

Scientific Computing Software. Maple

Lecture 3. Symbolic mathematics in Maple

Symbolic manipulation of mathematical expressions. Limits and series expansions. Function extrema. Symbolic differentiation and integration. Solving algebraic and transcendental equations and systems

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Symbolic manipulation of mathematical expressions

- Expression substitution commands

Expression substitution commands: subs

subs(x=a, expr) – substitution of x=a into the expression **expr**, depending on x

subs(eq1,... eqn , expr) – performing substitutions written as equations eq1,..., eqn , into the expression expr

> subs(x=-2, x² + x + 1)

3

> subs(sin(x) = t, $\frac{\sin(x)}{\sqrt{1 - \sin(x)}}$)

$\frac{t}{\sqrt{1 - t}}$

> subs(x = 1, y = 2, x - 5·y)

-9

> subs([x = 1, y = 2], x - 5·y)

-9

> subs($x = \frac{\text{Pi}}{2}$, y = Pi, sin(x) + cos(y))

$\sin\left(\frac{1}{2}\pi\right) + \cos(\pi)$

> simplify(%)

0

- Substitution is not evaluation and further simplification of an expression can be required

Methods for defining functions in Maple

- Function as Maple expression
- Function as a functional operator

Function as Maple expression

f:=expr , where **f** is function name, **expr** is Maple expression

To evaluate a function defined as Maple expression at a point, use

➤ **eval** command in the form **eval (expr , x = a)** for a function of one variable or **eval (expr , [x1 = a1,..., xn =an]** for a function of several variables

➤ or the **subs** command in the form **subs (x= a,expr)** for a function of one variable or **subs ([x1 = a1,..., xn =an] , expr)** or **subs (x1 = a1,..., xn =an , expr)** for a function of several variables

➤ **Important** : **subs** is substitution and not evaluation, that's why the result might need further simplification

```
> restart
```

```
> f := sin(x) + x^2 :
```

```
> eval(f, x = π)
```

π^2

```
> subs(x = π, f)
```

$\sin(\pi) + \pi^2$

```
> simplify(%)
```

π^2

```
> evalf(%)
```

9.869604404

```
> g := sin(x) + cos(y) :
```

```
> eval(g, [x = π/3, y = π/6])
```

$\sqrt{3}$

```
> subs(x = π/3, y = π/6, g)
```

$\sin\left(\frac{1}{3}\pi\right) + \cos\left(\frac{1}{6}\pi\right)$

```
> subs([x = π/3, y = π/6], g)
```

$\sin\left(\frac{1}{3}\pi\right) + \cos\left(\frac{1}{6}\pi\right)$

```
> simplify(%)
```

$\sqrt{3}$

Function as Maple expression

➤ **Important** : note what happens when we don't use **eval** or **subs** , but assign the value to x instead

```
> f := sin(x) + x^2 :
```

x is independent variable for f, but what happens when x has assigned value?

```
> x := π/4 : f
```

$$\frac{1}{2} \sqrt{2} + \frac{1}{16} \pi^2$$

```
> x := 10 : f
```

$$\sin(10) + 100$$

```
> g := 3·a + 5·b^3 : a := 1 : b := -1 : g
```

$$-2$$

Now x should be independent variable for g1, but x has assigned value:

```
> g1 := 3 x + 5 y^3 : x := 1 : y := -1 : g1
```

Explain the result

25

Function as functional operator

- Defining a function using an arrow operator
- The user-defined function is accessed as a Maple command.

2D Math-input mode is automatically converted from two characters entered from the keyboard to $\rightarrow$.

$\> f := x \rightarrow \sin(x) + x^2$

$$f := x \rightarrow \sin(x) + x^2$$

$\> f\left(\frac{\pi}{4}\right); f(10);$

$$\frac{1}{2} \sqrt{2} + \frac{1}{16} \pi^2$$

$$\sin(10) + 100$$

$\> g := (x, y) \rightarrow \sin(x) + \cos(y)$

$$g := (x, y) \rightarrow \sin(x) + \cos(y)$$

$\> g\left(\frac{\pi}{3}, \frac{\pi}{6}\right)$

$$\sqrt{3}$$

Direct and inert action commands

- For such mathematical operations as calculating the limit, derivative, integral, sum of a series and some other operations, there are two forms of recording: **active** (direct action command) and **inert** (deferred action command)
- **Direct action commands** begin with a lowercase letter and are executed immediately.
- **Inert action commands** begin with a capital letter and are used to record an operation in mathematical form.

To execute them, use the command **value**.

Some commands with two forms of writing

diff and **Diff**

int and **Int**

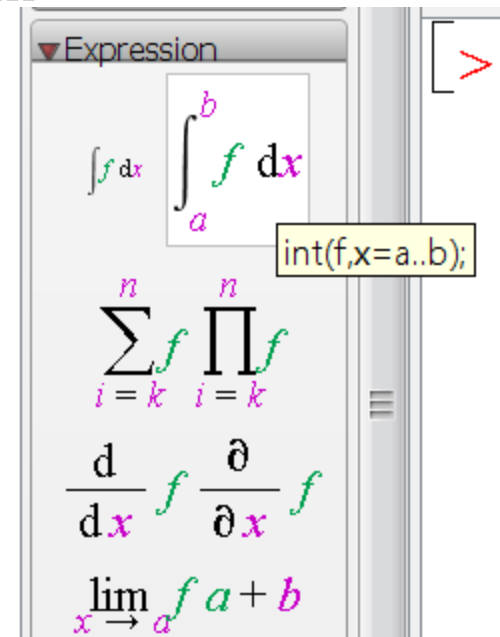
limit and **Limit**

normal and **Normal**

product and **Product**

sum and **Sum**

- The templates in the Expression panel are direct action commands.



Calculating Limits and Derivatives

- Calculating Limits
- Differentiation of a function of one variable
- Differentiation of a function of several variables
- Differential operator

Calculating Limits

- `limit ( f, x=a )`

`limit ( f, x=a, dir)`

a is a limit point, including **infinity** or **-infinity**

dir – limit type: **left** (left limit), **right** (right limit), **real** (real limit),
complex (complex limit)

- **Limit** - a similar inert command



```
> lim_{x \to \infty} e^x; lim_{x \to -\infty} e^x
```

∞

0

```
> Limit(exp(x), x = infinity)=limit(exp(x), x = infinity);
```

$\lim_{x \rightarrow \infty} e^x = \infty$

```
> Limit(exp(x), x = infinity)=limit(exp(x), x = -infinity);
```

$\lim_{x \rightarrow \infty} e^x = 0$

Differentiation of a function of one variable

- **diff (f, x)**
- **diff(f,x\$n)** – calculation of the **n** - th order derivative
diff(f(x),x\$2) is equivalent to **diff(f(x), x,x)**
- **Diff** - a similar inert command

$$\frac{d}{dx} f$$

```
> d/dx sin(x)
```

$$\cos(x)$$

```
> Diff(sin(x),x)=diff(sin(x),x);
```

$$\frac{d}{dx} \sin(x) = \cos(x)$$

```
> d^2/dx^2 (3 x^3 + 2 x^2 + 5 x + 10)
```

$$18x + 4$$

```
> diff(3 x^3 + 2 x^2 + 5 x + 10, x, x, x)
```

$$18$$

Differentiation of a function of several variables

- **diff (f, x1, ..., xj)** – calculation of partial derivatives
- **diff (f,x1\$n1,x2\$n2,..., xm\$nm)** – partial derivatives

$$\frac{\partial}{\partial x} f$$

higher orders

diff(diff (f(x1,x2), x1), x2) is equivalent to **diff(f(x1,x2), x1, x2)**

diff(g(x,y),x\$2,y\$3) is equivalent to **diff(g(x,y), x,x,y,y,y)**

- **Diff** is a similar inert command

$$\begin{aligned} &> \frac{\partial}{\partial y} (3x + y^2) \\ & \qquad \qquad \qquad 2y \\ &> \frac{\partial}{\partial x} (3x + y^2) \\ & \qquad \qquad \qquad 3 \\ &> \frac{\partial^2}{\partial x \partial y} (3x + y^2) \\ & \qquad \qquad \qquad 0 \end{aligned}$$

Differential operator

- $D(f)$

$D[i](f)$

$D[i](f)(x, y, \dots)$ or $D_i(f)(x, y, \dots)$ – differentiation of a function f depending on variables x, y (including for specific values of variables) ; i is the number of the variable for calculating the partial derivative :

1 - variable x , 2 - variable y etc.

