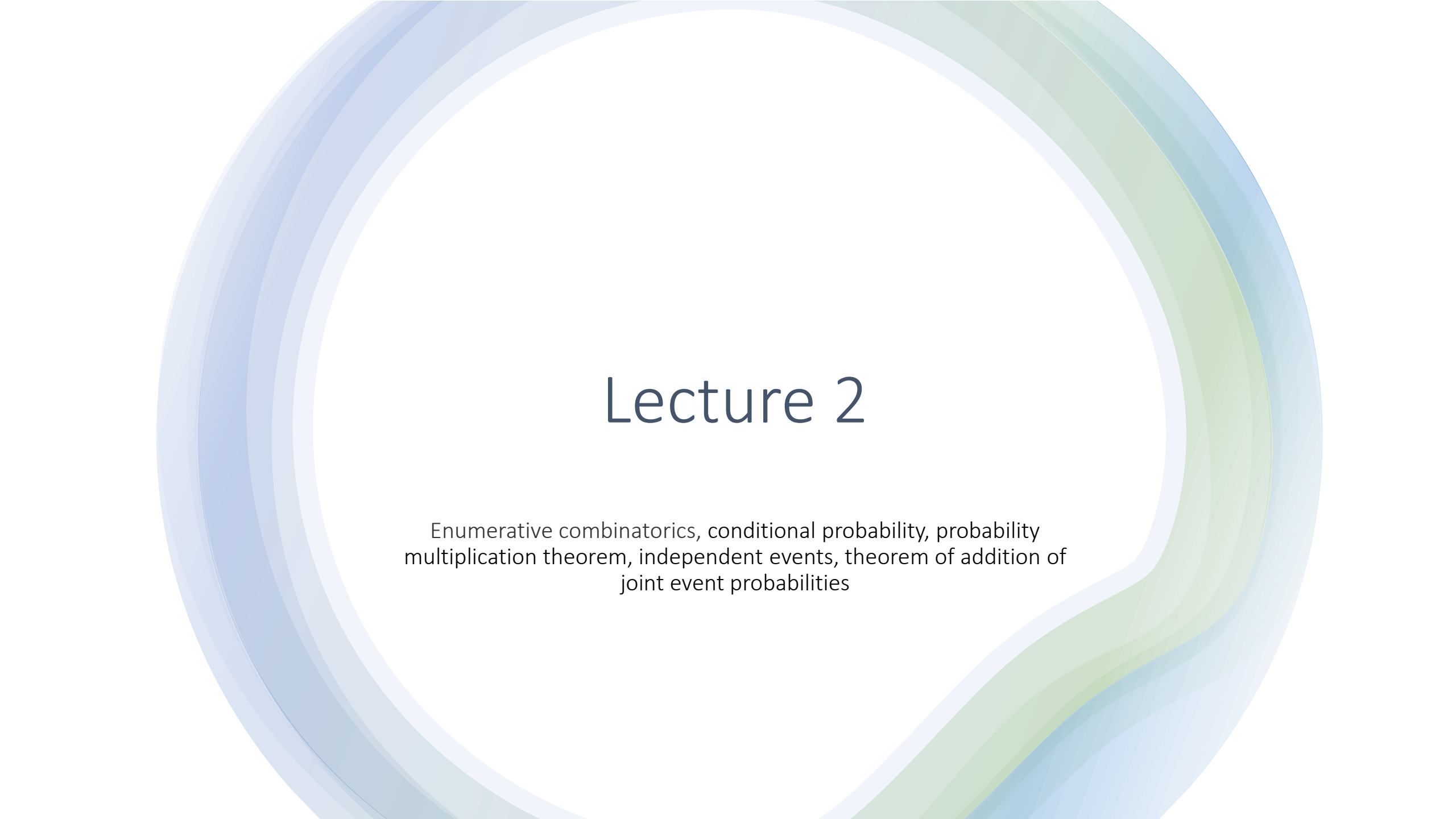




Lecture 2

Enumerative combinatorics, conditional probability, probability multiplication theorem, independent events, theorem of addition of joint event probabilities

Examples

1. A software company has 20 programming tasks in its database, 12 of which are marked as completed. A developer randomly selects 4 tasks for review. Find the probability that all selected tasks will be completed.

Solution.

The total number of possible selections is the number of combinations of 20 tasks taken 4 at a time. Let us recall that **combinations** are responsible for the number of ways to select a certain set of objects from some other, larger set. A combination is a way of selecting k objects from n objects without considering the order of selection. The number of combinations is denoted by $C(n, k)$ and represents the number of possible groups of k elements chosen from a set of n elements. Let's recall the formula for calculating combinations:

$$C(n, k) = \frac{n!}{k! (n - k)!}$$

where:

- n is the total number of objects;
- k is the number of selected objects;
- $!$ denotes the factorial operation.

