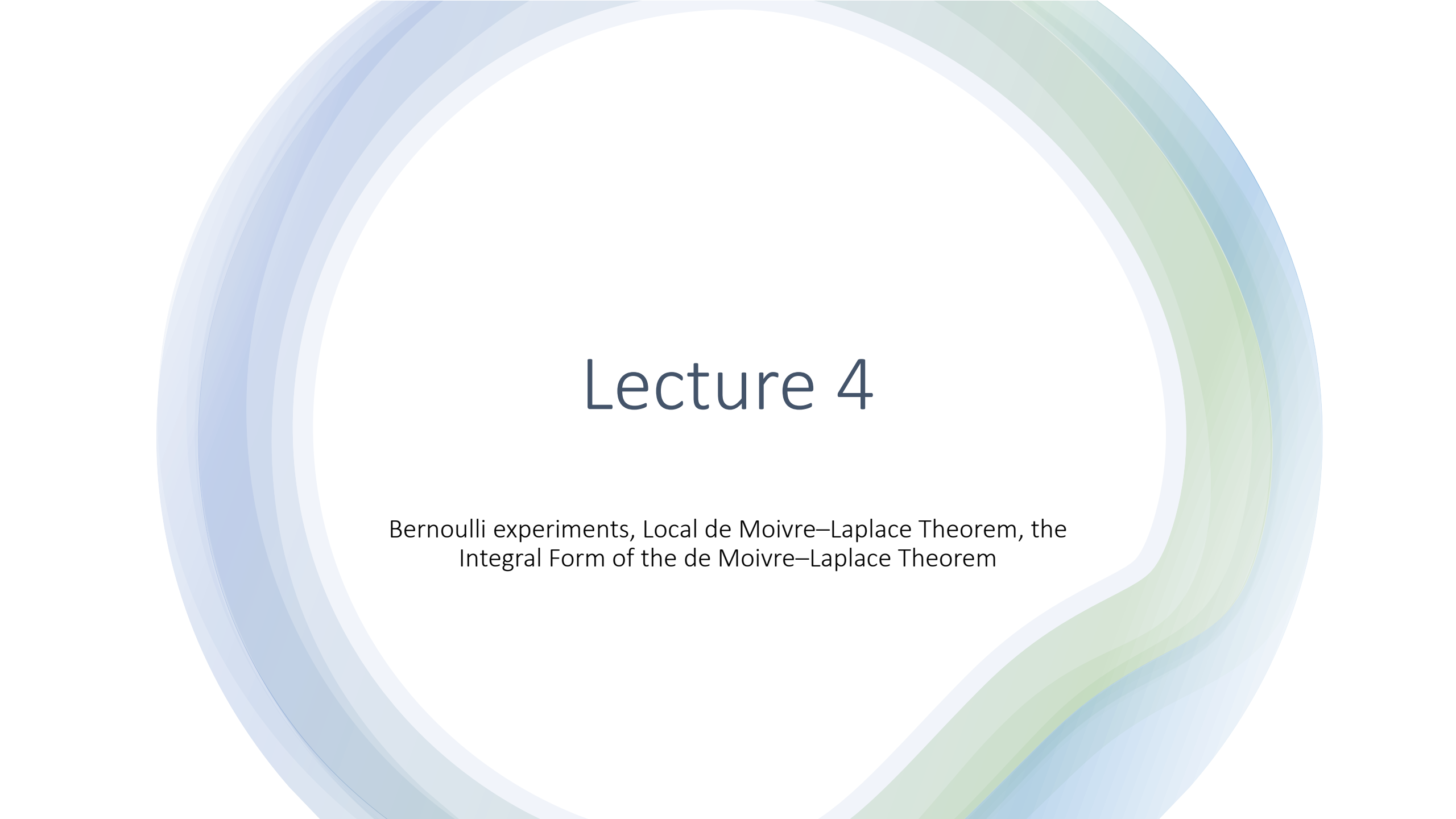




Lecture 4

Bernoulli experiments, Local de Moivre–Laplace Theorem, the
Integral Form of the de Moivre–Laplace Theorem

Bernoulli experiments

A **Bernoulli trial** (or **binomial trial**) is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

A **binomial experiment** consists of a fixed number n of Bernoulli trials and satisfies the following conditions:

- The number of trials, n , is fixed in advance.
- Each trial has exactly two possible outcomes: success and failure.
- The probability of success remains constant from trial to trial and is equal to p . Consequently, the probability of failure is $q = 1 - p$.
- The trials are independent. This means that the outcome of one trial does not affect the outcome of any other trials.

Bernoulli experiments

Let n independent trials be performed, in each of which event A can either appear with probability p or not appear with probability $q = 1 - p$. Let us determine the probability that in a series of n independent trials the event A will repeat exactly k times.

$p^k q^{n-k}$ — the probability that in n trials event A occurs k times and does not occur $n-k$ times; such events can be C_n^k

The probability of obtaining exactly k successes in n trials is given by **the binomial probability formula**:

$$P_n(k) = C_n^k p^k q^{n-k}$$

Bernoulli experiments

Example. A fair coin is tossed ten times. Find the probability of obtaining exactly six heads.

