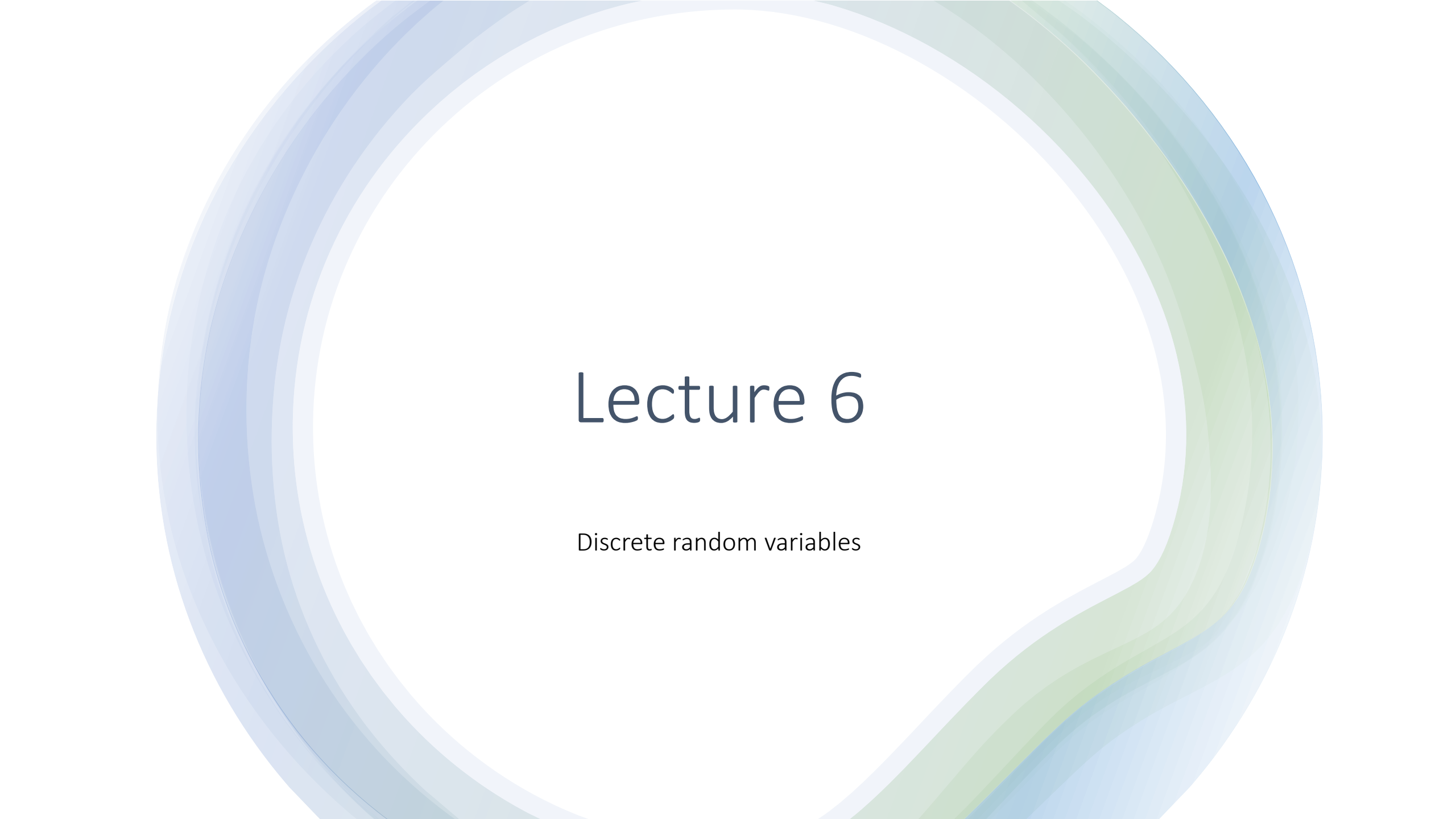


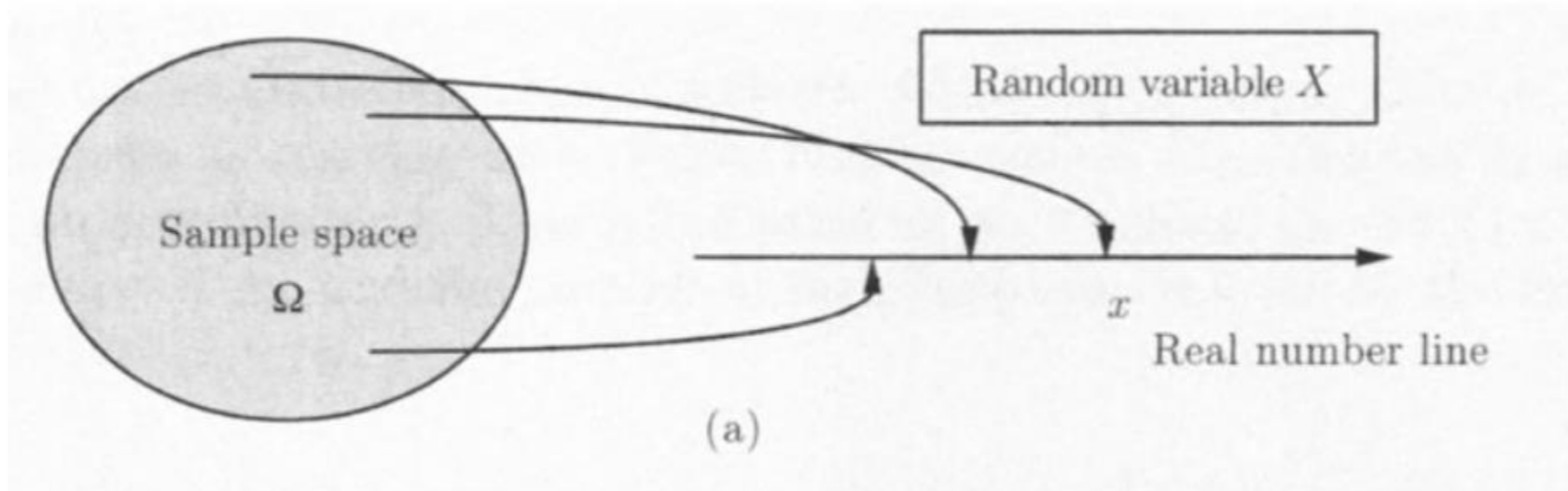


Lecture 6

Discrete random variables

Random variable

Given an experiment and the corresponding set of possible outcomes, a **random variable** associates a particular number with each outcome. We refer to this number as the numerical value or simply the value of the random variable. Mathematically, a random variable is a real-valued function of the experimental outcome.



Random variable

Example. Let X be the number of points that comes up when a dice is rolled. The result of this experiment is the loss of one and only one face, which one is impossible to predict. Random variable X can take one of the following values $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6$.

Example. In an experiment involving two rolls of a die the following are examples of random variables:

- The sum of the two rolls.
- The number of sixes in the two rolls.

Example. In an experiment involving a sequence of 5 tosses of a coin the number of heads in the sequence is a random variable.

We will use upper case characters to denote random variables, and lower case characters to denote real numbers such as the numerical values of a random variable.

Random variable

The set of all values that a random variable X can take is called **the range of X** , or its **set of possible values**.

An event involving a random variable is determined by the outcomes for which the value of the random variable satisfies a specified condition. For example, the event $\{X = 2\}$ consists of all outcomes in which exactly two heads are obtained: {HHT, HTH, THH}.

Similarly, the event $\{X \geq 2\}$ consists of all outcomes in which at least two heads are obtained: {HHT, HTH, THH, HHH}.

Discrete random variable

A random variable is called **discrete** if its range (the set of values that it can take) is either finite or countably infinite.

