Algorithms and Data Structures

Module 1

Lecture 4
Efficient algorithms for DAGs, part 1

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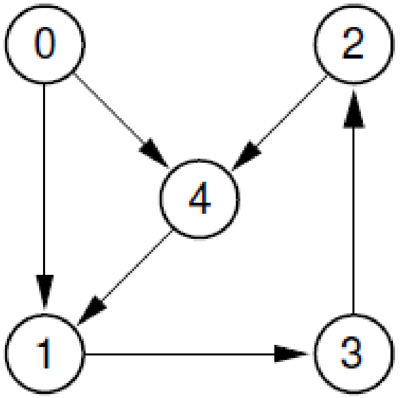
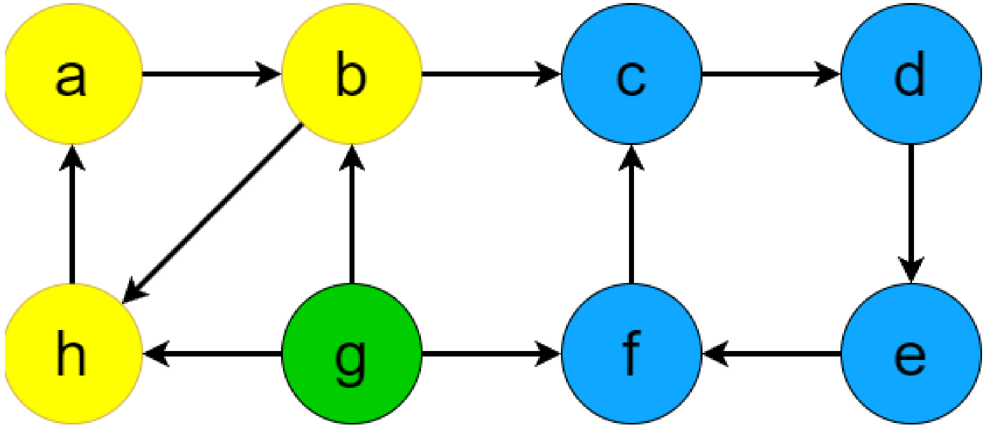
Condensation

- A digraph is called *strongly connected* iff for each pair of vertices $\{u, v\}$ there is a directed path between u and v.
- The maximum strongly connected subgraphs of *G* are called *strong connected components*.

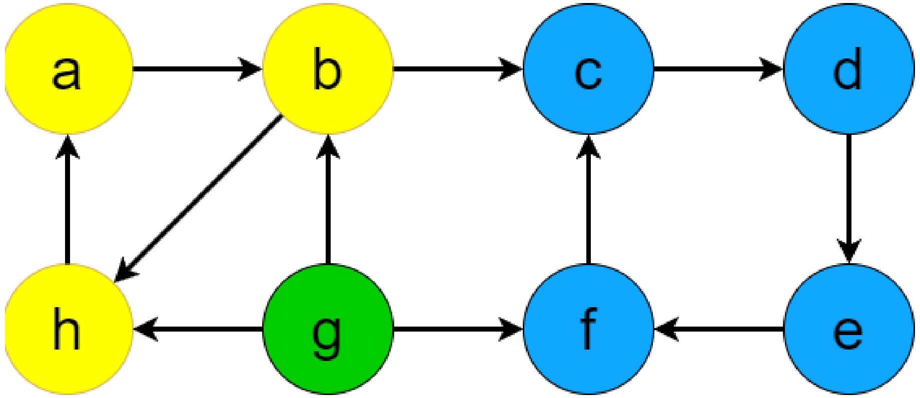


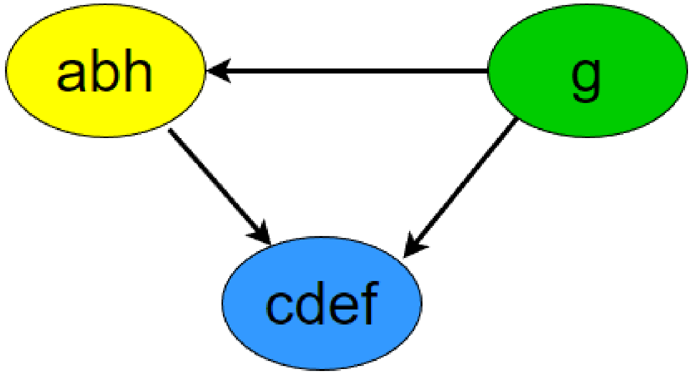


Condensation

Definition. A ***condensation*** of a digraph $G(V, E)$ is a graph $C(V_C, E_C)$ such that

- Vertices of *C* are strong components of *G*.
- There is an arc (directed edge) (*u*,*v*) iff for *G* there is an arc from a certain vertex of the strong component *u* to a certain vertex of the strong component *v*.





