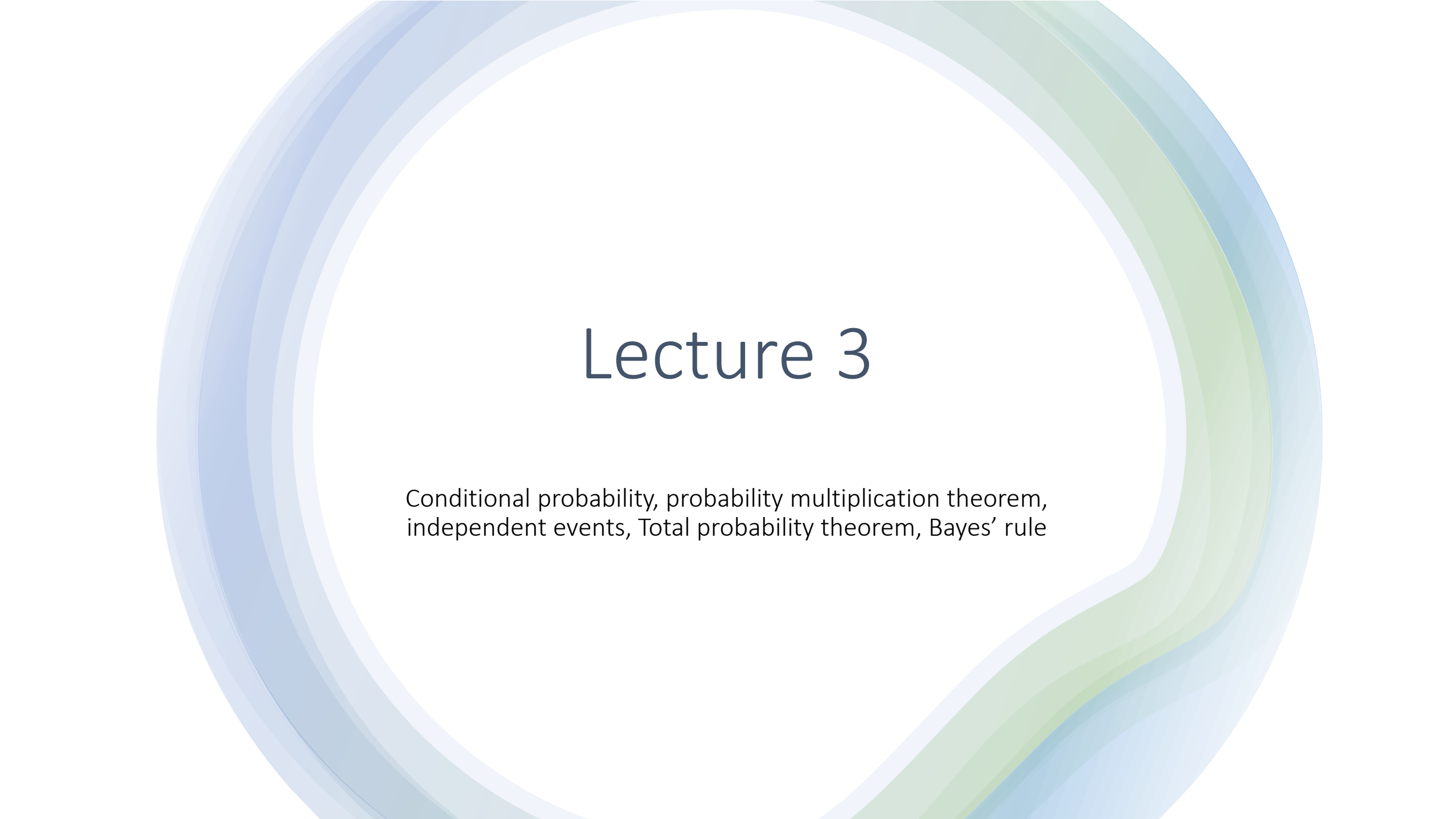




Lecture 3

Conditional probability, probability multiplication theorem,
independent events, Total probability theorem, Bayes' rule

Conditional probability

The conditional probability is the probability that one event will occur given that another event has already occurred.

It is known that event B has occurred. Knowing this, calculate the probability of event A. The conditional probability of event A relative to event B is calculated by the formula:

$$P(A|B) = \frac{P(AB)}{P(B)}, P(B) \neq 0$$

Example. Two fair six-sided dice are rolled. What is the probability that the sum of the points is 10 if it is known that at least one die shows 6.