Concepts Related to Discrete Random Variables

Starting with a probabilistic model of an experiment:

- A **discrete random variable** is a real-valued function of the outcome of the experiment that can take a finite or countably infinite number of values.
- A discrete random variable has an associated **probability mass function (PMF)**, which gives the probability of each numerical value that the random variable can take.
- A **function of a discrete random variable** defines another discrete random variable, whose PMF can be obtained from the PMF of the original random variable.

Discrete random variable

Example. In an experiment involving two rolls of a six-sided die, the following quantities are examples of random variables:

- the sum of the numbers obtained;
- the product of the numbers obtained;
- the larger of the two numbers;
- the number of times a six is obtained.

Example. Let X be the sum of the numbers obtained when two fair dice are rolled. The smallest possible value of X is 2, and the largest possible value is 12. Therefore, X can take the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12.

Probability mass function

For a discrete random variable X , **probability mass function (PMF)** $p_X(x)$ characterize a random variable is through the probabilities of the values that it can take:

$$p_X(x) = \mathbf{P}(X = x)$$

Example. Let the experiment consist of two independent tosses of a fair coin, and let X be the number of heads obtained. Then the PMF of X is

$$p_X(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 0 \text{ or } x = 2 \\ \frac{1}{2}, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the distribution of X .

Probability mass function

The probability mass function satisfies the following properties:

- $p_X(x) \geq 0$ for every x
- the sum of the probabilities over all possible values of X is equal to 1: $\sum_x p_X(x) = 1$

Thus, if a discrete random variable X can take the values $x_1, x_2, \dots$ with probabilities $p_1, p_2, \dots$, then

$$p_X(x) = P(X=x_i) = p_i$$

where $p_i \geq 0$ and $p_1 + p_2 + \dots = 1$.

Probability mass function

Calculation of the PMF of a Random Variable X

For each possible value x of X :

1. Collect all the possible outcomes that give rise to the event $\{X = x\}$.
2. Add their probabilities to obtain $p_X(x)$.

Example. A four-sided dice is tossed 2 times. Let a, b – number of points on the first and second tosses. Find distribution for random variables $X = \max\{a, b\}, Y = a + b$.

Example. An examination consists of three problems. A student solves the first, second, and third problems correctly with probabilities 0.8, 0.5, and 0.2, respectively. Assume that the results for the three problems are independent. Find the distribution of the number of problems solved correctly.

The Bernoulli random variable

Consider the toss of a coin, which comes up a head with probability p , and a tail with probability $1 - p$.

A Bernoulli random variable is a discrete random variable that takes only two possible values: 0 and 1. The value 0 represents a failure, while the value 1 represents a success.

$$X = \begin{cases} 1, & \text{if a head} \\ 0, & \text{if a tail} \end{cases}$$

$$p_X(k) = \begin{cases} p, & \text{if } k = 1 \\ 1 - p, & \text{if } k = 0 \end{cases}$$

Probability distribution of a Bernoulli random variable

X	0	1
p	$1 - p$	p

The Bernoulli random variable

Example. An AI system generates a response to a user's question. Let X indicate whether the response meets the required quality standard:

- $X = 1$ if the response meets the quality standard;
- $X = 0$ if the response does not meet the quality standard.

Suppose that the probability that the response meets the quality standard is $p = 0.9$. Then $P(X = 1) = 0.9$, $P(X = 0) = 0.1$. Thus, X is a Bernoulli random variable with parameter $p = 0.9$:

$$X \sim \text{Bernoulli}(0.9)$$

The binomial random variable

A coin is tossed n times. At each toss, the coin comes up a head with probability p , and a tail with probability $1 - p$, independent of prior tosses. Let X be the number of heads in the n -toss sequence. We refer to X as a binomial random variable with parameters n and p . The PMF of X consists of the binomial probabilities:

$$p_X(k) = P(X = k) = C_n^k p^k q^{n-k}$$

A binomial random variable is a discrete random variable that represents the number of successes obtained in a fixed number n of independent Bernoulli trials, where each trial has the same probability of success p .