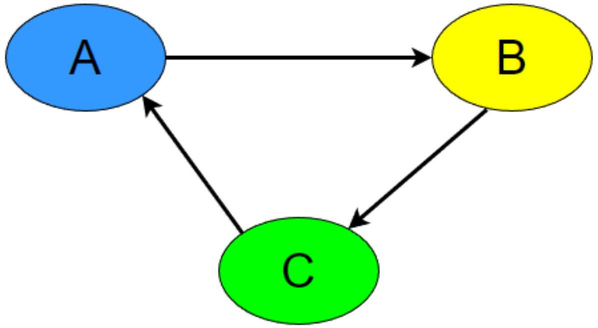
Condensation

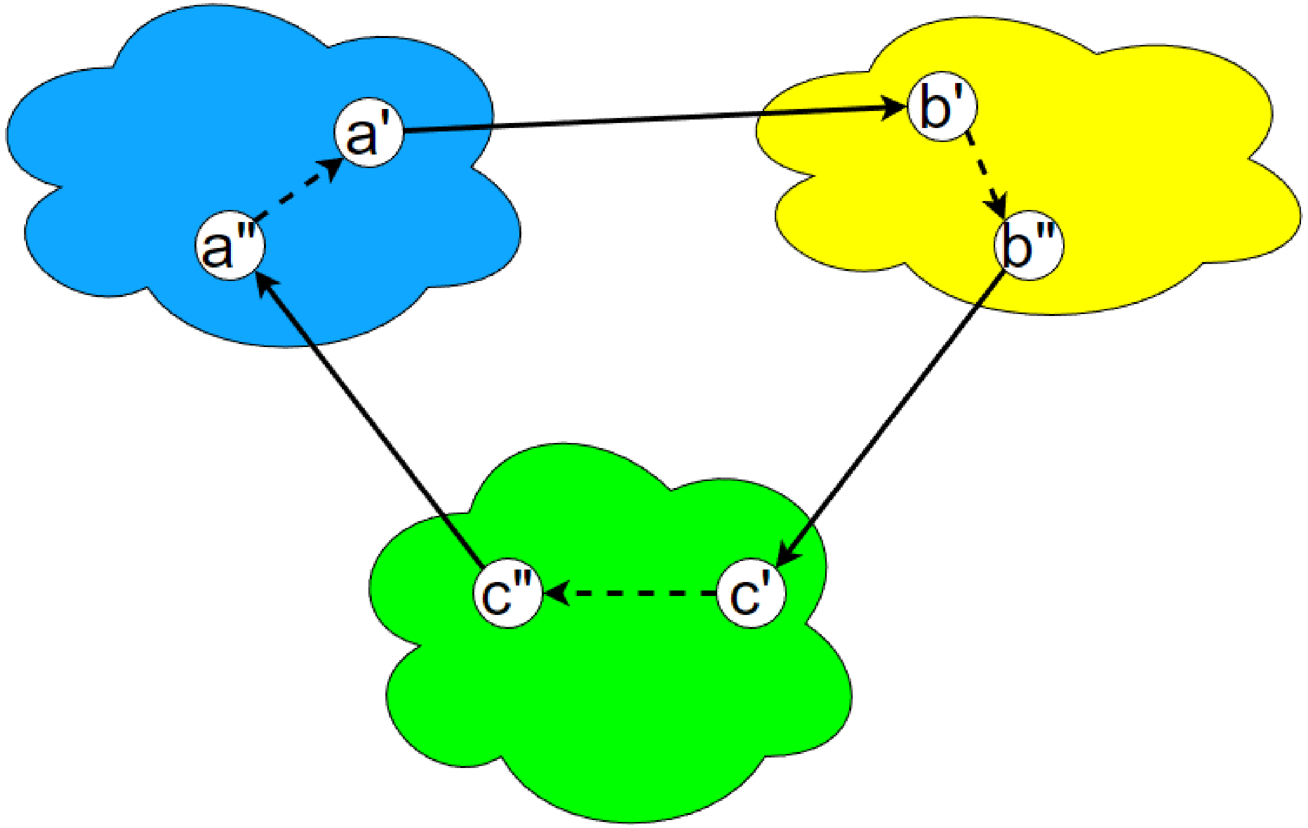
Theorem. For any digraph $G(V, E)$, its condensation $C(V_C, E_C)$ is a DAG.