Examples

A developer selects four distinct tasks uniformly at random.

So, the total number of possible selections is $C(20,4)$. The number of favorable selections (all selected tasks are completed) is $C(12,4)$.

$$\text{Therefore, the required probability is } P(A) = \frac{C(12,4)}{C(20,4)} = \frac{12!}{4!(12-4)!} \div \frac{20!}{4!(20-4)!} = \frac{12!}{4!8!} \cdot \frac{4!16!}{20!} = \frac{8! \cdot 9 \cdot 10 \cdot 11 \cdot 12}{4!8!} \cdot \frac{4!16!}{16! \cdot 17 \cdot 18 \cdot 19 \cdot 20} = \frac{9 \cdot 10 \cdot 11 \cdot 12}{17 \cdot 18 \cdot 19 \cdot 20} = \frac{33}{323} \approx 0.102.$$

The following representation was used in the calculations:

$$12! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot 12 = 1 \cdot 2 \cdot \dots \cdot 8 \cdot \dots \cdot 12 = 8! \cdot 9 \cdot 10 \cdot 11 \cdot 12$$

$$20! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot 20 = 1 \cdot 2 \cdot \dots \cdot 16 \cdot \dots \cdot 20 = 16! \cdot 17 \cdot 18 \cdot 19 \cdot 20$$

Examples

2. The shop employs six men and four women. Seven people were randomly selected by personnel numbers. Find the probability that there will be three women among the selected persons.

Enumerative combinatorics

- **Enumerative combinatorics** studies methods for counting the number of possible outcomes in different types of arrangements and selections. It provides the basic tools for solving problems involving combinations and permutations, which are widely used in probability theory.
- **Sum rule.** If two actions A and B are mutually exclusive, and action A can be performed in m ways, and B in n ways, then any one of these actions (either A or B) can be performed in $m + n$ ways.

Example. There are 16 boys and 10 girls in the class. In how many ways can one attendant be appointed?

Enumerative combinatorics

- **The Fundamental Counting Principle**, also known as the **Multiplication Principle**, states that if a process consists of k successive steps, and the i -th step can be performed in n_i ways for every possible result of the preceding steps, then the entire process can be completed in $\mathbf{N = n_1 \cdot n_2 \cdot ... \cdot n_k}$ different ways

Example. There are 16 boys and 10 girls in the class. In how many ways can two attendants be appointed?

Enumerative combinatorics

- **A permutation of n distinct elements** is an arrangement of all n elements in a specific order. Two permutations are different if the elements occur in different orders. The number of permutations of n distinct elements is **$P(n) = n!$**
- **A permutation of a multiset** is an arrangement of elements in which some elements are identical and appear multiple times. If an element a_1 occurs n_1 times, an element a_2 occurs n_2 times, and so on, with the element a_s occurring n_s times, then the total number of distinct permutations is calculated as:

$$P(n, n_1, \dots, n_s) = \frac{n!}{n_1! n_2! \dots n_s!}$$

Enumerative combinatorics

Example 1.

a) Find the number of words that can be obtained by permutation the letters of the word “HORSE”.

$$P(n) = n!; P(5) = 5! = 120$$

6) Find the number of words that can be obtained by permutation the letters of the word “FINANCE”.

$$P(n, n_1, \dots, n_s) = \frac{n!}{n_1!n_2!\dots n_s!}; P(7,2) = \frac{7!}{2!} = 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 2520$$

Enumerative combinatorics

- **An ordered selection with replacement** is a method of choosing k elements from a set of n elements, where the same element can be selected more than once. The number of possible ordered selections is $N = n^k$ where n is the number of available elements and k is the number of selections.
- **A k -permutation of n distinct elements** is an ordered selection of k elements from a set of n elements, where each element can be chosen at most once. Since the order of selected elements is important, different arrangements of the same elements are considered different outcomes. The number of possible k -permutations of n elements is:

$$P(n, k) = \frac{n!}{(n - k)!}$$

Enumerative combinatorics

- **A combination of n elements taken k at a time** is a selection of k elements from a set of n elements, where the order of selected elements does not matter and each element can be chosen at most once. The number of possible combinations of n elements taken k at a time is given by:

$$C(n, k) = \frac{n!}{k! (n - k)!}$$

Enumerative combinatorics

Example 2. From a deck of 52 cards, 3 cards are drawn at random. Find the number of ways this can be done if the cards are drawn

- a) with a return to the deck
- b) without returning to the deck

a) The number of distinct samples from n elements by k with return is n^k . $52^3 = 140608$

b) The number of distinct samples from n elements by k without return is $P(n, k) = \frac{n!}{(n-k)!}$.

$$P(52, 3) = \frac{52!}{(52-3)!} = \frac{52!}{49!} = 52 \cdot 51 \cdot 50 = 132600$$

Example 3. Determine the number of groups of 3 people that can be formed from 8 people

$$C(n, k) = \frac{n!}{k!(n-k)!}; C(8, 3) = \frac{8!}{3!(8-3)!} = 56$$

Conditional probability

The conditional probability is the probability that one event will occur given that another event has already occurred.

