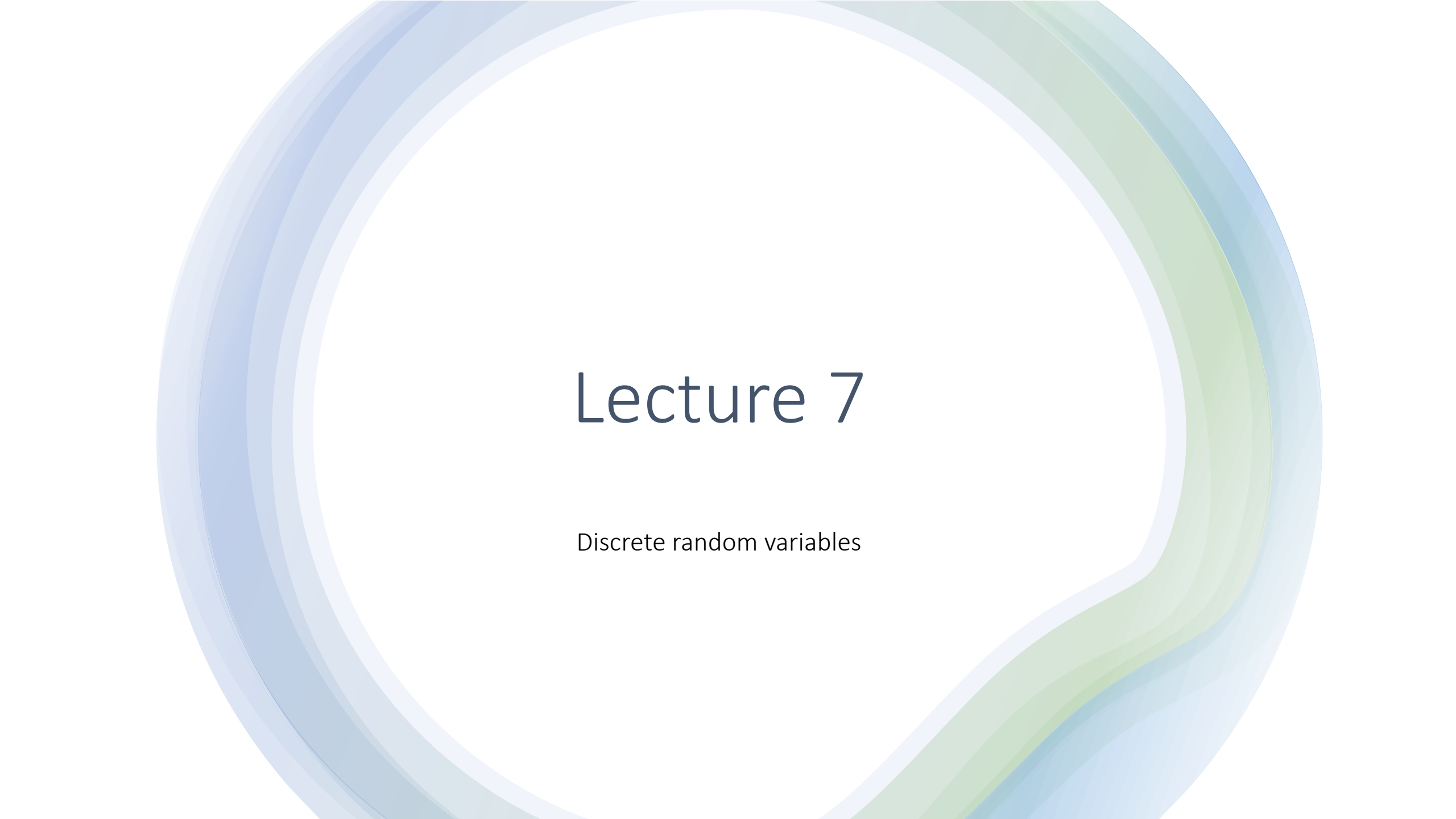




Lecture 7

Discrete random variables

Expectation

Let X be a discrete random variable with possible values $x_1, x_2, x_3, \dots$ and corresponding probabilities $p_i = P(X = x_i)$. The **expected value**, or **expectation**, of X is defined by:

$$E(X) = x_1p_1 + \dots + x_np_n + \dots = \sum_i x_ip_i$$

where the summation is taken over all possible values of X .

