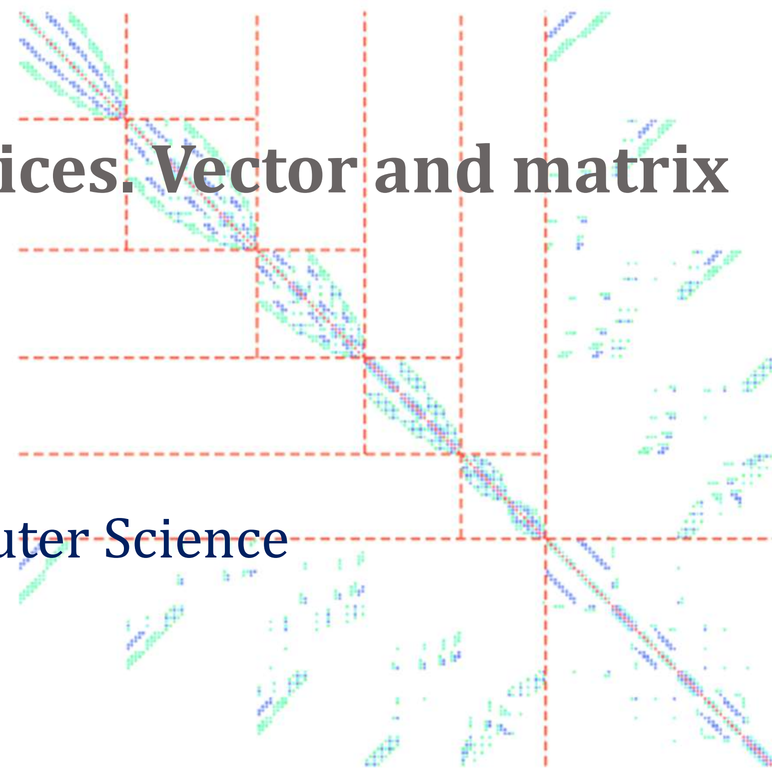
Numerical Methods of Linear Algebra for Sparse Matrices

Lecture 2

Types and structures of square matrices. Vector and matrix norms



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Outline

- Types of square matrices
- Structure of square matrices with respect to location of zero entries
- Vector and matrix norms

Types of square matrices: symmetric and Hermitian

- The choice of solution method often depends on the structure of a matrix. For example, symmetry has an impact on the eigenstructure

- Symmetric matrix

$$A^T = A$$

- Hermitian matrix

$$A^H = A, \text{ where } A^H = \overline{A^T}$$

- Skew-symmetric matrix

$$A^T = -A$$

- Skew-Hermitian matrix

$$A^H = -A$$

Properties of Hermitian matrices

1) $\det A \in \mathbb{R}$ (determinant is a real scalar)

2) if A is Hermitian, then A^{-1} (if exists) is Hermitian

3) if A, B are Hermitian, then AB is Hermitian $\Leftrightarrow$ $AB = BA$

$\Leftrightarrow$ stands for "if and only if"

4) if A, B are Hermitian, then $A + B$ is Hermitian

Types of square matrices: positive and negative, normal, unitary and orthogonal

- Positive matrix $a_{ij} > 0, \quad i, j = \overline{1, n}$
- Non-positive matrix $a_{ij} \le 0, \quad i, j = \overline{1, n}$
- Negative matrix $a_{ij} < 0, \quad i, j = \overline{1, n}$
- Non-negative matrix $a_{ij} \ge 0, \quad i, j = \overline{1, n}$

- Normal matrix

$A^H A = AA^H$ (or $A^T A = AA^T$ for real matrices)

- Unitary matrix (usually denoted as Q)

$Q^H Q = QQ^H = I$, where I is identity matrix

Property: $Q^{-1} = Q^H$

For nonsquare $Q \in \mathbb{C}^{n \times m}$

either $Q^H Q = I_m$ (orthonormal columns)

or $QQ^H = I_n$ (orthonormal rows)

- Orthogonal (orthonormal) matrix is a unitary real matrix $Q \in \mathbb{R}^{n \times n}$