- D operator is a generalization of the **diff command** : it can compute derivatives at specific points and differentiate functional operators and procedures.
- $D(f)(x) = \text{diff}(f(x), x)$
- For D transformations the **convert** command is used
- $(D@@n)(f)(x)$ or $D^{(n)}(f)(x)$ – calculation of the n -th derivative with respect to x
- $(D@@n)[i](f)(x, y, \dots)$ or $D_{i\$n}(f)(x, y, \dots)$ or $(D[i\$n](f))(x, y, \dots)$ – calculation of the n -th partial derivative with respect to i -th variable .
- $((D@@n)[i])((D@@m)[j])(f)(x, y, \dots)$ or $D_{i\$n, j\$m}(f)(x, y, \dots)$ or $(D[i\$n, j\$m](f))(x, y, \dots)$ – calculation of the n -th partial derivative with respect to i -th variable and the m -th partial derivative with respect to the j -th variable

Examples of using the differential operator

> D(sin)

cos

Derivative at a point

> D(sin)(π)

-1

Functional operator of one variable

> $f := x \rightarrow \ln(x^2) + \exp(3 \cdot x) :$

> $g := D(f)$

$g := x \rightarrow \frac{2}{x} + 3e^{3x}$

> D(f)(x)

$\frac{2}{x} + 3e^{3x}$

Functional operator of two variables

> $f := (x, y) \rightarrow x^3 \cdot y :$

> D[1](f)(x, y)

$3x^2y$

> D[2](f)(x, y)

x^3

D operator and diff command

> D(ln)(x)

$\frac{1}{x}$

> diff(ln(x), x)

$\frac{1}{x}$

> D[1, 1, 2](h)(x, y)

$D_{1, 1, 2}(h)(x, y)$

> convert(%, diff)

$\frac{\partial^3}{\partial y \partial x^2} h(x, y)$

Examples of high order derivatives

> diff(z(x), x\$5); convert(%, D);

$\frac{d^5}{dx^5} z(x)$

$D^{(5)}(z)(x)$

> (D@@5)(z)(x)

$D^{(5)}(z)(x)$

> diff(p(x, y), x\$2, y\$5); convert(%, D);

$\frac{\partial^7}{\partial y^5 \partial x^2} p(x, y)$

$D_{1, 1, 2, 2, 2, 2, 2}(p)(x, y)$

> $D_{1\$2, 2\$5}(p)(x, y)$

$D_{1, 1, 2, 2, 2, 2, 2}(p)(x, y)$

> (D[1\$2, 2\$5])(p)(x, y);

$D_{1, 1, 2, 2, 2, 2, 2}(p)(x, y)$

> ((D@@2)[1])((D@@5)[2])(p)(x, y);

$D^{(2)}_1(D^{(5)}_2)(p)(x, y)$

Series expansion

- Expansion of a function into a power series
- Expansion of a function in a Taylor series

Expansion of a function into a power series

- **series (f (x), x=a , n)** – expansion of the function f in a series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots + a_n (x - x_0)^n$$

where **x0** is the point in the neighborhood of which the expansion is performed, **n** is the order of the decomposition. If the order is not specified, it is determined by the value of the constant **Order**

```
> series(e-x√(x+1), x=0)
```

$$1 - \frac{1}{2}x - \frac{1}{8}x^2 + \frac{13}{48}x^3 - \frac{79}{384}x^4 + \frac{503}{3840}x^5 + O(x^6)$$

```
> Order := 10 :
```

```
> series(e-x√(x+1), x=0)
```

$$1 - \frac{1}{2}x - \frac{1}{8}x^2 + \frac{13}{48}x^3 - \frac{79}{384}x^4 + \frac{503}{3840}x^5 - \frac{3953}{46080}x^6 + \frac{39317}{645120}x^7 - \frac{479071}{10321920}x^8 + \frac{6886063}{185794560}x^9 + O(x^{10})$$

```
> series( x / (1 - x - x^2), x=0, 5 )
```

$$x + x^2 + 2x^3 + 3x^4 + O(x^5)$$

```
> convert(%, polynom)
```

$$x + x^2 + 2x^3 + 3x^4$$

Expansion of a function in a Taylor series

- **taylor (f(x), x=a, n)** – expansion of a function of one variable in a Taylor series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

- **coeftayl (f(x), vars , n)** – coefficient of the term of order **n** by variable decomposition **vars**
- **mtaylor (f(x), [x1= a1 ,..., xn =an], n)** – expansion of a function of several variables in a Taylor series

```
[> f:= sin(x) + cos(x) :
> taylor(f, x=0)
1 + x - 1/2 x^2 - 1/6 x^3 + 1/24 x^4 + 1/120 x^5 + O(x^6)
> taylor(f, x=0, 4)
1 + x - 1/2 x^2 - 1/6 x^3 + O(x^4)
> coeftayl(f, x=0, 3)
-1/6
```

Разложение функции многих переменных

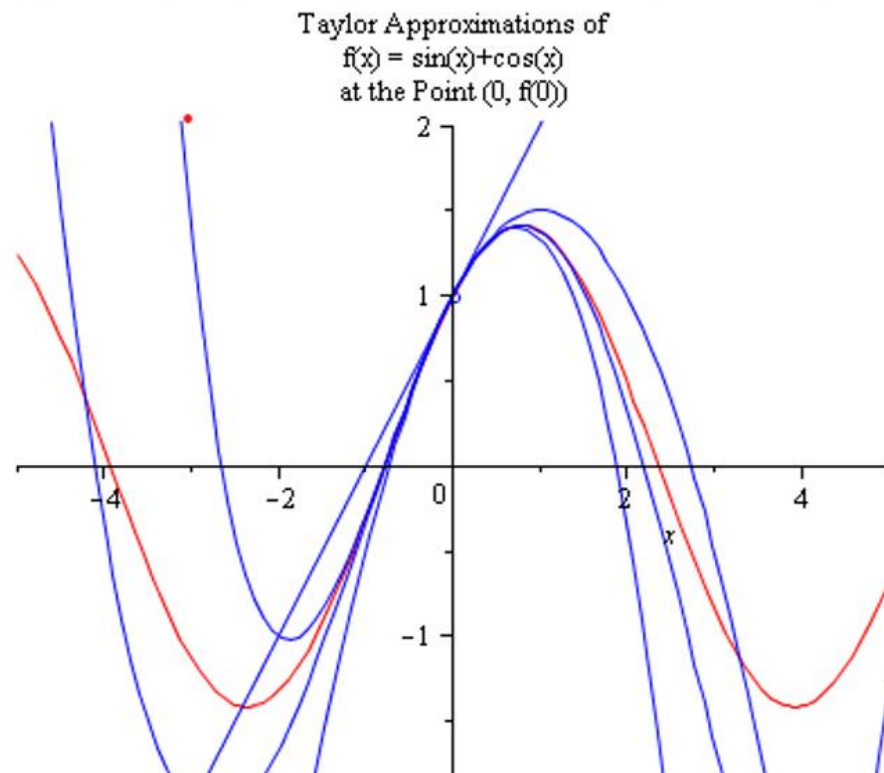
```
> mtaylor(sin(x^2 + y^2), [x=0, y=0], 10);
x^2 + y^2 - 1/6 x^6 - 1/2 y^2 x^4 - 1/2 y^4 x^2 - 1/6 y^6
> coeftayl(sin(x^2 + y^2), [x, y] = [0, 1], [0, 0]);
sin(1)
```