Probability multiplication theorem

Theorem. The probability of the joint occurrence of two events A and B is equal to the product of the probability of one of them by the conditional probability of the other, calculated under the condition that the first event has already occurred:

$$P(AB) = P(A) \cdot P(B|A)$$

Proof: by definition of conditional probability $P(B|A) = \frac{P(AB)}{P(A)}$, so $P(AB) = P(A)P(B|A)$

Corollary. Consider an event A that occurs if and only if each of the events $A_1, \dots, A_n$ happened, that is $A = A_1A_2 \dots A_n$, so

$$P(A_1A_2 \dots A_n) = P(A_1)P(A_2|A_1) \dots P(A_n|A_1A_2 \dots A_{n-1})$$

Probability multiplication theorem

Example. There is a deck of 36 cards. 3 cards are drawn consecutively. What is the probability that there are no spades among these cards?

Consider events : $A_1 = \{the\ first\ card\ is\ not\ a\ spade\}$;

$A_2 = \{the\ second\ card\ is\ not\ a\ spade\}$; $A_3 = \{the\ third\ card\ is\ not\ a\ spade\}$. Then, according to the condition of the problem, one should find $P(A_1A_2A_3)$. According to the probability multiplication theorem, we have: $P(A_1A_2A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)$.

$$P(A_1) = \frac{27}{36}; P(A_2|A_1) = \frac{26}{35}; P(A_3|A_1A_2) = \frac{25}{34}$$
$$P(A_1A_2A_3) \approx 0,41$$

Independent events

Corollary. The probability of the joint occurrence of several events that are independent in the aggregate is equal to the product of the probabilities of these events:

$$P(A_1 A_2 \dots A_n) = P(A_1)P(A_2) \dots P(A_n)$$

Proof: Consider 3 events A, B, C . $P(ABC) = P(AB \cdot C)$. The independence of the events A, B, C implies the independence of AB and C , as well as A and B . We use the probability multiplication theorem for two independent events:

$$P(AB \cdot C) = P(AB)P(C) \text{ и } P(AB) = P(A)P(B)$$

Finally we get: $P(ABC) = P(A)P(B)P(C)$

Example. A company has three warehouses containing computer components. Each warehouse contains 20 components. The first warehouse contains 16 working components; the second warehouse contains 18 working components; the third warehouse contains 15 working components. One component is selected independently and uniformly at random from each warehouse. Find the probability that all selected components are working.

Examples

Consider an example of the joint application of the addition and multiplication theorems.

Example. Probabilities of occurrence of each of three independent events A_1, A_2, A_3 respectively equal p_1, p_2, p_3 . Find the probability of occurrence of only one of these events.

Example. The printing house has 4 flatbed printing presses. For each machine, the probability that it is running at a given time is 0.9. Find the probability that at least one machine is running at the moment.

Theorem of addition of joint event probabilities

Two events are called **joint** , if the appearance of one of them does not exclude the appearance of the other in the same experience.

Example. The event A is the appearance of four points when throwing a die, B is the appearance of an even number of points.

Theorem. The probability of the occurrence of at least one of the two joint events is equal to the sum of the probabilities of these events without the probability of their joint occurrence:

$$P(A + B) = P(A) + P(B) - P(AB)$$

Total probability theorem

Theorem. Probability of event B , which can occur only together with one of the events $A_1, \dots, A_n$, forming a complete group:

$$P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$$

Example. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1). 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

Let $A_i = \{\text{playing with the opponent of type } i\}$, we have:

$$P(A_1) = 0,5; P(A_2) = 0,25; P(A_3) = 0,25$$

Let $B = \{\text{winning}\}$, we have $P(B|A_1) = 0,3; P(B|A_2) = 0,4; P(B|A_3) = 0,5;$

Thus, by the total probability theorem, the probability of winning is:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) = 0,375$$

Total probability theorem

Example. A software company has three development teams:

- Team A develops 40% of all software modules;
- Team B develops 35% of all software modules;
- Team C develops 25% of all software modules.

The probability that a module contains a programming error is:

- 2% for modules developed by Team A;
- 4% for modules developed by Team B;
- 5% for modules developed by Team C.
- A module is selected at random.

Find the probability that the selected module contains a programming error.

Bayes' rule

Let event B occur as a result of the implementation of one of the events $A_1, \dots, A_n$. We determine the probability that one or another hypothesis took place.

The probability of event B is determined by the total probability theorem:

$$P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$$

Suppose that a test was performed, as a result of which an event appeared B . Define the conditional probability $P(A_1|B)$. By the multiplication theorem, we get :

$$P(BA_1) = P(B)P(A_1|B) = P(A_1)P(B|A_1)$$

So $P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(B)}$, write $P(B)$ according to the total probability formula :

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

Bayes' rule

Let event B occur as a result of the implementation of one of the hypotheses $A_1, \dots, A_n$. We determine the probability that one or another hypothesis took place:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$

Example. Let's consider the first example. Let the player win. What is the probability that he played a game with a player of type 1?