$Q^T Q = QQ^T = I$

Property: $Q^{-1} = Q^T$

Structures of square matrices with respect to location of zero entries: diagonal, triangular, Hessenberg

- Diagonal matrix $a_{ij} = 0 \quad \text{for } \forall j \neq i$

$\forall$ stands for "for every" or "for any"

Notation: $A = diag(a_{11}, a_{22}, ..., a_{nn})$

- Upper-triangular matrix

$a_{ij} = 0, \quad \text{for } \forall i > j$

$$\begin{matrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{matrix}$$

- Lower-triangular matrix

$a_{ij} = 0, \quad \text{for } \forall i < j$

$$\begin{matrix} \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \end{matrix}$$

- Upper-Hessenberg matrix

$a_{ij} = 0, \quad \text{for } \forall i, j: \ i > j + 1$

$$\begin{matrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{matrix}$$

- Lower-Hessenberg matrix

$a_{ij} = 0, \quad \text{for } \forall i, j: \ i < j + 1$

$$\begin{matrix} \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{matrix}$$

Structures of square matrices with respect to location of zero entries: bidiagonal, tridiagonal, banded

- Upper-bidiagonal matrix

$a_{ij} = 0, \quad \text{for } \forall j \neq i \text{ or } j \neq i+1$

- Lower-bidiagonal matrix

$a_{ij} = 0, \quad \text{for } \forall j \neq i \text{ or } j \neq i-1$

- Tridiagonal matrix

$a_{ij} = 0, \quad \text{for } \forall i, j : \ |j-i| > 1$

Notation: $A = tridiag(a_{i,i-1}, a_{ii}, a_{i,i+1})$

- Banded matrix

$a_{ij} \neq 0 \ \Leftrightarrow \ i - m_l \leq j \leq i + m_u$, where m_l, m_u are positive integers

$m_l + m_u + 1$ is the bandwidth

m_l is the lower bandwidth: $a_{ij} = 0, \quad \text{for } \forall i > j + m_l$

m_u is the upper bandwidth: $a_{ij} = 0, \quad \text{for } \forall j > i + m_u$

Structures of square matrices with respect to location of zero entries: block, block diagonal and tridiagonal

- Block matrix

$$A = \begin{pmatrix} A_{11} & \dots & A_{1l} \\ \dots & \dots & \dots \\ A_{k1} & \dots & A_{kl} \end{pmatrix}, \quad \text{where } k \times l \text{ are block sizes of } A$$

- Block-diagonal matrix

$$A_{ij} = 0, \quad \text{for } \forall j \neq i \text{ or } j \neq i-1$$

- Tridiagonal matrix

$$A_{ij} = 0, \quad \text{for } \forall i, j: \ |j - i| > 1$$

Notation: $A = tridiag(A_{i,i-1}, A_{ii}, A_{i,i+1})$

- Permutation matrix is the identity matrix with its rows (or columns) permuted

Example

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Inner product

Let's consider a complex space $X \subset \mathbb{C}^n$ (X is a subspace of $\mathbb{C}^n$).

Note that $\mathbb{C}^n$ is the space of vector-columns.

For the vectors $x, y \in X$ an inner product

is a mapping $s(x, y): X \times X \to \mathbb{C}$, which satisfies the following properties:

1) $s(x, y)$ is linear by x

$$s(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 s(x_1, y) + \lambda_2 s(x_2, y) \quad \forall x_1, x_2 \in X, \ \lambda_1, \lambda_2 \in \mathbb{C}$$

2) $s(x, y)$ is Hermitian

$$s(y, x) = \overline{s(x, y)} \quad \forall x, y \in X, \text{ therefore } s(x, x) \in \mathbb{R}$$

3) $s(x, y)$ is positive definite: $s(x, x) \geq 0$ and $s(x, x) = 0 \Leftrightarrow x = 0$

$$s(x, 0) = 0 \quad \forall x \in X, \ s(0, y) = 0 \quad \forall y \in X.$$

Proof. $s(x, 0) = s(x, 0 \cdot y) = 0 \cdot s(x, y) = 0$

Cauchy - Schwartz inequality

$$|s(x, y)|^2 \leq s(x, x) \cdot s(y, y)$$

Euclidean inner product

When $X = \mathbb{C}^n$, the inner product $s(x, y)$ is called Eucledian and denoted by (x, y).

