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Numerical Methods of Linear Algebra for Sparse Matrices

Lecture 11

Classical iterative methods for linear systems

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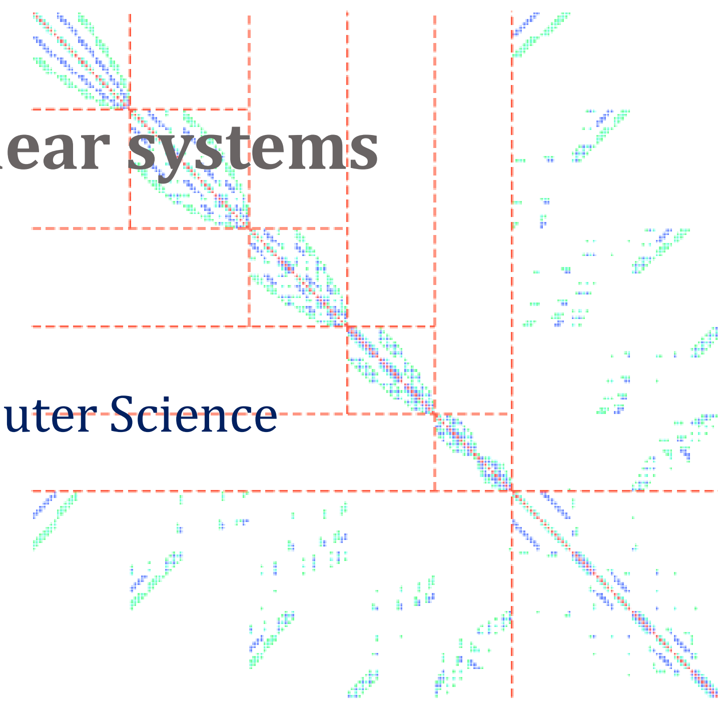


Classical iterative methods for linear systems

Simple iteration method

Jacobi method

Gauss-Seidel method and its variations

SOR, SSOR, USOR methods

Matrix splitting methods and preconditioners

Formulation of classical iterative method

Idea of an iterative method to solve the system $Ax = b$:

1. Take *initial guess* $x^{(0)}$

2. Apply iterative process until convergence

$x^{(k+1)} = G_k x^{(k)} + g_k$, where

$x^{(k)}$ is *approximate solution* at k-th iteration

G_k is iteration (transition) matrix, g_k is iteration vector

For stationary process $G_k = G,\ g_k = g:\ x^{(k+1)} = Gx^{(k)} + g$

At every iteration $x^{(k)}$ is improved by the modification of one or several components of it, until convergence is reached.

This is called a *relaxation step*: $\left\| x^{(k+1)} - x^{(k)} \right\| < \varepsilon$

Also the goal is to make the norm of the residual vector $\left\| r_k \right\| = \left\| b - Ax^{(k)} \right\|$ smaller at every iteration.

The simple iteration (Richardson) method

Consider a linear system: $Ax = b,\ A \in \mathbb{C}^{n\times n},\ b \in \mathbb{C}^n$

Idea of the method: assume $x^{(k+1)} - x^{(k)} = r_k = b - Ax^{(k)}$

$$x^{(k+1)} = x^{(k)} + r_k = x^{(k)} + b - Ax^{(k)} = (I - A)x^{(k)} + b$$

$$x^{(k+1)} = (I - A)x^{(k)} + b$$

Hence $G = I - A,\ \ g = b$

Simple iteration method converges, when $M(A) < 2$,
where $M(A)$ is maximal eigenvalue of A

Jacobi method

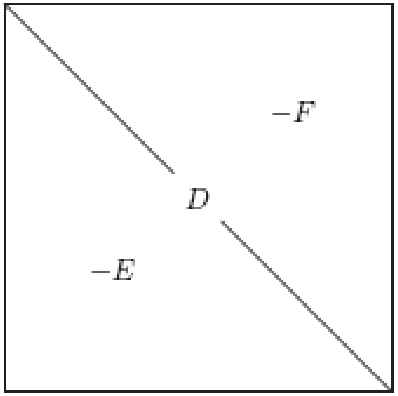
Consider splitting of matrix A:

$$A = D - E - F$$

$-E$ is a strict lower part of A, $E = \{-a_{ij}\}, j < i$

$-F$ is a strict upper part of A, $F = \{-a_{ij}\}, j > i$

D is a diagonal part of A, $D = \{a_{ii}\}, i = 1, \dots, n$



$$Ax = b,\ A \in \mathbb{C}^{n \times n},\ b \in \mathbb{C}^{n}$$

$$(D - E - F)x = b$$

$$Dx = (E + F)x + b$$

Iterative process:

$$Dx^{(k+1)} = (E + F)x^{(k)} + b$$

Matrix form:

$$x^{(k+1)} = D^{-1}(E + F)x^{(k)} + D^{-1}b$$

$$G_J = D^{-1}(E + F) = I - D^{-1}A,\ \ g_J = D^{-1}b$$

Component - wise form:

$$x_i^{(k+1)} = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k)} + \frac{b_i}{a_{ii}}$$