The expected value is a probability-weighted average of the possible values of X . Under repeated independent realizations of the same experiment, it can be interpreted as the long-run average value of X . The expected value need not be one of the possible values of X .

Expectation. Properties

- 1. Expected Value of a Constant.** If C is a constant, then $E(C) = C$.
- 2. Multiplication by a Constant.** For any constant C , $E(C \cdot X) = C \cdot E(X)$.
- 3. Additivity.** For any random variables X and Y , $E(X + Y) = E(X) + E(Y)$.
More generally, $E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$.
- 4. Expected Value of a Product of Independent Random Variables.** If X and Y are independent, then $E(XY) = E(X) \cdot E(Y)$. More generally, if $X_1, X_2, \dots, X_n$ are mutually independent, then $E(X_1 X_2 \dots X_n) = E(X_1) \cdot E(X_2) \cdot \dots \cdot E(X_n)$. Independence is essential for this property.

Expectation. Example

Find the expected value of the random variable X with distribution:

x	0	1	2	3
p	0.2	0.4	0.3	0.1

Expected Values of Common Discrete Distributions

1. Bernoulli Random Variable.

Let $X \sim \text{Bernoulli}(p)$. The random variable X takes the values 0 and 1 with probabilities $P(X = 0) = 1 - p$, $P(X = 1) = p$. By the definition of expected value, $E(X) = 0 \cdot (1 - p) + 1 \cdot p = p$.

Thus, **$E(X) = p$** .

2. Binomial Random Variable.

Let $X \sim \text{Binomial}(n, p)$, where X represents the number of successes in n independent Bernoulli trials. Let $q = 1 - p$.

$$E(X) = \sum_{k=0}^n k \cdot C(n, k) p^k q^{n-k} = np \cdot \sum_{k=1}^n C(n-1, k-1) p^{k-1} q^{n-k}$$

Let $j = k - 1$. Then j takes the values $0, 1, \dots, n - 1$. Hence,

$$E(X) = np \cdot \sum_{j=0}^{n-1} C(n-1, j) p^j q^{n-1-j}$$

By the binomial theorem, $\sum_{j=0}^{n-1} C(n-1, j) p^j q^{n-1-j} = (p + q)^{n-1}$. Since $p + q = 1$, $(p+q)^{n-1} = 1$.

Therefore, **$E(X) = np$** .

Expected Values of Common Discrete Distributions

3. Geometric Random Variable.

Let $X \sim \text{Geometric}(p)$, where X represents the number of trials required to obtain the first success, including the successful trial. The possible values of X are 1, 2, 3, ... Let $q = 1 - p$. By the definition of expected value,

$$E(X) = \sum_{k=1}^{\infty} k \cdot q^{k-1} p = p \cdot \sum_{k=1}^{\infty} k \cdot q^{k-1}$$

For $|q| < 1$, the geometric series is:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

Differentiating both sides with respect to q gives:

$$\sum_{k=1}^{\infty} k \cdot q^{k-1} = \frac{1}{(1-q)^2}$$

Therefore,

$$E(X) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

Thus,

$$\mathbf{E(X) = \frac{1}{p}}$$

The expected number of trials required to obtain the first success is the reciprocal of the probability of success.

Expected Values of Common Discrete Distributions

4. Poisson Random Variable.

Let $X \sim \text{Poisson}(\lambda)$, where $\lambda > 0$. By the definition of expected value,

$$E(X) = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

Let $j = k - 1$. Then

$$E(X) = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

Using the exponential-series expansion,

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda}$$

Therefore,

$$E(X) = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Thus, $E(X) = \lambda$.

The parameter λ is both the parameter of the Poisson distribution and the expected number of events occurring in the specified interval.

Expected Values of Common Discrete Distributions

5. Hypergeometric Random Variable.

Let $X \sim \text{Hypergeometric}(N, K, n)$. The expected value can be derived using indicator random variables. For each selected object, define $I_i = 1$ if the i th selected object is a success, $I_i = 0$ otherwise.

Then $X = I_1 + I_2 + \dots + I_n$. For every selection position, the probability of selecting a success is $P(I_i = 1) = K / N$.

Therefore, $E(I_i) = 1 \cdot K/N + 0 \cdot (1 - K/N) = K/N$.