Taylor series expansion: illustration in the Student package

> `with(Student[Calculus1]): f := sin(x) + cos(x) : TaylorApproximation(f, x = 0, order = 4)`

$$1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4$$

> `TaylorApproximation(f, x = 0, output = plot, order = 1..4, view = [-5..5, -2..2]);`



Example of Maclaurin series expansion

```
> restart; f := x^3 * arccot(x);
```

$$f := x^3 \arccot(x)$$

конвертация в полином объекта типа series (для иллюстрации приведены типы)

```
> pp5 := taylor(f, x = 0, 5); p5 := convert(%, polynomial);  
whattype(pp5); whattype(p5)
```

$$pp5 := \frac{1}{2} \pi x^3 - x^4 + O(x^6)$$

$$p5 := \frac{1}{2} \pi x^3 - x^4$$

series

'+'

```
> taylor(f, x = 0, 7); p7 := convert(%, polynomial);
```

$$\frac{1}{2} \pi x^3 - x^4 + \frac{1}{3} x^6 + O(x^8)$$

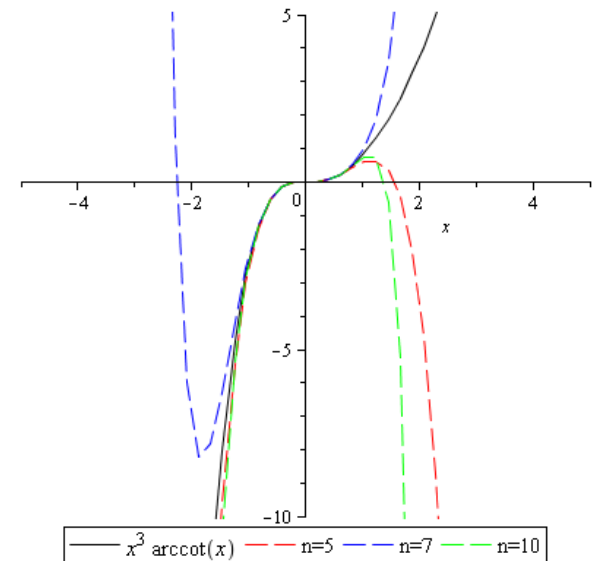
$$p7 := \frac{1}{2} \pi x^3 - x^4 + \frac{1}{3} x^6$$

```
> taylor(f, x = 0, 10); p10 := convert(%, polynomial);
```

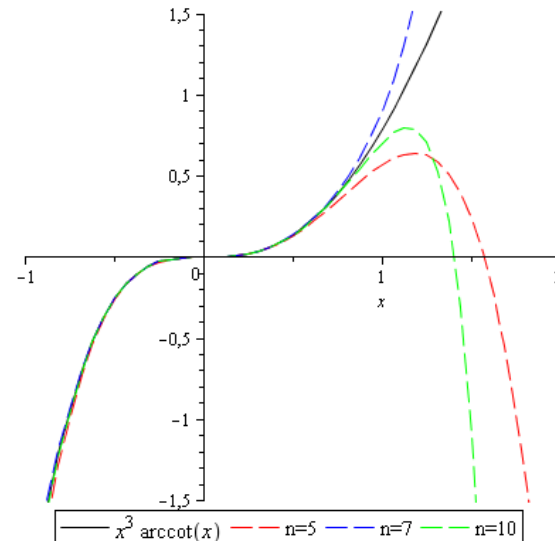
$$\frac{1}{2} \pi x^3 - x^4 + \frac{1}{3} x^6 - \frac{1}{5} x^8 + O(x^{10})$$

$$p10 := \frac{1}{2} \pi x^3 - x^4 + \frac{1}{3} x^6 - \frac{1}{5} x^8$$

```
> plot([f, p5, p7, p10], x = -5 .. 5, color = [black,  
red, blue, green], linestyle = [solid, dash, dash,  
dash], legend = [f, "n=5", "n=7", "n=10"])
```



```
> plot([f, p5, p7, p10], x = -1 .. 2, color = [black,  
red, blue, green], linestyle = [solid, dash, dash,  
dash], legend = [f, "n=5", "n=7", "n=10"])
```



Functional investigation

- Finding extrema : **extrema, minimize, maximize**

Critical point, necessary and sufficient conditions for extremum

Critical point. Point x_0 is called *critical point* of a function $f(x)$, if the first derivative is zero $f'(x_0) = 0$ or the first derivative $f'(x_0)$ doesn't exist.

Essential condition of extremum. Function $f(x)$ has an *extremum* at the point x_0 , if it is defined in the range $(x_0 - \delta_0, x_0 + \delta_0)$ and for all the points in this range:

$f(x) < f(x_0)$ (*maximum*) or $f(x) > f(x_0)$ (*minimum*)

Sufficient condition of extremum.

1) *Investigation by first derivative.* If the function $f(x)$ is defined and continuous at the range $(x_0 - \delta_0, x_0 + \delta_0)$ of a critical point x_0 , there exists $f'(x)$ in $(x_0 - \delta_0, x_0 + \delta_0)$ and $f'(x)$ does not change the sign at the left and right of x_0 , then:

Derivative sign $f'(x_0)$		Conclusion
$x < x_0$	$x > x_0$	
+	+	no extremum
-	-	no extremum
+	-	maximum
-	+	minimum

The largest and smallest value of a function, global and local extrema

Sufficient condition of extremum.

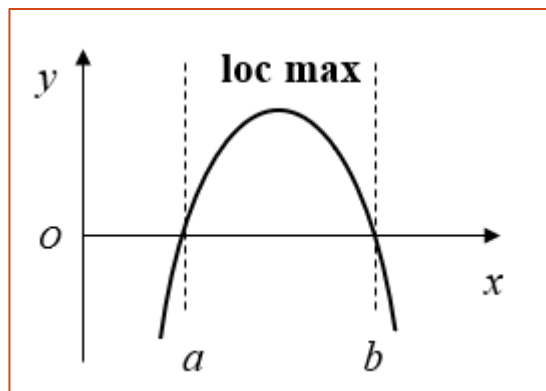
2) *Investigation by second derivative.* If the function $f(x)$ has the second derivative $f''(x)$ and $f'(x_0) = 0$ and $f''(x_0) \neq 0$, then:

Derivative sign $f''(x_0)$	Conclusion
-	maximum
+	minimum

The largest and the smallest values.

The largest (or the smallest) value at the closed interval $[a, b]$ of a continuous function $f(x)$ can take place either at a critical point of this function or at the end points $x = a$ and $x = b$ of this interval.

Global and local extremum. The extremum is called *global*, if it takes place at the whole real range otherwise it is called *local*.



Here the points $x=a$ and $x=b$ are **not** extrema, but at these points the function has the smallest value on the segment $[a, b]$

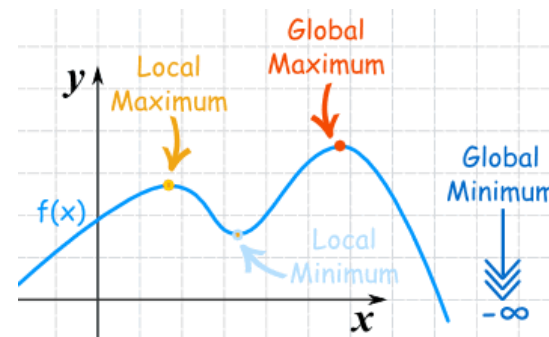


Image: <https://indicator.ru/label/lokalnyj-ekstremum>

Commands for finding extrema

- **extrema (f,{ cond },x, `s`)** – search for extrema satisfying the constraints **cond** , for a function of variables x, y
- **s** – optional parameter name for the coordinates of the extremum points
- **extrema (f,{ },x, `s`)** – search for extrema functions at the whole real range

Search for global minima

- **minimize (f) ; minimize (f , location)** – search for global minima
- **location** - keyword for outputting the x-coordinates of minimum points