When we compute $x_i^{(k+1)}$, we already know previous entries $x_{i-1}^{(k+1)},\ x_{i-2}^{(k+1)},\ \dots,\ x_1^{(k+1)}$

Gauss-Seidel method

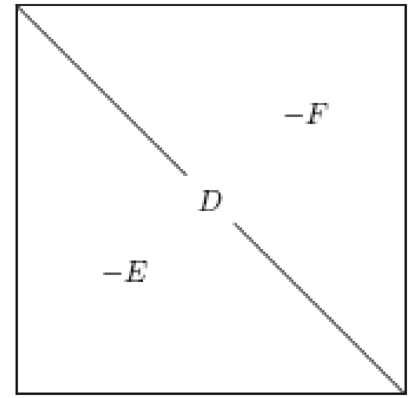
Consider splitting of matrix A:

$$A = D - E - F$$

$-E$ is a strict lower part of A, $E = \{-a_{ij}\}, j < i$

$-F$ is a strict upper part of A, $F = \{-a_{ij}\}, j > i$

D is a diagonal part of A, $D = \{a_{ii}\}, i = 1, \dots, n$



$$Ax = b, \; A \in \mathbb{C}^{n\times n}, \; b \in \mathbb{C}^n \;\Rightarrow\; (D - E - F)x = b$$

$$(D - E)x = Fx + b$$

Iterative process:

$$(D - E)x^{(k+1)} = Fx^{(k)} + b$$

Matrix form:

$$x^{(k+1)} = (D - E)^{-1} F x^{(k)} + (D - E)^{-1} b$$

$$G_{GS} = (D - E)^{-1} F = I - (D - E)^{-1} A, \quad g_{GS} = (D - E)^{-1} b$$

Component - wise form:

$$x_i^{(k+1)} = -\frac{1}{a_{ii}} \underbrace{\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)}} - \frac{1}{a_{ii}} \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} + \frac{b_i}{a_{ii}}$$

$$x^{(k+1)} = \begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \dots \\ x_i^{(k+1)} \\ \dots \\ x_n^{(k+1)} \end{pmatrix}$$

Use these entries

Variations of Gauss-Seidel method: backward and symmetric

Backward GS ($E \leftrightarrow F$):

Matrix form:

$$x^{(k+1)} = (D-F)^{-1}Ex^{(k)} + (D-F)^{-1}b$$

Component - wise form:

$$x_i^{(k+1)} = -\frac{1}{a_{ii}}\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \underbrace{\frac{1}{a_{ii}}\sum_{j=i+1}^{n} a_{ij}x_j^{(k+1)}} + \frac{b_i}{a_{ii}}$$

$$x^{(k+1)} = \begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \cdots \\ x_i^{(k+1)} \\ \cdots \\ x_n^{(k+1)} \end{pmatrix}$$

Use these entries

Symmetric GS: combine one iteration half-step of GS with one iteration half-step of backward GS

$$x^{\left(k+\frac{1}{2}\right)} = (D-E)^{-1}Fx^{(k)} + (D-E)^{-1}b$$

$$x^{(k+1)} = (D-F)^{-1}Ex^{\left(k+\frac{1}{2}\right)} + (D-F)^{-1}b$$

By eliminating $x^{\left(k+\frac{1}{2}\right)}$, we get $x^{(k+1)} = G_{SGS}x^{(k)} + g_{SGS}$, where

$$G_{SGS} = (D-F)^{-1}E(D-E)^{-1}F,\; g_{SGS} = (D-F)^{-1}D(D-E)^{-1}b$$

Diagonal dominance

From component-wise formulas of Jacobi and GS methods it can be seen, that convergence is related to diagonal elements.

Row diagonal dominance of matrix $A \in \mathbb{C}^{n \times n}$

- weak diagonal dominance

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \; i = \overline{1, n}$$

- strict diagonal dominance

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \; i = \overline{1, n}$$

- irreducible diagonal dominance

$$\forall i = \overline{1, n} \;\; |a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ and } \exists i : \;\; |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|,$$

where A is irreducible, i.e. cannot be reduced to block-diagonal form:

there is no P: $PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$

Proposition. For any norm $|\lambda_i| \leq \|A\|$, where λ_i is eigenvalue of A

Gershgorin theorem

Diagonal dominance is related to Gershgorin circles and location of eigenvalues.