Why the Binomial Formula Can Be Applied

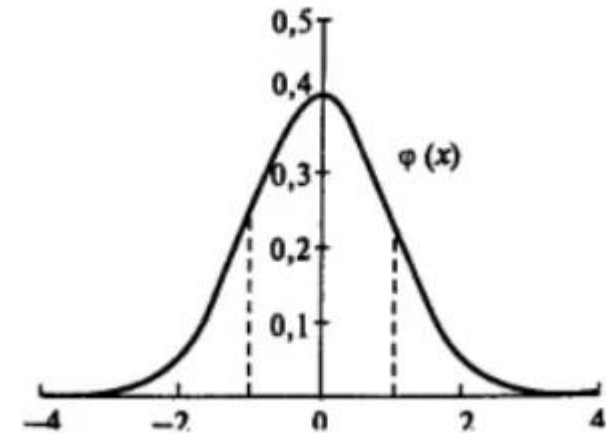
- The Binomial Formula can be used because all the necessary conditions are satisfied:
- The number of trials is fixed: the coin is tossed exactly 10 times.
- Each trial has exactly two possible outcomes: heads or tails.
- The probability of success remains constant in every trial: $p = 0.5$.
- The trials are independent: the result of one toss does not affect the results of the other tosses.
- We are interested in the probability of obtaining exactly 6 successes.

Local de Moivre–Laplace Theorem

The local de Moivre–Laplace theorem is used to approximate the probability of obtaining exactly k successes in a **large number** of independent Bernoulli trials.

Theorem. If the probability p of the occurrence of the event A in each trial is constant and different from zero and one, then the probability $P_n(k)$ that the event A in n trials will occur k times is approximately equal to the value of the function $y = \frac{1}{\sqrt{npq}} \varphi(x)$ where $x = \frac{k-np}{\sqrt{npq}}$, and $\varphi(x)$ – **standard normal probability density function**.

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



Local de Moivre–Laplace Theorem

The standard normal density function has the following properties:

- $\varphi(x)$ is defined for all real x .
- $\varphi(x) > 0$ for all x .
- **$\varphi(-x) = \varphi(x)$** , so the function is even.
- The graph is symmetric about the y -axis.
- The maximum value is attained at $x = 0$.
- The function increases on $(-\infty, 0)$ and decreases on $(0, +\infty)$.
- $\varphi(x)$ approaches zero as x approaches $\pm\infty$.
- Although $\varphi(x)$ is positive for every finite value of x , its values become negligibly small for large $|x|$. In most practical calculations, $\varphi(x)$ may be taken **to be approximately zero when $|x| \geq 4$** .

Local de Moivre–Laplace Theorem

That is, the probability $P_n(k)$ that the event A in n trials will occur k times is approximately equal to $P_n(k) \approx \frac{1}{\sqrt{npq}} \varphi(x)$ where $x = \frac{k-np}{\sqrt{npq}}$.

Example. A cloud service processes 500 independent user requests. The probability that a single request results in a server error is 0.12. Find the probability that exactly 50 of the 500 requests result in server errors.

Local de Moivre–Laplace Theorem

Example. A die is tossed 500 times. What is the probability that the number 1 will appear 50 times?

The Integral Form of the de Moivre–Laplace Theorem

The theorem can be applied under the following conditions:

- The experiment consists of n independent Bernoulli trials.
- Each trial has exactly two possible outcomes, conventionally called success and failure.
- The probability of success is the same in every trial and is denoted by p .
- The probability of failure is $q = 1 - p$.
- The number of trials n is sufficiently large.
- Both np and nq are sufficiently large. A commonly used practical guideline is: $np \geq 5$ and $nq \geq 5$.

The Integral Form of the de Moivre–Laplace Theorem

The integral form of the de Moivre–Laplace theorem is used to approximate the probability that the number of successes in a **large number** of independent Bernoulli trials lies within a specified interval.

Theorem. If the probability p of the occurrence of the event A in each trial is constant and different from zero and one, then the probability $P_n(k_1, k_2)$ that event A in n trials will come from k_1 to k_2 times is approximately equal to a definite integral

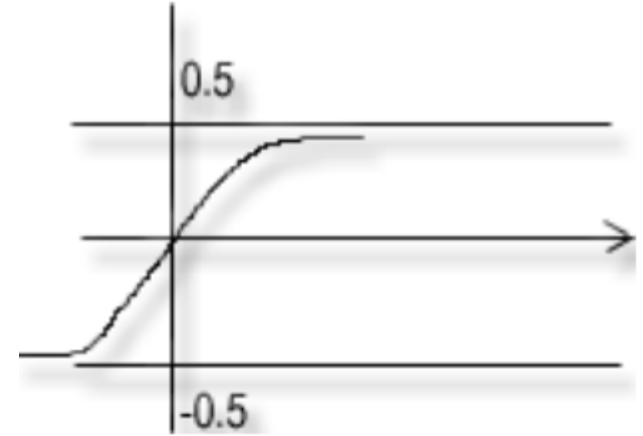
$$P_n(k_1, k_2) = \frac{1}{\sqrt{2\pi}} \int_{x'}^{x''} e^{-\frac{z^2}{2}} dz, \text{ где } x' = \frac{k_1 - np}{\sqrt{npq}}, x'' = \frac{k_2 - np}{\sqrt{npq}}$$