Search for local minima

- **minimize (f, x=x1..x2) ; minimize (f, x=x1..x2, location)** – search for the minimum values of a function on the segment $[x1,x2]$.
- Possible syntax: **minimize(f, x<a)**; **minimize(f, x>b)** – search on the semi-axis $(-\infty;a)$ or $(b,+\infty)$

Search for global and local maxima

Everything is similar for the **maximize** command.

```
> restart; y :=  $\frac{x^4}{(x+1)^3}$  :
```

```
> extrema(y, { }, x,'s'); evalf(%); s;
```

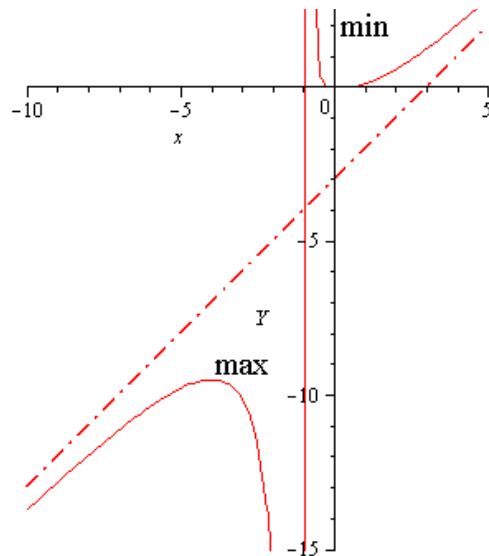
$\left\{ 0, -\frac{256}{27} \right\}$
 $\{-9.481481481, 0.\}$
 $\{\{x = -4\}, \{x = 0\}\}$

Example: Searching for extrema and extreme points

```

> y :=  $\frac{x^4}{(x+1)^3}$  :
> minimize(y, x < -1); maximize(y, x < -1, location); evalf(%)
                                     - ∞
                                     -  $\frac{256}{27}$ , { [ {x = -4}, - $\frac{256}{27}$  ] }
                                     -9.481481481, { [ {x = -4.}, -9.481481481 ] }
> maximize(y, x > -1); minimize(y, x > -1, location);
                                     ∞
                                     0, { [ {x = 0}, 0 ] }

```



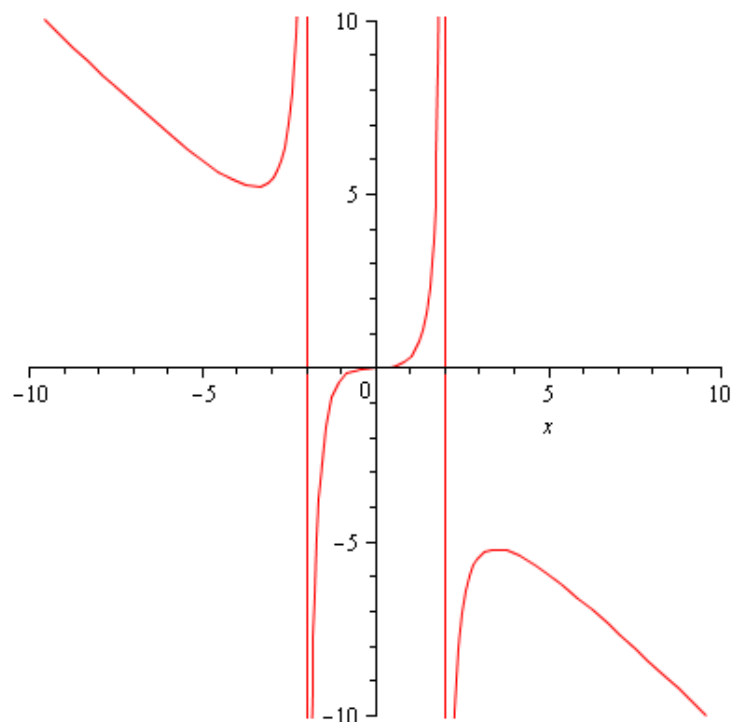
ANSWER

The function has

- 1) Local minimum at $x=0, y=0$
- 2) Local maximum at $x=-4, y= -256/27 \approx -9.4814$
- 3) No global extrema (they are infinite)

An example of finding extrema for a function with an inflection point

```
> restart; y :=  $\frac{x^3}{4 - x^2}$  :
> extrema(y, { }, x, 's'); s;
       $\{-3\sqrt{3}, 3\sqrt{3}\}$ 
       $\{\{x=0\}, \{x=-2\sqrt{3}\}, \{x=2\sqrt{3}\}\}$ 
> minimize(y)
       $-\infty$ 
> maximize(y)
       $\infty$ 
> plot(y, x=-10..10, -10..10)
```



```
> minimize(y, x < -2, location); evalf(%)
       $3\sqrt{3}, \{\{x=-2\sqrt{3}\}, 3\sqrt{3}\}$ 
      5.196152424,  $\{\{x=-3.464101616\}, 5.196152424\}$ 
> maximize(y, x > 2, location); evalf(%)
       $-3\sqrt{3}, \{\{x=2\sqrt{3}\}, -3\sqrt{3}\}$ 
      -5.196152424,  $\{\{x=3.464101616\}, -5.196152424\}$ 
> minimize(y, x=-2..2, location)
       $-\infty, \{\{x=-2\}, -\infty\}$ 
> maximize(y, x=-2..2, location)
       $\infty, \{\{x=2\}, \infty\}$ 
```

Exclude the points of discontinuity from the search: we get minimal and maximal values at the end points (not extremas!)

```
> minimize(y, x=-1.9..1.9, location)
      -17.58717949,  $\{\{x=-1.9\}, -17.58717949\}$ 
> maximize(y, x=-1.9..1.9, location)
      17.58717949,  $\{\{x=1.9\}, 17.58717949\}$ 
> dl := diff(y, x);
       $dl := \frac{3x^2}{4-x^2} + \frac{2x^4}{(4-x^2)^2}$ 
> eval(dl, x=0.1)
      0.007531359727
> eval(dl, x=-0.1)
      0.007531359727
```

The first derivative does not change the sign when getting over the point $x=0$, hence this point is the inflection point

Symbolic integration

- Integration: indefinite and definite integrals
- Iterated, double and triple integrals
- Graphical approximation of a definite integral

Integration of a function of one variable

- **int (f, x)** – indefinite integral
- **int (f , x = a .. b)** – definite integral
- **Int** - a similar inert command

$$\int f \, dx$$

$$\int_a^b f \, dx$$

$$> \int \sin(x) \, dx$$

$$-\cos(x)$$

$$> \int_0^{\pi} \sin(x) \, dx$$

$$2$$

$$> \text{Int}((1 + \cos(x))^2, x=0..\text{Pi}) = \text{int}((1 + \cos(x))^2, x=0..\text{Pi})$$

$$\int_0^{\pi} (1 + \cos(x))^2 \, dx = \frac{3}{2} \pi$$

Integration of a function of several variables

- Iterated Integral - Nesting of **Int Commands**

1) **Doubleint** commands And **Tripleint** outdated **student package**

- **Doubleint (f (x , y), D)** is a double integral , where **D** is the domain of integration
- **Tripleint (f (x , y , z), x , y , z , V)** – triple integral , where **V** is the domain of integration

MultiInt command **Student** package [**MultivariateCalculus**]

- **MultiInt (f(x,y), x= a..b , y= c..d , opts)**
- **MultiInt (f(x,y,z), x= a..b , y= c..d , z= e..f , opts)**
- The order of the integration limits is important!

