Numerical methods of Linear Algebra for Sparse Matrices
Spring 2026

Lecture topics
(16 lectures in total together with introduction)

Date	Contents
Module 1. Background in matrix theory and sparse linear systems	
18.02	**Introduction to the course. Outline, schedule and assessment. Matrices and linear systems, applications to real-world problems.**
25.02	**Lecture 1. Fundamentals of Linear Algebra. Introduction of elementary notation.** Basic matrix operations, transposition, inversion, identity matrix, determinant and its properties, eigenvalues and eigenvectors, spectral radius and trace
04.03	**Lecture 2. Types and structures of square matrices. Vector and matrix norms.** Types of square matrices: symmetric, Hermitian, normal, unitary, orthogonal. Structures of square matrices with respect to the location of zero entries: diagonal, bi-and diagonal, triangular, banded, Hessenberg, block. Inner product, Eucledian inner product. Definition of vector norm. Vector norms: Eucledian norm, infinity norm, Holder norms. Definition of matrix norm. Matrix norms: max column, max row, spectral, Frobenius, l1-norm, M-norm. Operator norms. Consistency of vector and matrix norms.
11.03	**Lecture 3. Subspace, range and kernel. Existence of solution. Orthogonality, Gram-Schmidt process and QR-factorization.** Subspace, linear independence, basis, dimension, direct sum, range and kernel, rank, invariant subspace, eigenspace. Existence of a solution. Orthogonal and orthonormal vectors and subspaces, Gram-Schmidt orthogonalization (classic and modified). QR-factorization. Thin and full QR.
18.03	**Lecture 4. Matrix factorizations (QR, LU, Cholesky).** QR-factorization. Thin and full QR. Existence and uniqueness of QR-factorization. Solving linear systems using QR-factorization. LU- and PLU-factorization. Existence and uniqueness of LU- and PLU-factorizations. Using LU to solve linear systems. Cholesky factorization, its existence and uniqueness.
01.04	**Lecture 5. Multiplicities of eigenvalues. Canonical forms of matrices. Diagonal form, jordan form, schur form, singular value decomposition (svd), relation between schur and svd.** Eigenvalues and their multiplicities. Algebraic multiplicity, geometric multiplicity. Main definitions: simple, multiple, semisimple, defective eigenvalue. Simple, semisimple and deragotory matrix. Similarity transformation. Canonical forms of matrices: diagonal form, Jordan form, Schur form (Schur and quasi-Schur form), SVD. Existence and uniqueness. Relation between Schur and SVD
04.04	**Lecture 6. Positive definite matrices. Normal and Hermitian matrices. Powers of matrices. Perturbation analysis. Errors and costs.** Positive definite matrices. Normal and Hermitian matrices. Powers of matrices. Perturbation analysis, absolute and relative error. Condition number and its properties. Types of numerical errors. Well-conditioned and ill-conditioned problem. Stable and unstable algorithm. Computational costs.
11.04	**Lecture 7. Graph representations of sparse matrices. Permutations and reordering.** Definition of sparse matrix. Structured and unstructured sparse matrices. Matrix pattern, types of patterns. Graph representation of a matrix, matrix and its adjacency graph.