Proof

Suppose that there is a graph $G(V, E)$ whose condensation $C(V_C, E_C)$ has a directed cycle.

Condensation





Layered graph drawing

Definition. For a digraph $G(V, E)$, its layered drawing is a representation of *G* with vertex set partitioned into several disjoint numerated subsets, called ***layers*** (***tiers***) $V = \bigcup V_i$ and the following condition holds for each arc:

if arc starts at layer i and ends at layer j, then $i < j$.

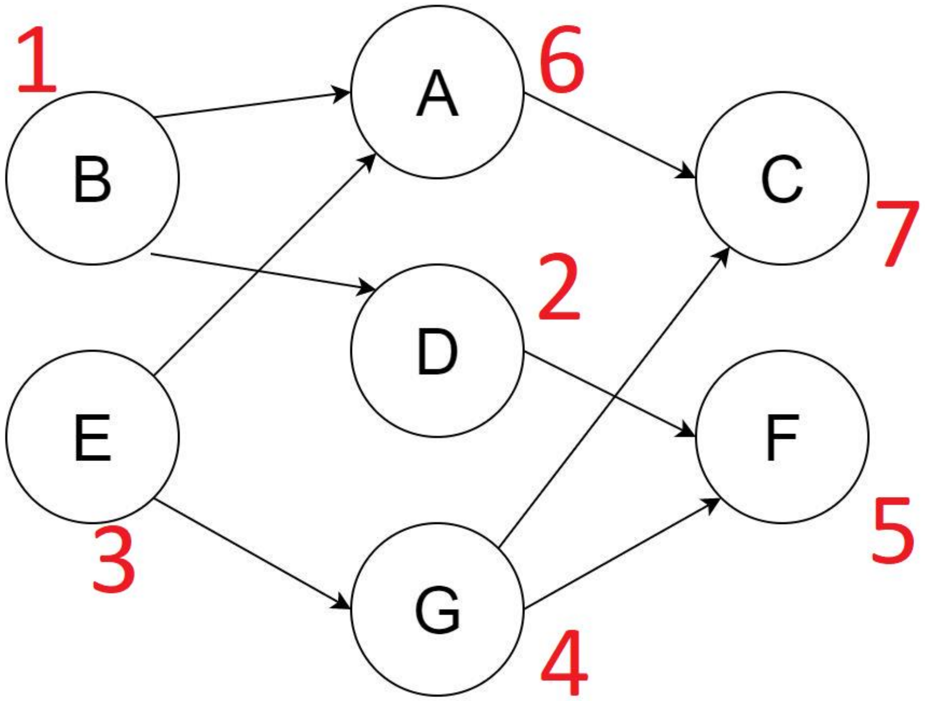
Layered graph drawing

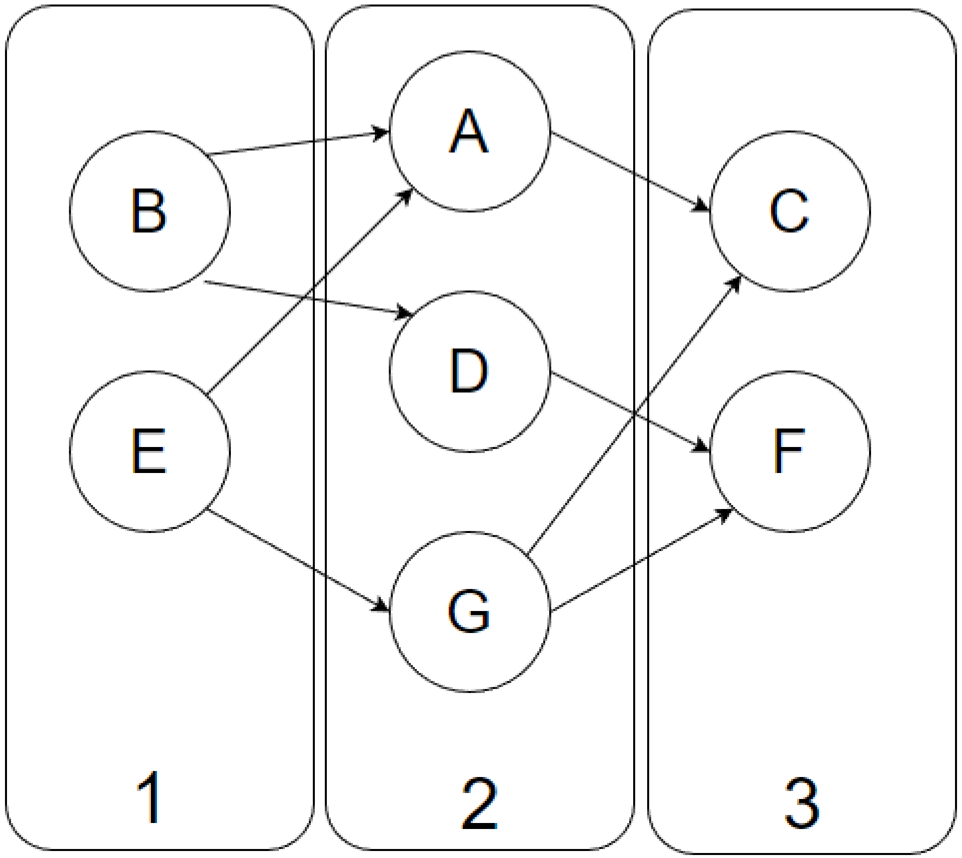
Theorem. All vertices of the same layer are mutually unreachable.