> $\text{Int}(\text{Int}(y^3 / (x^2 + y^2), x = 0 .. y), y = 2 .. 4) = \text{int}(\text{int}(y^3 / (x^2 + y^2), x = 0 .. y), y = 2 .. 4)$

$$\int_2^4 \int_0^y \frac{y^3}{x^2 + y^2} dx dy = \frac{14}{3} \pi$$

Integration of a function of several variables: examples

```
> with(student) :
```

```
> J2 := Doubleint( sin(x + 2·y), x = y ..  $\frac{\pi}{2}$  - y, y = 0 ..  $\frac{\pi}{2}$  )
```

$$J2 := \int_0^{\frac{1}{2}\pi} \int_y^{\frac{1}{2}\pi - y} \sin(x + 2y) \, dx \, dy$$

```
> value(J2)
```

$$\frac{2}{3}$$

```
> J3 := Tripleint( 4 + z, y = x^2 .. 1, x = -1 .. 1, z = 0 .. 2 )
```

$$J3 := \int_0^2 \int_{-1}^1 \int_{x^2}^1 (4 + z) \, dy \, dx \, dz$$

```
> value(J3)
```

$$\frac{40}{3}$$

```
> Doubleint(h·g, x, y, C)
```

$$\iint_C h g \, dx \, dy$$

```
> with(Student[MultivariateCalculus]) :
```

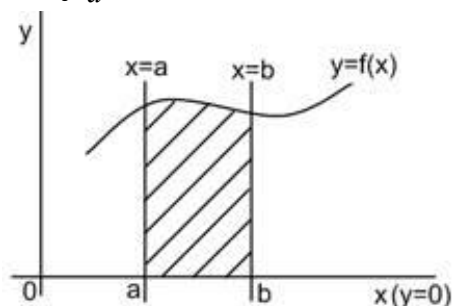
```
> MultiInt( 3 x^2 + 3 y^2, x = 1 .. 4, y = -1 .. 6 )
```

$$1092$$

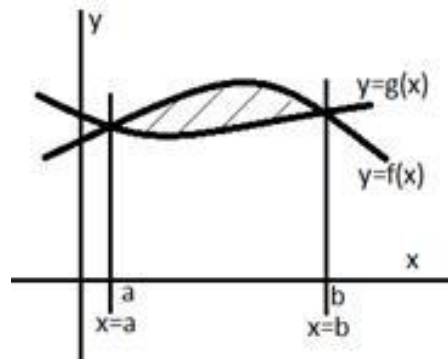
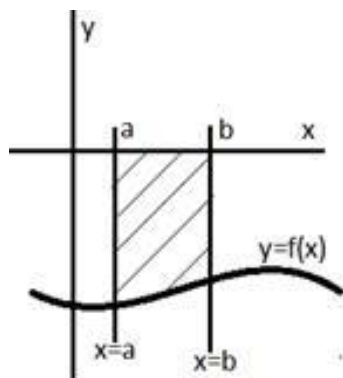
The geometric meaning of the definite integral

The geometric meaning of the definite integral $\int_a^b f(x)dx$: the integral value is the area S of the curvilinear trapezoid bounded by the graph of the nonnegative on $[a, b]$ function $f(x)$ (curve $y = f(x)$), the x-axis (the line $y = 0$) and the lines $x = a$ and $x = b$.

$$S = \int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$



If the curvilinear trapezoid is located under the x-axis, then the integral will be negative:



$$\int_a^b f(x)dx = -S$$

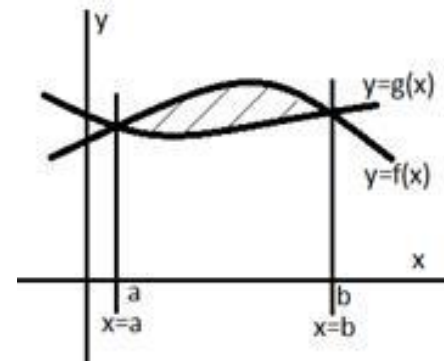
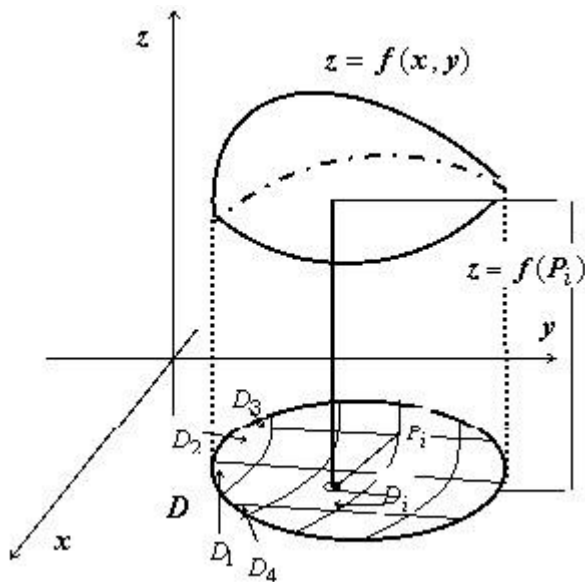
$$S = S_1 - S_2 = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

Geometrical meaning of the double integral

Geometrical meaning of the double integral $\iint_D f(x, y) dx dy$: the integral value is

the volume V of the cylindrical body bounded by the surface $z = f(x, y) \geq 0$ on the top, the area D on the bottom located in the Oxy plane, and cylindrical surface parallel to the z -axis on the sides.

$$V = \iint_D f(x, y) dx dy$$

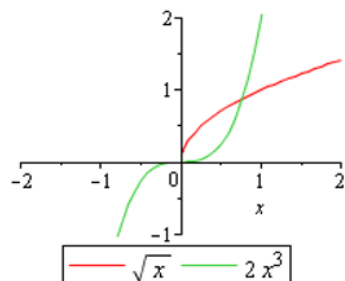


$$S = \iint_D dx dy = \int_a^b dx \int_{g(x)}^{f(x)} dy$$

Example: the geometric meaning of a definite integral

Find the area of the figure bounded by the graphs of the given functions

```
> restart; f1 := sqrt(x) : f2 := 2 * x^3 :
> with(plots) : plot([f1, f2], x = -2 .. 2, -1 .. 2, legend = [f1, f2])
```



```
> _EnvExplicit := true : solve([y = f1, y = f2], [x, y]); evalf(%)
[[x = 0, y = 0], [x = 1/2 * 2^(3/5), y = 1/2 * 2^(4/5)], [x = 1/2 * 2^(3/5) * (-1)^(4/5), y
= 1/2 * 2^(4/5) * (-1)^(2/5)], [x = -1/2 * 2^(3/5) * (-1)^(1/5), y = -1/2 * 2^(4/5) * (-1)^(3/5)
[[x = 0., y = 0.], [x = 0.7578582835, y = 0.8705505635], [x =
-0.6131202305 + 0.4454579224 I, y = 0.2690149186
+ 0.8279427860 I], [x = -0.6131202305 - 0.4454579224 I, y
= 0.2690149186 - 0.8279427860 I]]
```

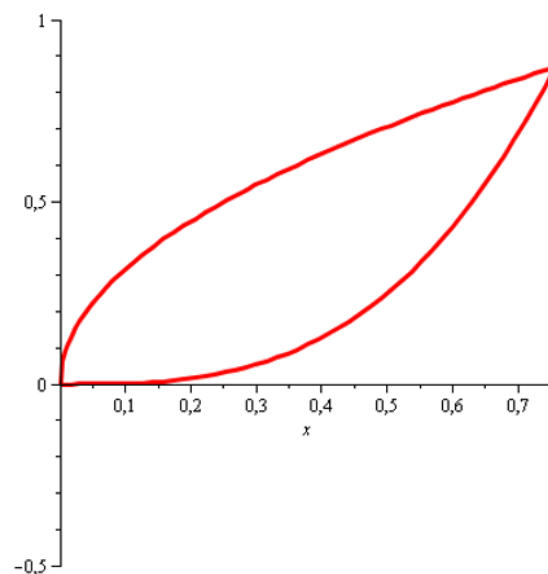
Way 1: calculate definite integral

```
> g := f1 - f2; int(g, x = 0 .. 1/2 * 2^(3/5)); evalf(%)
g := sqrt(x) - 2 * x^3
5/24 * 2^(2/5)
0.2748974814
```