Using linearity of expectation,

$$E(X) = E(I_1 + I_2 + \dots + I_n) = E(I_1) + E(I_2) + \dots + E(I_n) = K/N + K/N + \dots + K/N.$$

There are n terms, so **$E(X) = n \cdot K/N$.**

Expected Values of Common Discrete Distributions.

Example

Example. A software company tests 20 modules. Assume that the test outcomes are independent and that each module passes all quality checks with probability 0.85.

Variance

Let X be a discrete random variable with possible values $x_1, x_2, x_3, \dots$ and corresponding probabilities $p_i = P(X = x_i)$.

The **variance** of X is defined as the expected value of the squared deviation of X from its expected value:

$$\text{Var}(X) = E[(X - E(X))^2]$$

The variance can also be calculated using the computational formula

$$\text{Var}(X) = E(X^2) - [E(X)]^2,$$

where $E(X^2) = \sum_i x_i^2 \cdot p_i$.

The variance measures the degree to which the possible values of a random variable are spread around its expected value. A small variance indicates that the values of X tend to be close to $E(X)$, whereas a large variance indicates greater variability.

Variance

The variance is measured in the square of the original units. For this reason, the **standard deviation** of X is defined by

$$\sigma(X) = \sqrt{\text{Var}(X)}$$

The standard deviation is measured in the same units as the random variable.

Properties of Variance.

- 1. Nonnegativity.** For every random variable X , $\text{Var}(X) \geq 0$. This follows from the fact that $(X - E(X))^2 \geq 0$.
- 2. Variance of a Constant.** If C is a constant, then $\text{Var}(C) = 0$. A constant random variable has no variability because it always takes the same value.
- 3. Multiplication by a Constant.** For any constant C , $\text{Var}(C \cdot X) = C^2 \cdot \text{Var}(X)$. Thus, multiplying a random variable by a constant multiplies its variance by the square of that constant.
- 4. Variance of a Sum of Independent Random Variables.** If X and Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. More generally, if $X_1, X_2, \dots, X_n$ are mutually independent, then $\text{Var}(X_1 + X_2 + \dots + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n)$. Independence is essential for this property.

Example. Find the variance and standard deviation of the random variable X from previous Example.

Variance

1. *Bernoulli Random Variable.*

$$\mathbf{Var}(X) = p \cdot q$$

2. *Binomial Random Variable*

$$\mathbf{Var}(X) = n \cdot p \cdot q$$

3. *Geometric Random Variable*

$$\mathbf{Var}(X) = \frac{q}{p^2}$$

4. *Poisson Random Variable*

$$\mathbf{Var}(X) = \lambda$$

5. *Hypergeometric Random Variable*

$$\mathbf{Var}(X) = n \cdot \frac{K}{N} \cdot \left(1 - \frac{K}{N}\right) \cdot (N - n) / (N - 1)$$

Example

A software company tests 20 independently developed modules. Each module passes all quality checks with probability 0.85. Let X denote the number of modules that pass all quality checks. Find the variance and standard deviation of X .

Cumulative distribution function

The cumulative distribution function (CDF) of a random variable X is the function $F(x)$ defined by:

$$F(x) = P(X \leq x), x \in \mathbf{R}$$

For a discrete random variable X with possible values $x_1, x_2, \dots$, the cumulative distribution function is given by:

$$F_X(x) = \sum P(X = x_i)$$

where the summation is taken over all values x_i satisfying $x_i \leq x$.

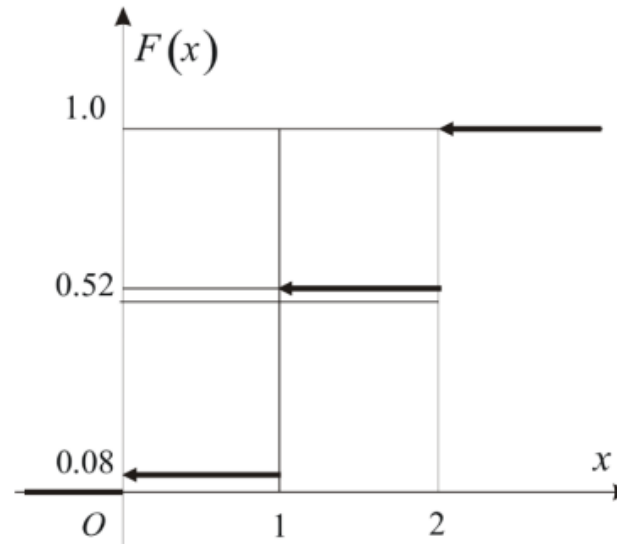
Cumulative distribution function

The cumulative distribution function (CDF) of a random variable X is the function $F(x)$ defined by:

$$F(x) = P(X \leq x), x \in \mathbf{R}$$

Example (discrete case):