It is known that event B has occurred. Knowing this, calculate the probability of event A. The conditional probability of event A relative to event B is calculated by the formula:

$$P(A|B) = \frac{P(AB)}{P(B)}, P(B) \neq 0$$

Example. 2 fair dice are tossed in turn. What is the probability that the sum of the points is 10 if it is known that the number 5 fell on one die.

$$B = \{a_1 = 5\}, A = \{a_1 + a_2 = 10\}$$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\{a_1=5, a_2=5\})}{P(\{a_1=5\})} = \frac{1}{36} : \frac{1}{6} = \frac{1}{6}$$

Probability multiplication theorem

Theorem. The probability of the joint occurrence of two events A and B is equal to the product of the probability of one of them by the conditional probability of the other, calculated under the condition that the first event has already occurred:

$$P(AB) = P(A) \cdot P(B|A)$$

Proof: by definition of conditional probability $P(B|A) = \frac{P(AB)}{P(A)}$, so $P(AB) = P(A)P(B|A)$

Corollary. Consider an event A that occurs if and only if each of the events $A_1, \dots, A_n$ happened, that is $A = A_1A_2 \dots A_n$, so

$$P(A_1A_2 \dots A_n) = P(A_1)P(A_2|A_1) \dots P(A_n|A_1A_2 \dots A_{n-1})$$

Probability multiplication theorem

Example. There is a deck of 36 cards. 3 cards are drawn consecutively. What is the probability that there are no spades among these cards?

Consider events : $A_1 = \{the\ first\ card\ is\ not\ a\ spade\}$;

$A_2 = \{the\ second\ card\ is\ not\ a\ spade\}$; $A_3 = \{the\ third\ card\ is\ not\ a\ spade\}$. Then, according to the condition of the problem, one should find $P(A_1A_2A_3)$. According to the probability multiplication theorem, we have: $P(A_1A_2A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)$.

$$P(A_1) = \frac{27}{36}; P(A_2|A_1) = \frac{26}{35}; P(A_3|A_1A_2) = \frac{25}{34}$$
$$P(A_1A_2A_3) \approx 0,41$$

Independent events

Two events are said to be independent if the occurrence of one of them makes it neither more nor less probable that the other occurs.

An event B is called **independent** of the event A if the occurrence of the event A does not change the probability of the event B , that is, if the conditional probability of the occurrence of the event B is equal to its unconditional probability:

$$P(B|A) = P(B)$$

If event B does not depend on A , then event A does not depend on B .

For independent events, the probability multiplication theorem has the form:

$$P(AB) = P(A)P(B)$$

Two events are called independent if the probability of their joint occurrence is equal to the product of the probabilities of these events, otherwise the events are called dependent.

Example. Find the probability of joint hitting the target by two guns, if the probability of hitting the target by the first gun (event A) is 0.8, the probability of hitting the target by the second gun (event B) is 0.7.

Tasks

Tasks

1. There are five identical products in the box, and three of them are painted. Two items were retrieved at random. Find the probability that among the two items taken out there will be: a) one painted item; b) two painted products; c) at least one painted product.
2. There are 100 details in a box, 10 of them are defective. Four details were randomly extracted. Find the probability that among the extracted details: a) there are no defective ones; b) no good ones.

Tasks

3. Find the number of words made up of the letters of the word:

-FATHER

-MATHEMATICS

4. Two boys are taking part in a tennis tournament. The winner is the one who wins 2 games in a row or 3 games in total. In how many ways can the tournament be played? (use tree diagram)

5. There are 8 boys and 6 girls in the class. Determine the number of ways to choose:

- a) 1 person from the class;
- b) 1 boy and one girl;
- c) 1 president and 1 vice president

Tasks

6. A farmer buys 3 cows, 2 pigs, 4 ducks from a man who has 6 cows, 5 pigs, 8 ducks. How many ways are there to make this purchase?
7. There are 7 blue and 5 red balls in a box. Determine the number of ways: a) choose 2 balls;
b) choose 2 balls of the same color
8. Mom has 2 apples and 3 pears. Every day for 5 days in a row, she gives out one piece of fruit. In how many ways can this be done?

Homework

Study guide: p.33-35 Tasks 1-16