Way 2: use double integral

```
> int(int(1, y = 2 * x^3 .. sqrt(x), x = 0 .. 1/2 * 2^(3/5)); evalf(%)
5/24 * 2^(2/5)
0.2748974814
> with(Student[MultivariateCalculus]) :
> MultiInt(1, y = 2 * x^3 .. sqrt(x), x = 0 .. 1/2 * 2^(3/5)); evalf(%)
5/24 * 2^(2/5)
0.2748974814
```

```
> plot([f1, f2], x = 0 .. 1/2 * 2^(3/5), -0.5 .. 1, color = red, thickness = 3)
```



Graphical approximation of a definite integral: commands from the legacy student package

- **leftbox** ($f(x)$, $x= a..b$, n , 'shading'=<color>, <plot options>) – graphical approximation of the integral in the range from a to b using the left Riemann sum , n – number of approximating rectangles (optional parameter) , shading – rectangle fill
- **rightbox** – a similar approximation of the integral using the right Riemann sum
- **middlebox** – a similar approximation of the integral using the average Riemann sum
- **leftsum** ($f(x)$, $x= a..b$, n) – the value of the left Riemann sum
- **rightsum** ($f(x)$, $x= a..b$, n) – the value of the right Riemann sum
- **averagesum** ($f(x)$, $x= a..b$, n) – the value of the average Riemann sum

Graphical approximation of a definite integral: commands from the Student[Calculus1] package

- **ApproximateInt (f(x), x = a..b , opts)**
- **ApproximateInt (f(x), a..b , opts)**
- **ApproximateInt (Int (f(x), x = a..b), opts)** – numerical or graphical (with **the output=plot** option) approximation of the integral within integration from a to b

Some options (see Help for details)

- **method = lower, upper, left, midpoint, right, trapezoid, simpson , simpson [3/8], boole , newtoncotes [posint], random or procedure** – integral approximation method (default is the average Riemann sum)
- **output = value, sum, plot, or animation** – the result (default is an approximate value)
- **partition = posint , list(algebraic), random[algebraic] or algebraic** – number of intervals (rectangles), default is 10

Examples: graphical approximation of a definite integral (Student[Calculus1] package commands)

```
> with(Student[Calculus1]):
```

```
> f := sin(x):
```

```
> int(f, x = 0 .. pi);
```

2

Approximate value of the integral

```
> ApproximateInt(f, x = 0 .. pi); evalf(%)
```

$$\frac{1}{20} \pi \sqrt{2} \sqrt{5} + \frac{3}{20} \pi \sqrt{2} + \frac{1}{10} \pi \sqrt{5 + \sqrt{5}}$$

2.008248408

Middle Riemann sum

```
> ApproximateInt(f, x = 0 .. pi, method = midpoint); evalf(%)
```

$$\frac{1}{20} \pi \sqrt{2} \sqrt{5} + \frac{3}{20} \pi \sqrt{2} + \frac{1}{10} \pi \sqrt{5 + \sqrt{5}}$$

2.008248408

Left Riemann sum

```
> ApproximateInt(f, x = 0 .. pi, method = left); evalf(%)
```

$$\frac{1}{10} \pi + \frac{1}{10} \pi \sqrt{5} + \frac{1}{20} \pi \sqrt{2} \sqrt{5 - \sqrt{5}} + \frac{1}{20} \pi \sqrt{2} \sqrt{5 + \sqrt{5}}$$

1.983523537

Right Riemann sum

```
> ApproximateInt(f, x = 0 .. pi, method = right); evalf(%)
```

$$\frac{1}{10} \pi + \frac{1}{10} \pi \sqrt{5} + \frac{1}{20} \pi \sqrt{2} \sqrt{5 - \sqrt{5}} + \frac{1}{20} \pi \sqrt{2} \sqrt{5 + \sqrt{5}}$$

1.983523537

Upper Riemann sum

```
> ApproximateInt(f, x = 0 .. pi, method = upper);
```

2.297682802

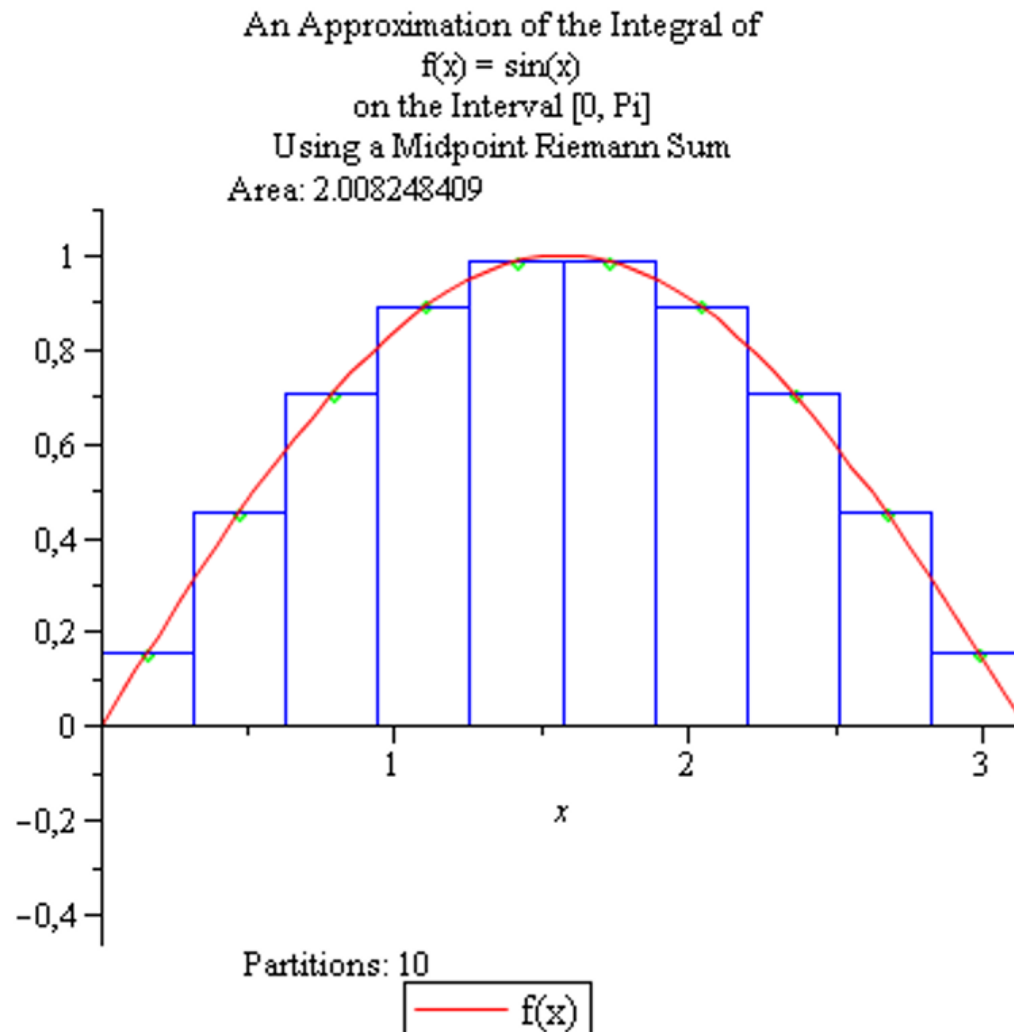
Lower Riemann sum

```
> ApproximateInt(f, x = 0 .. pi, method = lower);
```

1.669364272

Examples: Graphical approximation of a definite integral using an average sum (Student[Calculus1] package)