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)} = \frac{0,3 \cdot 0,5}{0,375} = 0,4$$

Bayes' rule

Example. A software company has three development teams:

- Team A develops 40% of all software modules;
- Team B develops 35% of all software modules;
- Team C develops 25% of all software modules.

The probability that a module contains a programming error is:

- 2% for modules developed by Team A;
- 4% for modules developed by Team B;
- 5% for modules developed by Team C.

It is known that the selected module contains a programming error. Find the probability that this module was developed by team A.

Bernoulli experiments

A **Bernoulli trial** (or **binomial trial**) is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

1. The experiment is repeated a fixed number of times (n times).
2. Each trial has only two possible outcomes, "success" and "failure". The possible outcomes are exactly the same for each trial.
3. The probability of success remains the same for each trial. We use p for the probability of success (on each trial) and $q = 1 - p$ for the probability of failure.
4. The trials are independent (the outcome of previous trials has no influence on the outcome of the next trial).

Bernoulli experiments

Let n independent trials be performed, in each of which event A can either appear with probability p or not appear with probability $q = 1 - p$. Let us determine the probability that in a series of n independent trials the event A will repeat exactly k times.

$p^k q^{n-k}$ — the probability that in n trials event A occurs k times and does not occur $n-k$ times; such events can be C_n^k

According to the theorem of addition of probabilities of incompatible events

$$P_n(k) = C_n^k p^k q^{n-k}$$

Bernoulli experiments

Example. Flip a coin 12 times, count the number of heads.

Here $n = 12$. Each flip is a trial. It is reasonable to assume the trials are independent.

Each trial has two outcomes heads (success) and tails (failure).

The probability of success on each trial is $p = 1/2$ and the probability of failure is $q = 1 - 1/2 = 1/2$.

This is an example of a Bernoulli Experiment with 12 trials.

Example. A shooter makes 6 shots at a target. The probability of hitting with one shot is 0.3. Find the probability that he hit 4 times.

Tasks

Tasks

1. A four-sided dice is tossed twice. Let X, Y be the results of the first and second throws, respectively. Determine $P(A|B)$, where $A=\{\max(X,Y)=4\}$, $B=\{\min(X,Y)=2\}$.
2. To signal an accident, two independently operating signaling devices are installed. The probability that the signaling device will work in case of an accident is 0.95 for the first signaling device and 0.9 for the second one. Find the probability that only one signaling device will work in case of an accident.
3. There are 15 textbooks randomly arranged on a library shelf, 5 of them bound. The librarian takes 3 textbooks at random. Find the probability that at least one of the selected textbooks will be bound.
4. Two shooters shoot at a target. The probability of hitting the target with one shot for the first shooter is 0.7, and for the second - 0.8. Find the probability that in one shot only one of the shooters hits the target.

Tasks

5. The device consists of three elements that work independently. The probabilities of no-failure operation (during time t) of the first, second and third elements, respectively, are equal to 0.6; 0.7; 0.8. Find the probabilities that during time t the following will work without fail: a) only one element; b) only two elements; c) all three elements.
6. A software company employs 14 programmers, including 8 backend developers and 6 frontend developers. Five programmers are randomly selected for a new project. Find the probability that exactly three of the selected programmers are frontend developers.

Tasks

7. A software company has three development teams, X, Y, and Z. Team X has developed 10 software modules, 4 of which contain programming errors. Team Y has developed 6 software modules, 1 of which contains a programming error. Team Z has developed 8 software modules, 3 of which contain programming errors. One module is selected uniformly at random from the 24 modules developed by the three teams.

- a) Find the probability that the selected module is error-free.
- b) Given that the selected module is error-free, find the probability that it was developed by Team Z.

Tasks

8. A factory produces two types of machine parts: Type A and Type B. Type A and Type B parts are produced in the ratio 4 : 3. The probability that a Type A part fails the quality inspection is 0.08, while for a Type B part this probability is 0.15. A randomly selected part has failed the quality inspection. Find the probability that it is a Type B part.
9. Research conducted by psychologists has shown that men and women may respond differently to certain life situations. The study found that: 75% of women respond positively to these situations. 35% of men respond negatively to these situations. A survey was completed by 18 women and 12 men. Determine the probability that a randomly selected questionnaire contains a positive response. A randomly selected questionnaire contains a negative response. Find the probability that it was completed by a man.

Tasks

11. Five students from Group 1, seven from Group 2, and four from Group 3 participate in a programming contest. The probabilities that a student from the respective groups solves a particular difficult problem are 0.6, 0.5, and 0.4.

a) What is the probability that a randomly selected contestant will solve the difficult problem?

b) A randomly selected contestant solved the difficult problem. What is the probability that he is from the second group?

Homework

Study guide: p.37-38 Tasks 1, 3, 5, 7; p.44-45 Tasks 1, 3, 7, 10, 12; p. 52-33 Tasks 2, 6, 7, 8