$$\begin{aligned} \text{Laplace function } \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{z^2}{2}} dz, \text{ then } P_n(k_1, k_2) = \frac{1}{\sqrt{2\pi}} \int_{x'}^{x''} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{x'}^0 e^{-\frac{z^2}{2}} dz + \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^{x''} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_0^{x''} e^{-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_0^{x'} e^{-\frac{z^2}{2}} dz = \Phi(x'') - \Phi(x') \end{aligned}$$

The Integral Form of the de Moivre–Laplace Theorem

The Laplace function has the following properties [6]:

- $\Phi(x)$ is defined for all real values of x .
- $\Phi(0) = 0$.
- The function $\Phi(x)$ is odd: $\Phi(-x) = -\Phi(x)$.
- The function is strictly increasing on the entire real line.
- The values of the Laplace range from -0.5 to 0.5.
- As x approaches positive infinity:
- $\lim_{x \rightarrow +\infty} \Phi(x) = \frac{1}{2}$.
- As x approaches negative infinity:
- $\lim_{x \rightarrow -\infty} \Phi(x) = -\frac{1}{2}$.
- $\Phi(x) \approx 0.5$ when $x \geq 4$, $\Phi(x) \approx -0.5$ when $x \leq -4$.



The Integral Form of the de Moivre–Laplace Theorem

The Integral Form of the de Moivre–Laplace Theorem). Suppose that n independent Bernoulli trials are performed and that the probability p of the occurrence of an event A is the same in each trial, where $0 < p < 1$ and $q = 1 - p$. If n is sufficiently large, then the probability that the event A occurs at least k_1 times and at most k_2 times can be approximated by the formula:

$$P_n(k_1, k_2) \approx \Phi(x'') - \Phi(x'), \text{ где } x' = \frac{k_1 - np}{\sqrt{npq}}, x'' = \frac{k_2 - np}{\sqrt{npq}}$$

The Integral Form of the de Moivre–Laplace Theorem

Example 1 . Find the probability that if you toss a coin 200 times, it will come up heads 90 to 110 times.

Example 2. An artificial intelligence system classifies 300 images independently. The probability that the system classifies any given image correctly is 0.8. Find the probability that between 230 and 250 images, inclusive, are classified correctly.

Tasks

Tasks

1. The factory uses 3 machines X, Y, Z. X produces 50% of the parts, with 3% of them defective. Y produces 30% of the details, and 4% of them are defective. Z produces 20% of the details, and 5% of them are defective. What is the probability that a randomly selected detail has a defect?
3. Consider the factory from Problem 8. Let a defective detail be produced. Find the probability that it was released by machine X.
4. The number of trucks driving along a highway on which there is a gas station is related to the number of cars driving along the same highway as 3:2. The probability that a truck will refuel is 0.1; for a car this probability is 0.2. A car drove up to a gas station to refuel. Find the probability that it is a truck.

Tasks

5. Two equal chess players play chess. Which is more likely: to win two games out of four or three games out of six (draws are not taken into account)?
6. The coin is tossed five times. Find the probability that the “head” will fall: a) less than two times; b) at least twice.
7. There are five children in the family. Find the probability that among these children: a) two boys; b) no more than two boys; c) more than two boys; d) not less than two and not more than three boys. The probability of having a boy is assumed to be 0.51.
8. The test consists of six problems, and the test is considered passed if the student has solved at least four of them. The student can solve each problem with a probability of 0.6. What is the probability that he will pass the test?

Tasks

9. An artificial intelligence system analyzes 200 images independently. The probability that the system classifies any given image correctly is 0.70. Find the probability that the number of correctly classified images is:

a) at least 130 and no more than 150;

b) at least 130;

c) no more than 129.

10. A software tester checks ten program modules. Each module independently contains an error with probability 0.15. Find the probability that exactly two of the modules contain errors.

11. A university administers an online test to 2500 students. Each student passes the test independently with probability 0.64. Find the probability that exactly 1550 students pass the test.

Tasks

12. An artificial intelligence system analyzes 600 medical images independently. The probability that the system classifies any given image correctly is 0.82. Using the integral form of the de Moivre–Laplace theorem and the continuity correction, find:

- a) the probability that between 475 and 510 images, inclusive, are classified correctly;
- b) the probability that the number of correctly classified images differs from 492 by no more than 15;
- c) the smallest positive integer d such that the probability that the number of correctly classified images differs from 492 by no more than d is at least 0.95.

Homework

Study guide. p. 63-64 Tasks 1-11

Prepare for the **Test:** study all the topics and problems in the study guide from pages 1 to 65