Examination program
Numerical Methods of Linear Algebra for Sparse Matrices
Spring 2026

1. Basic concepts of linear algebra: basic matrix operations, transposition, inversion, identity matrix, determinant and its properties, eigenvalues and eigenvectors, spectral radius and trace.
2. Types of square matrices: symmetric, Hermitian, normal, unitary, orthogonal. Structures of square matrices with respect to the location of zero entries: diagonal, bi-and diagonal, triangular, banded, Hessenberg, block.
3. Inner product, Eucledian inner product. Definition of vector norm. Vector norms: Eucledian norm, infinity norm, Holder norms.
4. Definition of matrix norm. Matrix norms: max column, max row, spectral, Frobenius (Eucledian), l1-norm, M-norm. Operator norms. Consistency of vector and matrix norms.
5. Subspace, linear independence, basis, dimension, direct sum, range and kernel, rank, invariant subspace, eigenspace. Existence of a solution.
6. Orthogonal and orthonormal vectors and subspaces, Gram-Schmidt orthogonalization (classic and modified). QR-factorization. Thin and full QR. Existence and uniqueness of QR-factorization. Solving linear systems using QR-factorization.
7. LU- and PLU-factorizations. Cholesky factorization. Existence and uniqueness of LU- , PLU-factorizations and Cholesky factorizations. Solving linear systems using LU- , PLU- and Cholesky factorizations.
8. Eigenvalues and their multiplicities. Algebraic multiplicity, geometric multiplicity. Main definitions: simple, multiple, semisimple, defective, derogatory eigenvalue. Simple, semisimple, defective and deragotory matrix. Similarity transformation. Canonical forms of matrices: reduction to diagonal form, its existence and uniqueness.
9. Similarity transformation. Canonical forms of matrices: diagonal form (Eigenvalue and Singular Value Decomposition), Jordan form, Schur form (Schur and quasi-Schur form). Existence and uniqueness of each factorization.
10. Reduction to Schur form and Singular Value Decomposition. Relation between Schur and SVD. Properties and theorems.
11. Positive definite matrices, properties and theorems. Normal and Hermitian matrices, properties and theorems. Powers of matrices.
12. Perturbation analysis, absolute and relative error. Condition number and its properties.

13. Types of numerical errors. Well-conditioned and ill-conditioned problem. Stable and unstable algorithm. Computational costs.
14. Definition of sparse matrix. Structured and unstructured sparse matrices. Matrix pattern, types of patterns. Graph representation of a matrix, matrix and its adjacency graph. Permutations and Reordering. Symmetric permutation. Fill-ins in direct solution methods. Examples of reordering algorithms.
15. Storage schemes for sparse matrices: compressed sparse row (CSR), compressed sparse column (CSC), modified CSR and CSC, diagonal format, Ellpack-Itpack format. Algorithms for matrix by vector multiplication for sparse formats.
16. Discretization of partial differential equations. Finite difference method (FDM). Basic finite differences. Descretization by FDM on an example of 1D Poisson's equation.
17. Discretization of partial differential equations. Finite difference method (FDM). Descretization by FDM on an example of 2D Poisson's equation. Node numbering in 2D.
18. Discretization of partial differential equations. Discretization by Finite element method. Weak formulation, finite element approximations. Assembly process. Mesh refinement.
19. Direct and iterative solution methods. Comparison of direct and iterative methods. Overview of direct solution methods. Direct sparse methods: Gaussian elimination without and with partial pivoting. Formulation of classic iterative method.
20. Classic iterative methods for linear systems. Convergence of iterative methods: necessary and sufficient conditions. Simple iteration method. Jacobi and Gauss-Seidel methods, their variations. Properties of diagonally dominant matrices, location of matrix eigenvalues. Convergence of Jacobi and Gauss-Seidel for diagonally dominant matrices.
21. Classic iterative methods for linear systems. Convergence of iterative methods: necessary and sufficient conditions. Successive Over Relaxation (SOR), Symmetric Successive Over Relaxation (SSOR), their variations. Convergence of SOR and SSOR for positive definite matrices.
22. General formulation of a projection method. Search subspace and subspace of constraints. Classification of projection methods. Matrix representation of a projection method. Properties of W^HAV matrix.
23. General formulation of a projection method. General formulation of 1D projection method. Steepest descent method, its modification and properties.
24. General formulation of a projection method. General formulation of 1D projection method. Minimal Residual Iteration method, its modification and

properties. Residual Norm Steepest Descent method, its modification and properties. Orthogonal and oblique projection methods