Gershgorin Theorem

Any eigenvalue λ of matrix $A \in \mathbb{C}^{n\times n}$ is located in one of the closed circles of the complex plane with the center in a_{ii}

and the radius $R_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|,\ i = \overline{1,n}$

$$\forall \lambda \in \sigma(A)\ \ \exists i:\ \ \left|\lambda - a_{ii}\right| \le \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|$$

Convergence of Jacobi and GS methods

Sufficient convergence condition

Theorem 1. If matrix A has strict diagonal dominance, then Jacobi and GS methods converge for any initial guess and

$$\|G_{GS}\|_\infty \le \|G_J\|_\infty < 1$$

(Recall that ∞ denotes the maximal row norm)

Theorem 2. If matrix A has irreducible diagonal dominance or strict diagonal dominance, then Jacobi and GS methods converge for any initial guess and $\rho(G_{GS}) < \rho(G_J) < 1$

Successive over relaxation (SOR) method

Consider splitting of matrix A: $A = D - E - F$

$Ax = b,\ A \in \mathbb{C}^{n\times n},\ b \in \mathbb{C}^n$

$\omega Ax = \omega b$

$\omega A = \omega D - \omega E - \omega F + D - D = (D - \omega E) - (D - \omega D) - \omega F =$

$= (D - \omega E) - (\omega F + (1-\omega)D)$

$\big((D - \omega E) - (\omega F + (1-\omega)D)\big)x = \omega b$

$(D - \omega E)x = (\omega F + (1-\omega)D)x + \omega b$

Iterative process:

$(D - \omega E)x^{(k+1)} = (\omega F + (1-\omega)D)x^{(k)} + \omega b$

Matrix form:

$x^{(k+1)} = (D - \omega E)^{-1}[\omega F + (1-\omega)D]x^{(k)} + \omega(D - \omega E)^{-1}b$

$G_{SOR} = (D - \omega E)^{-1}[\omega F + (1-\omega)D],\ \ g_{SOR} = \omega(D - \omega E)^{-1}b$

Successive over relaxation (SOR) method

Component - wise form:

$$x_i^{(k+1)} = \omega\left(-\frac{1}{a_{ii}}\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \frac{1}{a_{ii}}\sum_{j=i+1}^{n} a_{ij}x_j^{(k)} + \frac{b_i}{a_{ii}} \right) + (1-\omega)x_j^{(k)}$$

$$x_i^{(k+1)} = \omega x_i^{GS} + (1-\omega)x_i^{(k)}$$

ω is relaxation parameter, $\omega \in [0,2]$

For $\omega = 1$ we get Gauss-Seidel method.

Variations of SOR method: backward SOR

Backward SOR is derived in a similar way: $E \leftrightarrow F$

$$\omega A = (D - \omega F) - (\omega E + (1-\omega)D)$$

$$\big((D - \omega F) - (\omega E + (1-\omega)D)\big)x = \omega b$$

$$(D - \omega F)x = (\omega E + (1-\omega)D)x + \omega b$$

Iterative process:

$$(D - \omega F)x^{(k+1)} = (\omega E + (1-\omega)D)x^{(k)} + \omega b$$

Matrix form:

$$x^{(k+1)} = (D - \omega F)^{-1}[\omega E + (1-\omega)D]x^{(k)} + \omega(D - \omega F)^{-1} b$$

$$G_{SOR} = (D - \omega F)^{-1}[\omega E + (1-\omega)D], \quad g_{SOR} = \omega(D - \omega F)^{-1} b$$

Component-wise form:

$$x_i^{(k+1)} = \omega\left(-\frac{1}{a_{ii}}\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \frac{1}{a_{ii}}\sum_{j=i+1}^{n} a_{ij}x_j^{(k+1)} + \frac{b_i}{a_{ii}}\right) + (1-\omega)x_j^{(k)}$$

Symmetric successive over relaxation (SSOR) method

Consider splitting of matrix A: $A = D - E - F$

Idea of symmetric SOR (SSOR) is to combine one iteration half-step of SOR with one iteration half-step of backward SOR

$$x^{\left(k+\frac{1}{2}\right)} = (D-\omega E)^{-1}[\omega F + (1-\omega)D]x^{(k)} + \omega(D-\omega E)^{-1}b$$

$$x^{(k+1)} = (D-\omega F)^{-1}[\omega E + (1-\omega)D]x^{\left(k+\frac{1}{2}\right)} + \omega(D-\omega F)^{-1}b$$

