of Matlab or Maple)

Study materials

Course textbook

Yousef Saad. Iterative Methods for Sparse Linear Systems, 2nd edition. SIAM, 2003. 528 p.

Download from https://www-users.cs.umn.edu/~saad/IterMethBook_2ndEd.pdf

Yousef Saad webpage:

https://www-users.cs.umn.edu/~saad/

Study materials

Additional reading

1. Gene H. Golub, Charles F. Van Loan. Matrix Computations. The Johns Hopkins University Press; 3rd edition, 1996. 728 p.
2. James W. Demmel. Applied Numerical Linear Algebra. SIAM, 1997. 184 p.
3. Ole Osterby, Zahari Zlatev. Direct Methods for Sparse Matrices. Springer-Verlag, 1983.
4. Sergio Pissanetzky. Sparse Matrix Technology. Academic Press, 1984. 312 p.

Workload and assessment

Workload

- Lectures: 34 hours
- Practice (work on practical assignments): 52 hours
- Independent study (work on individual project): 58 hours

Assessment

- Total score: 100 points
  - Practical assignments: **60 points**
  - Final exam (oral exam or individual project defense): **40 points**

Grading: 2-5 (60-100 points) are pass grades

Grade scale in Southern Federal University

Score	SFedU grades	ECTS grades
85-100	5 (excellent)	A (excellent): 95-100 B (very good): 85-94
71-84	4 (good)	C (good)
60-70	3 (satisfactory)	D (satisfactory): 65-70 E (sufficient): 60-64
<60	2 (fail)	FX (some further work required): 31-59 F (re-study of the discipline required): <31

Lectures

Module 1. Background in sparse linear systems

- **Basic concepts of linear algebra and matrix theory.** Types and structures of square matrices.
- Vector and matrix norms. Range and kernel. Existence of Solution. Orthonormal vectors. Gram-Schmidt process.
- Eigenvalues and their multiplicities. **Matrix factorizations and canonical forms**: QR, diagonal form, Jordan form, Schur form.
- Matrix factorizations: SVD, LU, Cholessky. Properties of normal, Hermittian matrices and positive definite matrices
- Existence of solution. Perturbation analysis and condition number. Errors and costs.
- Structures and graph representations of sparse matrices. Storage formats for sparse matrices.

Lectures (continues)

Module 2. Direct, iterative and projection methods for sparse linear systems

- **Direct and iterative methods: comparison.** Direct solution methods (Gaussian elimination with partial pivoting). Direct sparse methods.
- **Iterative methods**: general idea and convergence criterion. **Classic iterative methods**: Jacobi, Gauss-Seidel, Successive Over Relaxation (SOR), Symmetric Successive Over Relaxation (SSOR). Convergence criteria for classic iterative methods.
- **Projection methods**: derivation and general formulation of a projection method. 1D and multidimensional projection methods.

Lectures (continues)

Module 3. Krylov subspace methods for sparse linear systems and preconditioning techniques

- **Krylov subspace methods.** Definition of Krylov subspace. General formulation of a Krylov subspace method. Process of Arnoldi orthogonalization.
- **Methods based on Arnoldi process:** Full Orthogonalization method (FOM). Generalized Minimal Residual method (GMRES). Comparison of FOM and GRMRES.
- **Idea of short-term recurrence methods**. Lanczos orthogonalization for symmetric systems and biorthogonalization for nonsymmetric systems. Methods based on Lacnzos orthogonalization and biorthogonalization. Overview and comparison of efficient and optimal methods.
- **Basic ideas of preconditioning technique**. Examples of preconditioners. Preconditioned Krylov Subspace methods.

Individual project (40 points)

- **Description of individual project**
  - Perform a discretization of a given boundary-value problem and obtain a linear system with varied size
  - Study the properties of the matrix in the resulting system, write the code for the given sparse format
  - Solve the system using given classic iterative methods, 1D projection methods, Krylov subspace methods
  - Solve the system applying preconditioning

Lectures

  - Compare the results for different system size and make conclusions
- Individual project is a part of final control together with oral exam. It can be done in a group of maximum two students. Students are expected to prepare a report containing the results of all project assignments.
- Formal presentation and defense of an individual project can be regarded as taking exam. In this case it will include oral discussion with each student about theoretical aspects of the used methods and approaches.

Practice (tentative)

Module 1. Background in sparse linear systems	
PA 1	Getting started with Matlab
PA 2	Matrix fundamentals: types and structures. Matrix norms. Matrix factorizations
PA 3	Solving linear systems in Matlab, computation time, conditioning of the problem.
PA 4	Discretization of PDEs. Permutations and reordering. Sparse formats.
Module 2. Direct, iterative and projection methods for sparse linear systems	
PA 5	Comparison of direct and iterative methods for different sparse systems
PA 6	Classic iterative methods and 1D projection methods • simple iteration, Jacobi, GaussSeidel, SOR, SSOR • SDM, MRIM, RNSD
Module 3. Krylov subspace methods for sparse linear systems and preconditioning techniques	
PA 7	Arnoldi process, FOM, restarted FOM
PA 8	GMRES. Convergence of GMRES and eigenvalue distribution
PA 9	Understanding preconditioning. Effects of preconditioning when solving the system with symmetric and nonsymmetric matrices