25. Definition of a Krylov subspace. Multi-dimensional projection methods and Krylov subpace projection methods. The process of Arnoldi orthogonalization to form a basis for Krylov subspace (exact and modified). Properties of Arnoldi process, Arnoldi relation.
26. Definition of a Krylov subspace. Derivation of Full Orthogonalization Method (FOM) for linear systems. FOM algorithm, its modifications (restarted version). Calculation of residual in FOM.
27. Definition of a Krylov subspace. Derivation of Generalized Minimal Residual method (GMRES). Givens rotations. GMRES algorithm, its modifications (restarted version). Calculation of residual in GMRES.
28. Definition of a Krylov subspace. Arnoldi relation. FOM and GMRES algorithms. Comparison of FOM and GMRES. Optimal and efficient methods.
29. Krylov subspace methods for symmetric (Hermitian) linear systems. Lanczos orthogonalization process for symmetric systems to form a basis for Krylov subspace. Classic Lanczos method for symmetric systems. Lanczos relation. Calculation of residual in Classic Lanczos method.
30. Krylov subspace methods for symmetric (Hermitian) linear systems. Lanczos orthogonalization for symmetric systems. Derivation of Direct Lanczos method. Properties of residuals and search directions in Direct Lanczos.
31. Krylov subspace methods for symmetric (Hermitian) linear systems. Lanczos orthogonalization for symmetric systems. Derivation of Conjugate Gradient method (CG). Properties of residuals and search directions in CG.
32. Krylov subspace methods for symmetric (Hermitian) linear systems. Lanczos orthogonalization for symmetric systems. Generalization of CG for systems with Hermitian and nonsymmetric matrices: Conjugate Residual (CR), Generalized Conjugate Residual (GCR). Properties of residuals and search directions in CR and GCR.
33. Krylov subspace methods for nonsymmetric linear systems. Bi-orthonormal sets of vectors. Lanczos biorthogonalization process to form two bases for Krylov subspaces. Properties of Lanczos biorthogonalization algorithm, Lanczos relation for biorthogonalization. Classic Lanczos method for nonsymmetric systems. Calculation of residual in Classic Lanczos method.
34. Lanczos biorthogonalization for nonsymmetric systems. Derivation of Biconjugate Gradient method (BiCG). Properties of residuals and search directions in BiCG.

35. Overview of Krylov subspace methods. Definition of Krylov subspace. Classification of Krylov subspace methods, main ideas. Optimal and efficient Krylov subpace methods.
36. Preconditioning technique. Types of preconditioning (left, right, split). Matrix splitting and preconditioners. Basic preconditioners (Jacobi, Gauss-Seidel, SOR, SSOR, incomplete LU).
37. Preconditioning technique. Types of preconditioning (left, right, split). Examples of preconditioned iterative algorithms: Preconditioned Conjugate Gradient method (PCG), Split Preconditioned Conjugate Gradient method (Split PCG).
38. Preconditioning technique. Types of preconditioning (left, right, split). Examples of preconditioned iterative algorithms: algorithms of GRMES with left and right preconditioning (LP GMRES and RP GMRES).

Main references

1) Lecture slides on the course webpage https://edu.mmcs.sfedu.ru/course/view.php?id=862
2) Lecture recordings (without Lecture 15) https://edu.mmcs.sfedu.ru/mod/url/view.php?id=42505
3) Yousef Saad. Iterative Methods for Sparse Linear Systems, 2nd edition. SIAM, 2003. 528 p. (available in electronic form on the course page https://edu.mmcs.sfedu.ru/pluginfile.php/186964/mod_resource/content/2/Yosef%20Saad%20IterMethBook_2ndEd%20-%20text%20book.pdf

Additional reading

4) Gene H. Golub, Charles F. Van Loan. Matrix Computations. The Johns Hopkins University Press; 3rd edition, 1996. 728 p.
5) James W. Demmel. Applied Numerical Linear Algebra. SIAM, 1997. 184 p.
6) Ole Osterby, Zahari Zlatev. Direct Methods for Sparse Matrices. Springer-Verlag, 1983.
7) Sergio Pissanetzky. Sparse Matrix Technology. Academic Press, 1984. 312 p.