Example. An AI system generates 5 responses independently. The probability that any given response meets the required quality standard is 0.8. Let X be the number of responses that meet the standard.

$$X \sim \text{Binomial}(5, 0.8).$$

The geometric random variable

A **geometric random variable** represents the number of independent Bernoulli trials required to obtain the first success, including the trial on which the first success occurs.

Suppose that we repeatedly and independently toss a coin with probability of a head equal to p , where $0 < p < 1$. The geometric random variable is the number X of tosses needed for a head to come up for the first time. Its PMF is given by:

$$p_X(k) = q^{k-1}p$$

Naturally, the use of coin tosses here is just to provide insight. More generally, we can interpret the geometric random variable in terms of repeated independent trials until the first «success».

Example. The shooter fires several shots at the target until the first hit, with only 4 rounds. The probability of hitting with one shot is 0.6. Find the distribution of X .

$$X \sim \text{Geometric}(p).$$

The Poisson random variable

To get a feel for the Poisson random variable think of a binomial random variable with very small p and very large n . For example, let X be the number of typos in a book with a total of n words. Then X is binomial, but since the probability p that any one word is misspelled is very small X can also be well- modeled with a **Poisson** PMF :

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \lambda = np$$

Example. The textbook was published in 100 000 copies. The probability that a textbook is bound incorrectly is 0.0001. Find the probability that the circulation contains exactly five defective books.

$$X \sim \text{Poisson}(\lambda).$$

Hypergeometric Random Variable

Definition. A **hypergeometric** random variable is a discrete random variable that represents the number of successes obtained when a fixed number of objects is selected **without replacement** from a finite population containing a known number of successes and failures.

Suppose that:

- N is the total number of objects in the population;
- K is the number of objects classified as successes;
- n is the number of objects selected;
- k is the number of successes among the selected objects.

Then X is said to have a hypergeometric distribution with parameters N , K , and n . This is denoted by

$$X \sim \text{Hypergeometric}(N, K, n).$$

The probability that X takes the value k is given by:

$$P(X = k) = \frac{C(K, k) \cdot C(N - K, n - k)}{C(N, n)}$$

Hypergeometric Random Variable

Example. An AI system has generated six responses, four of which meet the required quality standard. Three responses are selected uniformly at random without replacement. Let X denote the number of selected responses that meet the standard. Find the probability distribution of X .

Tasks

Tasks

1. There are four standard details in a batch of six details. Three details were selected at random. Find the distribution of a discrete random variable X - the number of standard parts among the selected.
2. There are three tasks in the examination paper. The probability of a correct solution by a student of the first problem is 0.9, the second - 0.6, the third - 0.1. Find the distribution of the number of correctly solved problems and find its mathematical expectation.
3. Two students are taking an exam in probability theory. For one of them, the probability of passing the exam is 0.9, for the other - 0.6. Construct a distribution of X - number of students who passed the exam. Find the expectation of the X .
4. The device consists of three independently operating elements. The probability of failure of each element in one experiment is 0.1. Find the distribution, expectation of number of failed elements in one experiment.
5. The probability that the shooter will hit the target with one shot is 0.8. The shooter is given cartridges until he misses. It is required: a) to draw up a distribution law for a discrete random variable X —the number of cartridges issued to the shooter; b) find the most probable number of cartridges issued to the shooter.

Tasks

6. The textbook was published in a print run of 100,000 copies. The probability that the textbook is incorrectly bound is 0.0001. Find the probability that the print run contains exactly five defective books.
7. The test consists of three questions. Each question has four possible answers, one of which is correct. Find the distribution of the number of correct answers with simple guessing, find the expectation and variance.
8. Write a binomial distribution law for a discrete random variable X —the number of occurrences of the “heads” in two tosses of a coin. Find expectation of random variable X .

Homework

Study guide. p. 74, Tasks 1-7