ξ	0	1	2
p	0.08	0.44	0.48



$$F(x) = \begin{cases} 0; & x \in (-\infty, 0] \\ 0.08; & x \in (0, 1] \\ 0.52; & x \in (1, 2] \\ 1; & x \in (2, \infty] \end{cases}$$

Cumulative distribution function

Example. Let X denote the number of critical errors found in a randomly selected software module. Its probability distribution is given in Table. Construct the cumulative distribution function of X .

x	0	1	2	3
p	0.2	0.4	0.3	0.1

Cumulative distribution function

Properties of CDF:

1. $0 \leq F(x) \leq 1$

2. $F(x)$ – non-decreasing function

Proof: let $x_2 > x_1$, then $F(x_2) = P(X < x_2) = P(X < x_1) + P(x_1 \leq X < x_2) \Rightarrow$

$P(x_1 \leq X < x_2) = P(X < x_2) - P(X < x_1)$, and $P(x_1 \leq X < x_2) = F(x_2) - F(x_1) \geq 0$

3. $P(a \leq X \leq b) = F(b) - F(a)$

4. $P(X = a) = 0$

5. If the possible values of X belong to the interval (a, b) , that $F(x) = 0, x \leq a$;

$F(x) = 1, x \geq b$

6. $\lim_{x \rightarrow -\infty} F(x) = 0, \lim_{x \rightarrow +\infty} F(x) = 1$

Example

A device consists of three independently operating components. The probability of failure of each component in a single trial is 0.1. Find the probability distribution of the number of failed components in a single trial, calculate its expected value and variance, and construct its cumulative distribution function (CDF).

Continuous random variable, PDF

Consider random variables with a continuous range of possible values.

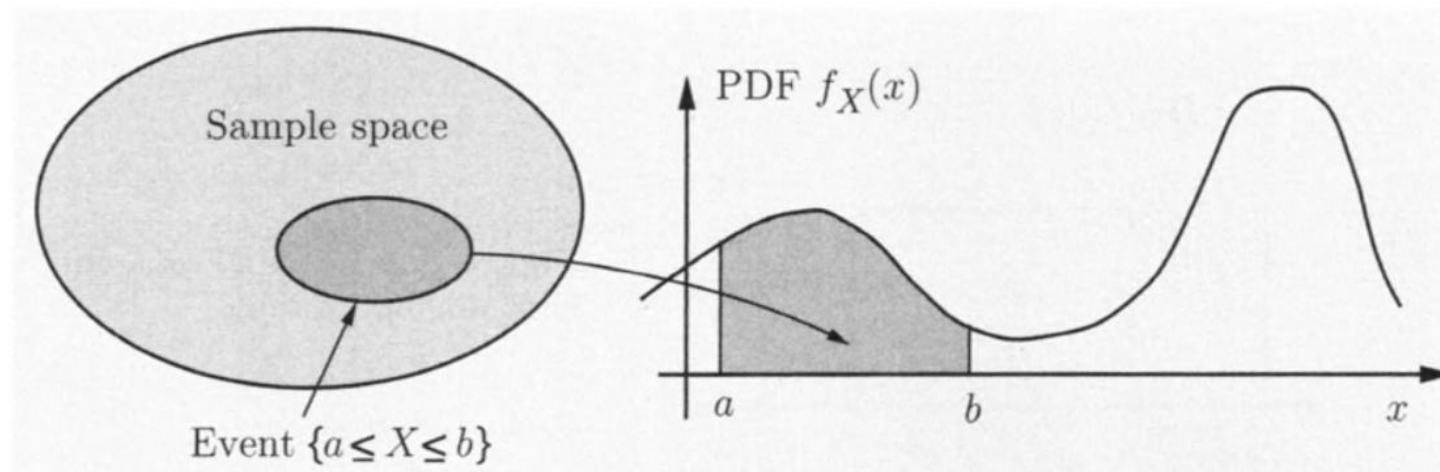
A random variable X is called **continuous** if there is a nonnegative function f , called **the probability density function** of X (**PDF**) such that:

$$P(X \in B) = \int_B f(x)dx$$

for every subset B of the real line.