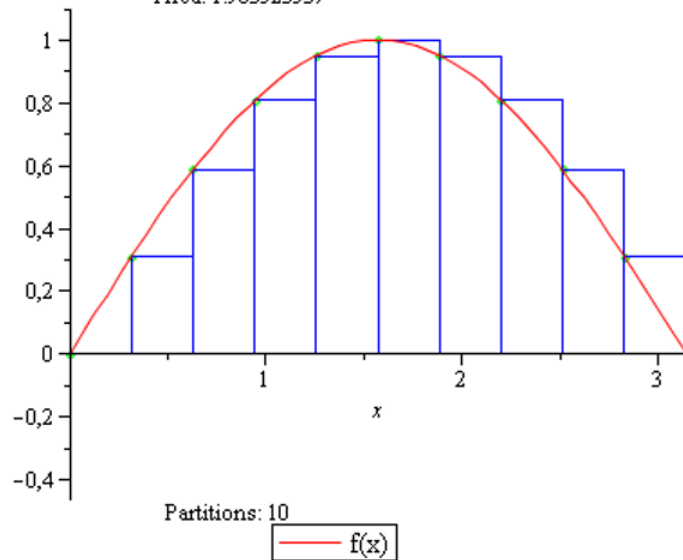
> *ApproximateInt*(*f*, *x* = 0 .. π , *method* = *midpoint*, *output* = *plot*)



Examples: Graphical approximation of a definite integral using left and right sums (Student[Calculus1] package)

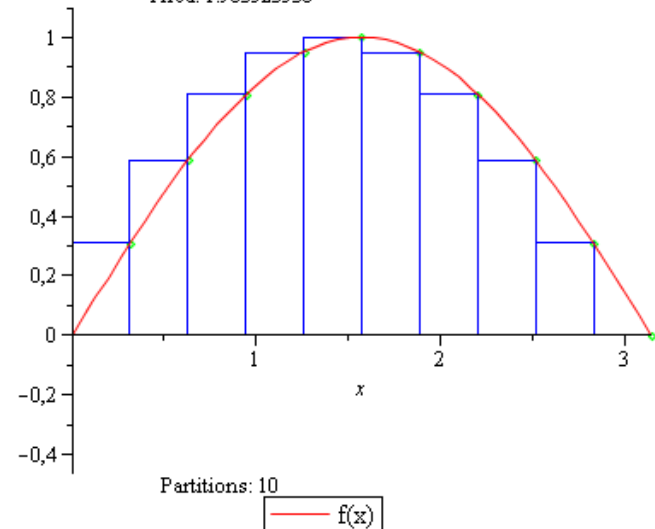
> `ApproximateInt(f, x = 0 ..  $\pi$ , method = left, output = plot)`

An Approximation of the Integral of
 $f(x) = \sin(x)$
on the Interval $[0, \pi]$
Using a Left-endpoint Riemann Sum
Area: 1.983523537



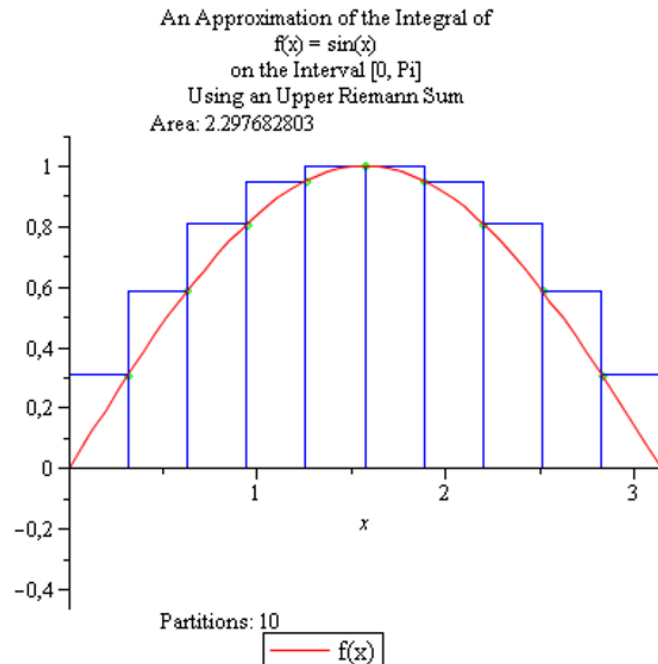
> `ApproximateInt(f, x = 0 ..  $\pi$ , method = right, output = plot)`

An Approximation of the Integral of
 $f(x) = \sin(x)$
on the Interval $[0, \pi]$
Using a Right-endpoint Riemann Sum
Area: 1.983523538

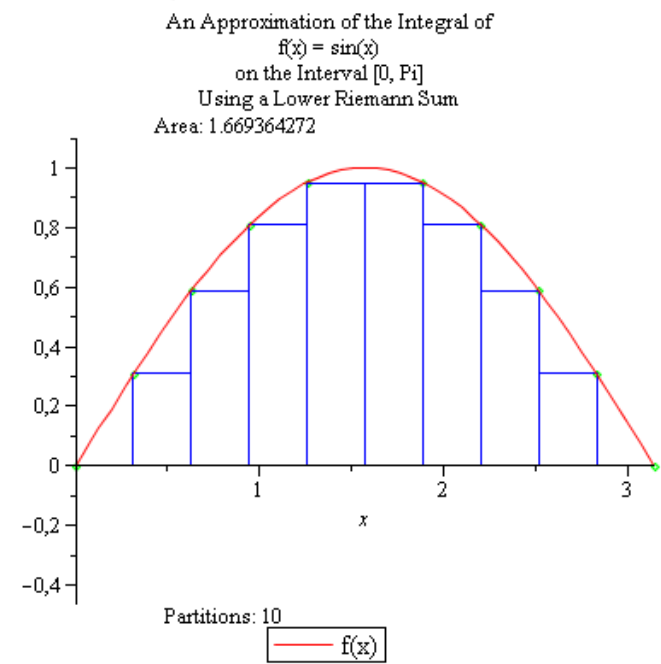


Examples: Graphical approximation of a definite integral using upper and lower sums (Student[Calculus1] package)

> `ApproximateInt(f, x = 0 ..  $\pi$ , method = upper, output = plot)`



> `ApproximateInt(f, x = 0 ..  $\pi$ , method = lower, output = plot)`



Solving algebraic and transcendental equations and systems

- Analytical solution of algebraic equations and inequalities, the solve command
- Special options for the solve command
- Numerical solution of equations and other special commands
- Searching for the equation roots

Solving algebraic equations: solve

solve (equations, variables) – solution of an equation (system of equations) with respect to a **variable** (set of variables) **variables**
solve (eq) – the set of variables can be omitted

> *solve*($x + 5$)

-5

> *solve*($x^2 - x = 2025$, x)

$\frac{1}{2} + \frac{1}{2} \sqrt{8101}, \frac{1}{2} - \frac{1}{2} \sqrt{8101}$

> *solve*($2y - x^2 = 4$, y)

$\frac{1}{2} x^2 + 2$

Find the sum of the roots

> $s := \textit{solve}(x^2 - 5x + 6)$

$s := 3, 2$

> $s[1] + s[2]$

5

Special options for the solve command: solving trigonometric equations

- **solve** command, used to solve a trigonometric equation, returns only the principal solutions, that is, solutions in the interval $[0, 2\pi]$. To obtain all solutions, you must first enter an additional option (will be valid until the **restart** command) **_EnvAllSolutions := true**

In 2D Math, when entering the `_` symbol, it may switch to lowercase. If this happens, you should type `\_`
In 1D Math-input the `_` symbol is typed in the usual way.