	Permutations and Reordering. Symmetric permutation. Fill-ins in direct solution methods. Examples of reordering algorithms.
15.04	**Lecture 8. Storage schemes for sparse matrices. Discretization of partial differential equations: overview of methods.** Storage schemes for sparse matrices: compressed sparse row (CSR), compressed sparse column (CSC), modified CSR and CSC, diagonal format, Ellpack-Itpack format. Algorithms for matrix by vector multiplication for sparse formats.
22.04	**Lecture 9. Discretization of PDEs. Finite difference method.** Discretization of partial differential equations (PDEs). Finite differences. 1D Poisson's equation. 2D Poisson's equation. Node numbering in 2D.
Module 2. Direct and iterative methods. Krylov subspace methods for sparse linear systems and preconditioning techniques	
29.04	**Lecture 10. Discretization of PDE: overview of FEM. Direct and iterative methods for sparse linear systems: comparison. General formulation of iterative methods and convergence criterion.** Discretization of partial differential equations. Discretization by Finite element method. Assembly process. Mesh refinement. Direct and iterative methods. Overview of direct solution methods. Direct sparse methods: Gaussian elimination without and with partial pivoting. Formulation of classic iterative method, convergence: necessary and sufficient conditions.
06.05	**Lecture 11. Classic iterative methods. Simple iteration, Jacobi, Gauss-Seidel, Successive Over Relaxation (SOR), Symmetric Successive Over Relaxation (SSOR) and their variations.** Formulation of classic iterative method. Simple iteration method. Jacobi and Gauss-Seidel methods, their variations. Properties of diagonally dominant matrices, location of matrix eigenvalues. Convergence of Jacobi and Gauss-Seidel for diagonally dominant matrices. Successive Over Relaxation (SOR), Symmetric Successive Over Relaxation (SSOR). Convergence of SOR and SSOR for positive definite matrices. Matrix splitting and preconditioners.
13.05	**Lecture 12. Projection methods, 1D projection methods. Steepest Descent Method (SDM), Minimal Residual Iteration Method (MRIM), Residual Norm Steepest Descent Method (RNSD).** General formulation of a projection method. Search subspace and subspace of constraints. Matrix representation of a projection method. Properties of W^HAV matrix. 1D projection methods. General formulation of 1D projection method. Steepest descent method, its modification and properties. Minimal Residual Iteration method, its modification and properties. Residual Norm Steepest Descent method, its modification and properties. Orthogonal and oblique projection methods
20.05	**Lecture 13. Krylov subspace methods based on Arnoldi's orthogonalization. Full Orthogonalization method (FOM).** Definition of a Krylov subspace. Multi-dimensional projection methods and Krylov subpace projection methods. Arnoldi orthogonalization process (exact and modified). Properties of Arnoldi process, Arnoldi relation. Derivation of Full Orthogonalization method (FOM). FOM algorithm, its modifications (restarted version). Calculation of residual in FOM.

23.05	**Lecture 14. Krylov subspace methods based on Arnoldi's orthogonalization. Generalized Minimal Residual method (GMRES).** Derivation of Generalized Minimal Residual method (GMRES). Givens rotations. GMRES algorithm, its modifications (restarted version). Calculation of residual in GMRES.
27.05	**Lecture 15. Comparison of FOM and GMRES. Methods based on Lacnzoc orthogonalization for symmetric (Hermitian) matrices. Methods based on Lacnzoc orthogonalization for nonsymmetric (non-Hermitian) matrices. Preconditioning techniques. Examples of preconditioners. Examples of preconditioned methods.** Lanczos orthogonalization process for symmetric systems. Classic Lanczos method for symmetric (Hermitian) systems. Lanczos relation. Calculation of residual in Classic Lanczos method. Derivation of Direct Lanczos method. Algorithm of Direct Lanczos (D-Lanczos) method. Residuals and search directions in D-Lanczos. Conjugate Gradient (CG) method. Conjugate Residual (CR) method. Generalized Conjugate Residual (GCR) method. Bi-orthonormal sets. Lanczos method of biorthogonalization, it's properties. Lanczos relation for biorthogonalization. Lanczos method for nonsymmetric systems. Derivation of Biconjugate gradient method. Algorithm of Biconjugate gradient (BiCG) method, it's properties. Types of preconditioning: left, right, two-sides. Examples of preconditioners: Jacobi, Gauss-Seidel, SOR and SSOR preconditioners, incomplete LU-factorization preconditionersPreconditioned Conjugate Gradient method (PCG), Split Preconditioned Conjugate Gradient method (Split PCG). Preconditioned Generalized Minimal Residual method, algorithms of GRMES with left and right preconditioning.