In particular, the probability that the value of X falls within an interval is

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$



Continuous random variable, PDF

1. $f(x) \geq 0$

2. $\int_{-\infty}^{+\infty} f(x)dx = 1$

Expectation: $E(X) = \int_{-\infty}^{+\infty} xf(x)dx$

Variance: $var(X) = E(X^2) - (E(X))^2$, where $E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx$

Standard deviation: $\sigma = \sqrt{var(X)}$

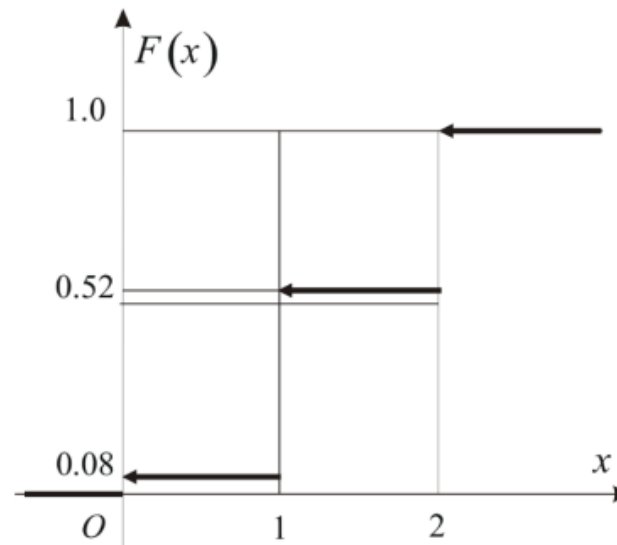
Cumulative distribution function

The cumulative distribution function, or CDF can describe all kinds of random variables with a single mathematical concept:

$$F(x) = P(X < x)$$

Example (discrete case):

ξ	0	1	2
p	0.08	0.44	0.48



$$F(x) = \begin{cases} 0; & x \in (-\infty, 0] \\ 0.08; & x \in (0, 1] \\ 0.52; & x \in (1, 2] \\ 1; & x \in (2, \infty] \end{cases}$$

Cumulative distribution function

Properties of CDF:

1. $0 \leq F(x) \leq 1$

2. $F(x)$ – non-decreasing function

Proof: let $x_2 > x_1$, then $F(x_2) = P(X < x_2) = P(X < x_1) + P(x_1 \leq X < x_2) \Rightarrow$

$P(x_1 \leq X < x_2) = P(X < x_2) - P(X < x_1)$, and $P(x_1 \leq X < x_2) = F(x_2) - F(x_1) \geq 0$

3. $P(a \leq X \leq b) = F(b) - F(a)$

4. $P(X = a) = 0$

5. If the possible values of X belong to the interval (a, b) , that $F(x) = 0, x \leq a$;

$F(x) = 1, x \geq b$

6. $\lim_{x \rightarrow -\infty} F(x) = 0, \lim_{x \rightarrow +\infty} F(x) = 1$

Cumulative distribution function

$F(x) = P(X < x) = P(-\infty < X < x) = \int_{-\infty}^x f(t)dt$, that is:

$$F(x) = \int_{-\infty}^x f(t)dt$$
$$f(x) = F'(x)$$

Example. The CDF of the random variable X has the form:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{33}(2x^2 + 5x), & 0 \leq x \leq 3 \\ 1, & x > 3 \end{cases}$$

Find PDF; $E(X)$; $\text{var}(X)$; $\sigma(X)$, $P(1 < X < 2)$; plot graphs of CDF and PDF.