$$G_{SSOR} = (D-\omega F)^{-1}[\omega E + (1-\omega)D](D-\omega E)^{-1}[\omega F + (1-\omega)D],$$

$$g_{SOR} = \omega(2-\omega)(D-\omega F)^{-1}D(D-\omega E)^{-1}b$$

Component-wise form:

$$x_i^{\left(k+\frac{1}{2}\right)} = \omega\left(-\frac{1}{a_{ii}}\sum_{j=1}^{i-1}a_{ij}x_j^{\left(k+\frac{1}{2}\right)} - \frac{1}{a_{ii}}\sum_{j=i+1}^{n}a_{ij}x_j^{(k)} + \frac{b_i}{a_{ii}}\right) + (1-\omega)x_j^{(k)},\ i = 1, 2, \ldots, n;$$

$$x_i^{(k+1)} = \omega\left(-\frac{1}{a_{ii}}\sum_{j=1}^{i-1}a_{ij}x_j^{\left(k+\frac{1}{2}\right)} - \frac{1}{a_{ii}}\sum_{j=i+1}^{n}a_{ij}x_j^{(k+1)} + \frac{b_i}{a_{ii}}\right) + (1-\omega)x_j^{\left(k+\frac{1}{2}\right)},\ i = n, n-1, \ldots, 1.$$

Variation of SSOR method: unsymmetric SOR (USOR)

In USOR two different relaxation parameters ω_1 and ω_2 are used:

$$x^{\left(k+\frac{1}{2}\right)} = (D-\omega_1 E)^{-1}[\omega_1 F + (1-\omega_1)D]x^{(k)} + \omega_1 (D-\omega_1 E)^{-1} b$$

$$x^{(k+1)} = (D-\omega_2 F)^{-1}[\omega_2 E + (1-\omega_2)D]x^{\left(k+\frac{1}{2}\right)} + \omega_2 (D-\omega_2 F)^{-1} b$$

Convergence of SOR method

Theorem 1. If matrix A is symmetric positive definite, then SOR method converges for any initial guess for any $\omega \in (0,2)$.

Theorem 2. If matrix A is symmetric and all its diagonal elements are positive $a_{ii}>0$, then SOR method converges for any initial guess for any $\omega \in (0,2) \Leftrightarrow A$ is positive definite.

Matrix splitting

Consider matrix splitting: $A = M - N$,

where M is nonsingular, system $Mx = b$ may be easy to solve,

M may be good approximation to A.

$Ax = b$

$(M - N)x = b$

$Mx = Nx + b$

Iterative process:

$Mx^{(k+1)} = Nx^{(k)} + b$

$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b = Gx^{(k)} + g$

Iteration matrix $G = M^{-1}N,\ g = M^{-1}b$

$G = M^{-1}N = M^{-1}(M - A) = I - M^{-1}A$

Preconditioners

The iteration $x^{(k+1)} = Gx^{(k)} + g$ can be viewed as technique for solving

$(I - G)x = g$

Since $G = I - M^{-1}A,\ g = M^{-1}b$

$(I - I + M^{-1}A)x = M^{-1}b$

$M^{-1}Ax = M^{-1}b$

System $M^{-1}Ax = M^{-1}b$ is a preconditioned system for original system $Ax = b$. It has the same solution.

Matrix M is called a preconditioning matrix or preconditioner

The preconditioned system $M^{-1}Ax = M^{-1}b$ may be less sparse than initial system $Ax = b$. Preconditioner M may be sparse, but its inverse M^{-1} not.

Iteration matrix and preconditioning matrices for the main iterative methods

In general for the iterative method we have: $G = M^{-1}N = I - M^{-1}A$.

Jacobi method:

$$G_J = D^{-1}(E+F) = I - D^{-1}A$$

$$M_J = D,\ N_J = D - A$$

Gauss-Seidel method:

$$G_{GS} = (D-E)^{-1}F = I - (D-E)^{-1}A$$

$$M_{GS} = D - E,\ N_{GS} = F$$

SOR method:

$$G_{SOR} = (D-\omega E)^{-1}[\omega F + (1-\omega)D]$$

$$M_{SOR} = \frac{1}{\omega}D - E,\ N_{SOR} = \frac{1}{\omega}\left[(1-\omega)D + \omega F\right]$$

SSOR method:

$$G_{SSOR} = (D-\omega F)^{-1}[\omega E + (1-\omega)D](D-\omega E)^{-1}[\omega F + (1-\omega)D],$$

$$M_{SSOR} = \frac{1}{\omega(2-\omega)}(D-\omega E)D^{-1}(D-\omega F),\ N_{SSOR} = \frac{1}{\omega(2-\omega)}(D+\omega F)D^{-1}(D+\omega E)$$