For the vector-columns $x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \in \mathbb{C}^n$, $y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix} \in \mathbb{C}^n$

$$(x, y) = \sum_{i=1}^{n} x_i \overline{y}_i = x_1 \overline{y}_1 + x_2 \overline{y}_2 + \dots + x_n \overline{y}_n$$

In a matrix form (using matrix multiplication): $(x, y) = y^H x$ (y^H is vector-row)

For $X = \mathbb{R}^n$ $(x, y) = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$, in a matrix form: $(x, y) = y^T x$

$$(x, x) = \sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

Property

$(Ax, y) = (x, A^H y) \geq 0 \quad \forall x \in \mathbb{C}^n, y \in \mathbb{C}^m, \ A \in \mathbb{C}^{m \times n}$

Proposition. Unitary matrices preserve inner product:

$(Qx, Qy) = (x, y) \quad \forall Q \in \mathbb{C}^{n \times n}$

$\vartriangle$ $(Qx, Qy) = (x, Q^H Q y) = (x, Iy) = (x, y)$ $\square$

$\vartriangle$ is the beginning of proof

$\square$ is the end of proof

Vector norms

- Norms are needed to measure lengths of vectors and closeness of two vectors (or matrices).
- Examples of using vector norms: estimate convergence rate of an iterative method; estimate the error of an approximation to a given solution

Vector norm on $X \subset \mathbb{C}^n$ is a function $X \to \mathbb{R}$, denoted by $\|x\|$,

which satisfies the following properties:

1) $\|x\| \geq 0 \quad \forall x \in X$ (the norm is non-negative)

$\|x\| = 0 \Leftrightarrow x = 0$

2) $\|\alpha x\| = |\alpha| \cdot \|x\| \;\; \forall x \in X, \;\; \alpha \in \mathbb{C}$

3) triangle inequality

$\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

Particular cases of vector norms

When $X = \mathbb{C}^n$, the Eucledian norm of $x \in X$ is denoted by $\|x\|_2$ and defined by

$$\|x\|_2 = \sqrt{(x,x)}$$

Hence $(x,x) = \|x\|_2^2$

Proposition. Unitary matrices preserve the Eucledian norm:

$$\|Qx\|_2 = \|x\|_2 \quad \forall Q \in \mathbb{C}^{n \times n}$$

- Hölder norms are most commonly used

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

Cauchy - Schwartz inequality

for Eucledian inner product and norm

$$|(x,y)| \le \|x\|_2 \cdot \|y\|_2$$

1) $p = 1 \;\Rightarrow\; \|x\|_1 = \sum_{i=1}^{n} |x_i| = |x_1| + |x_2| + \ldots + |x_n|$

2) $p = 2 \Rightarrow$ Eucledian norm $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$

3) $p = \infty \;\Rightarrow$ infinity norm $\|x\|_\infty = \max_{i=\overline{1,n}} |x_i|$

Matrix norms

- Matrix norms can be defined by vector norms, if we represent n by m matrices as vectors of the size nm

For matrix $A \in \mathbb{C}^{n\times m}$ $\|A\|$ is the matrix norm, if it satisfies the following properties:

1) $\|A\| \geq 0 \quad \forall A \in \mathbb{C}^{n\times m}$ (the norm is non-negative)

$\|A\| = 0 \Leftrightarrow A = 0$

2) $\|\alpha A\| = |\alpha| \cdot \|A\| \;\; \forall A \in \mathbb{C}^{n\times m}, \;\; \alpha \in \mathbb{C}$

3) triangle inequality

$\|A + B\| \leq \|A\| + \|B\| \quad \forall A, B \in \mathbb{C}^{n\times m}$

For matrices, which admit matrix product, there is extra property:

4) multiplicative property

$\|AB\| \leq \|A\| \cdot \|B\|$

Operator matrix norms

- Often the term "matrix norm" refers to an operator matrix norm. These operator norms are considered as proper matrix norms.