> `solve(sin(x) = cos(x), x)`

$$\frac{1}{4}\pi$$

`_EnvAllSolutions := true :`

> `solve(sin(x) = cos(x), x)`

$$\frac{1}{4}\pi + \pi_Z2\sim$$

Special options for the solve command : present solutions explicitly

- **RootOf** function appears in the answer when solving complex equations with powers or systems of equations , this means that Maple cannot express roots as radicals.

```
> eq := x^4 - 2·x^3 + 2 = 0 :  
> solve(eq, x)  
RootOf(_Z^4 - 2_Z^3 + 2, index = 1), RootOf(_Z^4 - 2_Z^3 + 2, index = 2),  
RootOf(_Z^4 - 2_Z^3 + 2, index = 3), RootOf(_Z^4 - 2_Z^3 + 2, index = 4)
```

- In this case (to represent the solution explicitly) the following methods can be used:
 - 1) Before the solve command, you should enter an additional option (it will be valid until the **restart** command) **_EnvExplicit := true ;**
 - 2) use command **allvalues** to the solution result (non-computable root)
 - 3) use command **convert** with the **radical** parameter to the solution result (non-computable root)

Explicit representation of solutions: examples

```
> eq := x^4 - 2·x^3 + 2 = 0 :  
Maple cannot calculate radicals directly; RootOf the solve result  
> solve(eq, x)  
RootOf(_Z^4 - 2 _Z^3 + 2, index = 1), RootOf(_Z^4 - 2 _Z^3 + 2, index = 2),  
RootOf(_Z^4 - 2 _Z^3 + 2, index = 3), RootOf(_Z^4 - 2 _Z^3 + 2, index = 4) (1)
```

```
1) _EnvExplicit  
> restart; _EnvExplicit := true :  
> eq := x^4 - 2·x^3 + 2 = 0 : solve(eq, x)  
 $\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}\sqrt{4 - 2i}, \frac{1}{2} + \frac{1}{2}i - \frac{1}{2}\sqrt{4 - 2i}, \frac{1}{2} - \frac{1}{2}i + \frac{1}{2}\sqrt{4 + 2i},$   
 $\frac{1}{2} - \frac{1}{2}i - \frac{1}{2}\sqrt{4 + 2i}$  (2)
```

```
2) allvalues  
> restart : eq := x^4 - 2·x^3 + 2 = 0 : s := solve(eq, x) :  
> allvalues(s) #мак бyдем ошибка  
Error, (in allvalues) redundant option RootOf(_Z^4-2*_  
_Z^3+2, index = 3), if there is more than one RootOf to  
avoid, they should be put in a set or list
```

Apply allvalues to a specific root

> $s := \text{solve}(eq, x) : \text{allvalues}(s[1])$

$$\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}\sqrt{4 - 2i} \quad (3)$$

Obtain a list or a set of roots and apply allvalues to the list or set

> $ss := \text{solve}(eq, [x])$

$$ss := [[x = \text{RootOf}(_Z^4 - 2_Z^3 + 2, \text{index} = 1)], [x = \text{RootOf}(_Z^4 - 2_Z^3 + 2, \text{index} = 2)], [x = \text{RootOf}(_Z^4 - 2_Z^3 + 2, \text{index} = 3)], [x = \text{RootOf}(_Z^4 - 2_Z^3 + 2, \text{index} = 4)]] \quad (4)$$

> $\text{allvalues}(\%)$

$$\left[\left[x = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}\sqrt{4 - 2i} \right], \left[x = \frac{1}{2} + \frac{1}{2}i - \frac{1}{2}\sqrt{4 - 2i} \right], \left[x = \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}\sqrt{4 + 2i} \right], \left[x = \frac{1}{2} - \frac{1}{2}i + \frac{1}{2}\sqrt{4 + 2i} \right] \right] \quad (5)$$

3) Apply convert with the option radical

> $\text{restart};$

> $eq := x^4 - 2 \cdot x^3 + 2 = 0 : s := \text{solve}(eq, x) :$

> $\text{convert}(s[1], \text{radical})$

$$\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}\sqrt{4 - 2i} \quad (6)$$

> $ss := \text{solve}(eq, [x]) : \text{convert}(ss, \text{radical})$

$$\left[\left[x = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}\sqrt{4 - 2i} \right], \left[x = \frac{1}{2} + \frac{1}{2}i - \frac{1}{2}\sqrt{4 - 2i} \right], \left[x = \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}\sqrt{4 + 2i} \right], \left[x = \frac{1}{2} - \frac{1}{2}i + \frac{1}{2}\sqrt{4 + 2i} \right] \right] \quad (7)$$

Numerical solution of equations and other special commands

fsolve (eq) – numerical solution of an equation, search for real roots

isolve (eq) – integer solution of the equation

rsolve (eq) – solution of a recurrence equation

Numerical solution of an equation that has no analytical solution

```
[ > fsolve( cos(x) = x)
                                0.7390851332
[ > solve( cos(x) = x)
                                RootOf(_Z - cos(_Z) )
[ > evalf( solve( cos(x) = x) )
                                0.7390851332
```

Search for an integer solution. If there is no integer solution, NULL is returned (no output).

```
[ > isolve( {3 x - 4 y = 7, x + y = 14} )
                                {x = 9, y = 5}
[ > isolve( {4 x - y = 7, x + 2 y = 8} )
[ > solve( {4 x - y = 7, x + 2 y = 8} )
                                {x = 22/9, y = 25/9}
```

Finding Real Roots with fsolve

fsolve (eq , x=a..b) – search for real roots on the interval [a,b]

```
> eqn:=x^3+3-exp(x)=0;
```

$$eqn := x^3 + 3 - e^x = 0$$

```
> solve(eqn,x);
```

$$\text{RootOf}(-e^{-Z} + _Z^3 + 3)$$

Here Maple cannot find the roots even with EnvExplicit option

```
> _EnvExplicit:=true:solve(eqn,x);
```

$$\text{RootOf}(-e^{-Z} + _Z^3 + 3)$$

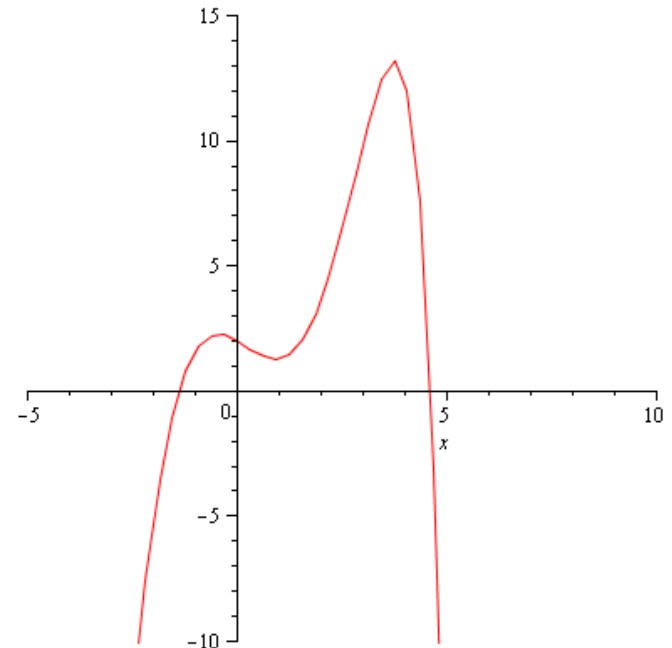
Attempt to find numerical solution (float point)

```
> fsolve(eqn);
```

4.623116246

Are there any other real roots?

```
> plot(lhs(eqn),x=-5..10,-10..15);
```



fsolve without additional options can find only one root

```
> fsolve(eqn);
```

4.623116246

The first real root

```
> fsolve(eqn,x=-2..-1);
```

-1.401667024

The second real root

```
> fsolve(eqn,x=-4..5);
```

4.623116246

Finding Roots with the RootFinding Package

with(RootFinding):

Analytic (eq , re=a..b, im =c..d) – search for the roots in the given intervals for the real and imaginary parts

```
> eqn:=x^3+3-exp(x)=0;
```

$$eqn := x^3 + 3 - e^x = 0$$

```
> with(RootFinding); Analytic(eqn, re = -10 .. 10,  
    im = -10 .. 10);
```

*[Analytic, AnalyticZerosFound, BivariatePolynomial, Homotopy, Isolate,
 NextZero]*

7.3410916840555 – 8.9305861665020 I, 4.6231162462848,
1.0348073103420 – 0.85239542694970 I, –1.4016670238656,
1.0348073103420 + 0.85239542694965 I, 7.3410916840555
+ 8.9305861665020 I