Continuous random variable, PDF

Expectation: $E(X) = \int_{-\infty}^{+\infty} xf(x)dx$

Variance: $var(X) = E(X^2) - (E(X))^2$, where $E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx$

Standard deviation: $\sigma = \sqrt{var(X)}$

$$F(x) = \int_{-\infty}^x f(t)dt$$
$$f(x) = F'(x)$$

Example

Example. The PDF of the random variable X has the form:

$$f(x) = \begin{cases} \frac{C}{x^4}, & x \geq 2 \\ 0, & x < 2 \end{cases}$$

Find C ; CDF; $E(X)$; $\text{var}(X)$; $\sigma(X)$; $P(2 < X < 3)$, plot graphs of CDF and PDF.

Example

Example. The CDF of the random variable T, the service life of the device, is known

$$F(t) = \begin{cases} 0, & t < 0 \\ kt^2, & 0 \leq t \leq \frac{3}{4} \\ 1, & t > \frac{3}{4} \end{cases}$$

Find k, average device life, and variance.

Tasks

Tasks

1. The exam paper contains three tasks. The probability of a student correctly solving the first problem is 0.9, the second – 0.6, and the third – 0.1. Find the distribution of the number of correctly solved problems and find its:
 - expectation
 - variance
 - standard deviation
 - CDF
2. Two students are taking an exam in probability theory. For one of them, the probability of passing the exam is 0.9, for the other - 0.6. Construct a distribution of X - number of students who passed the exam. Find the expectation of the X , standard deviation and CDF.

Tasks

3. The distribution of the random variable X has the form:

X	-2	1	2	5	7
p	0.3	0.1	0.1	0.3	0.2

Find its expectation, variance and standard deviation. Find the CDF of the random variable X and plot it.

4. Discrete random variable X takes three possible values : $x_1 = 4$ with a probability $p_1 = 0.5$; $x_2 = 6$ with a probability $p_2 = 0.3$ and x_3 with a probability p_3 . Find x_3 and p_3 knowing that $E(X) = 8$.

5. Find the variance of a discrete random variable X —the number of occurrences of event A in five independent trials, if the probability of occurrence of events A in each trial is 0.2.

6. If a die shows an even number, a prize of 1 ruble is awarded; if it shows a three, 3 rubles are awarded; in all other cases, 2 rubles are awarded. Find the probability distribution of the total prize amount when three dice are rolled. Calculate the expected value of the total prize amount.

Tasks

7. A discrete random variable X takes two possible values, x_1 and x_2 , $x_2 > x_1$. X taking the value x_1 with probability p_1 . Find the probability distribution of X if $E(X) = 1.4$ and $\text{Var}(X) = 0.24$.
8. Determine whether the given function $F(x)$ is cumulative distribution function (CDF). If not, propose a way to correct it. Plot the graph.

$$F(x) = \begin{cases} 0, & x \leq 0, \\ 0.3, & x \in (0; 1], \\ 0.5, & x \in (1; 2], \\ 1, & x > 2; \end{cases}$$

Tasks

9. A discrete random variable X has the following probability distribution:

X	-2	1	4
$P(X = x)$	a	b	$1 - a - b$

It is known that $E(X)=1, \text{Var}(X)=4.5$.

Find:

- The unknown probabilities a and b .
- The complete probability distribution of X .
- The standard deviation of X .
- The CDF of X and plot its graph.
- The probability $P(|X-E(X)| \geq 2)$.

Tasks

10. A fair six-sided die is rolled three times independently. For each roll, the following prize is awarded:

- 0 rubles if the die shows 1 or 2;
- 2 rubles if the die shows 3, 4, or 5;
- 5 rubles if the die shows 6.

Let X be the total prize amount obtained after three rolls.

Find:

- All possible values of X .
- The probability distribution of X .
- The expectation, variance, and standard deviation of X .
- The CDF of X and plot its graph.
- The probability that the total prize exceeds 7 rubles.

Tasks

11. A factory uses two machines, A and B, to produce electronic components.

Machine A is selected with probability 0.4, and machine B is selected with probability 0.6. The probability of producing a defective component is 0.1 for machine A and 0.3 for machine B. A machine is selected randomly, and three components produced by that machine are inspected. Conditional on the selected machine, the components are defective independently.

Let X be the number of defective components among the three inspected components.

Find:

- The probability distribution of X .
- The expectation, variance, and standard deviation of X .
- The CDF of X and plot its graph.
- The probability that machine B was selected, given that exactly two defective components were found.

Homework

Study guide: p. 83-84, Tasks 1-8