The set of matrix norms $\|\bullet\|_{pq}$ are called operator matrix norms,

if they are induced by two vector norms $\|\bullet\|_p$ and $\|\bullet\|_q$

$$\|A\|_{pq} = \max_{x\in\mathbb{C}^m,\ x\neq 0} \frac{\|Ax\|_p}{\|x\|_q} \quad \text{for } A\in\mathbb{C}^{n\times m}$$

When $p = q$ we have $\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$

- Consistency of norms

Matrix p-norm is consistent with a vector p-norm, if

$$\|Ax\|_p \le \|A\|_p \cdot \|x\|_p$$

Property. $\forall A\in\mathbb{C}^{n\times m}$ $\|A^k\|_p = \|A\|_p^k$

Particular cases of non-operator matrix norms

- Hölder matrix norms

$$\|x\|_{l_p} = \left(\sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ij}|^p \right)^{1/p}, \quad \forall A \in \mathbb{C}^{n\times m},\ p \ge 1$$

1) $p = 1 \;\Rightarrow\; l_1$-norm $\|A\|_{l_1} = \sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ij}| \quad \forall A \in \mathbb{C}^{n\times m}$

2) $p = 2 \Rightarrow$ Frobenius norm $\|A\|_F = \sqrt{\sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ij}|^2}$

Also $\|A\|_F = \sqrt{tr(A^H A)} = \sqrt{tr(AA^H)}$

Property. Frobenius norm is consistent with the vector norm $\|\cdot\|_2$

For square matrices $A \in \mathbb{C}^{n\times n}$ this norm is called Eucledian norm $\|A\|_E = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|^2}$

- M-norms $\|A\|_M = \sqrt{mn} \cdot \max\limits_{i,j=\overline{1,n}} |a_{ij}|$ for rectangular matrices $A \in \mathbb{C}^{n\times m}$

$\|A\|_M = n \cdot \max\limits_{i,j=\overline{1,n}} |a_{ij}|$ for square matrices $A \in \mathbb{C}^{n\times n}$

Particular cases of operator matrix norms

1) maximal column norm

$$\|A\|_1 = \max_{j=1,m} \sum_{i=1}^{n} |a_{ij}| \quad \forall A \in \mathbb{C}^{n\times m}$$

2) maximal row norm

$$\|A\|_\infty = \max_{i=1,n} \sum_{j=1}^{m} |a_{ij}| \quad \forall A \in \mathbb{C}^{n\times m}$$

3) spectral norm

$$\|A\|_2 = \sqrt{\rho(AA^H)} = \sqrt{\rho(A^H A)} \quad \forall A \in \mathbb{C}^{n\times m}$$

where ρ is the spectral radius (maximal eigenvalue by modulus)

It is known that all eigenvalues (the spectrum) of $A^H A$

are nonnegative: $\sigma(A^H A) \geq 0$

Also $\|A\|_2 = \max_{i=1,n} \sigma_i$ where σ_i is the singular value,

i.e. σ_i^2 is an eigenvalue of $A^H A$

Important relations

1) Matrix spectral norm $\|\bullet\|_2$ is consistent with vector norm $\|\bullet\|_2$, maximal column norm $\|\bullet\|_1$ is consistent with vector norm $\|\bullet\|_1$, maximal row norm $\|\bullet\|_\infty$ is consistent with vector norm $\|\bullet\|_\infty$

2) Matrix M-norm $\|\bullet\|_2$ is consistent with vector norms $\|\bullet\|_1$, $\|\bullet\|_2$, $\|\bullet\|_\infty$

3) Maltiplicative property $\|AB\| \le \|A\| \cdot \|B\|$ holds for any operator norm and for Frobenius norm.

4) $\rho(A) \le \|A\|$ for any matrix norm (the spectral radius is not greater than the norm).

Compute the p-norm for $p = 1, 2, \infty, F$ for the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$$