Theorem. A digraph has a layered drawing iff the digraph is a DAG.

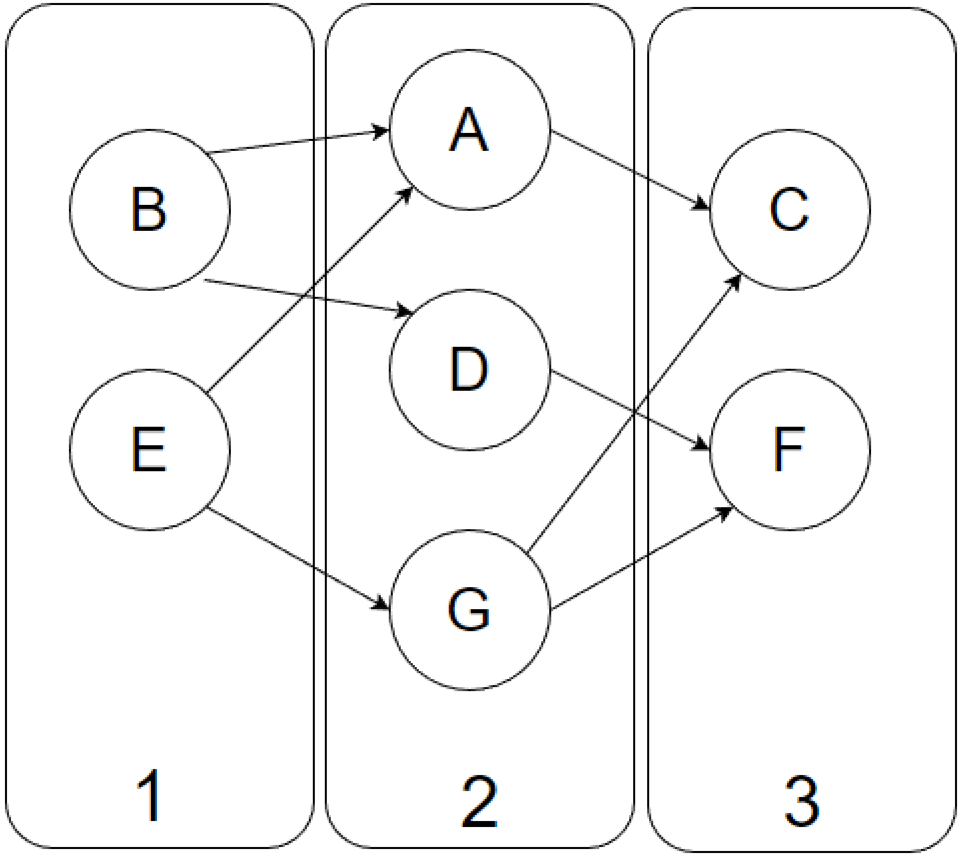
The proof is analogous to that of the theorem about topological ordering (topological sorting).

Layered graph drawing





Layered graph drawing



Minimum possible number of layers for a digraph *G* is called the ***depth*** of *G*.

Layered graph drawing

Problem. For the given DAG $G(V, E)$, build its layered drawing with the minimum possible number of layers.

This problem arises as a part of planning process when parallel execution of some tasks is possible.

Layered graph drawing

Algorithm

```
1. Create a layer counter and initialize it with 1.
2. While graph is not empty:
    2.1. Find all sources of the graph and put them to
         the current layer.
    2.2. Increment the layer counter.
    2.3. Remove all sources from the graph.
```

Layered graph drawing

<u>**Theorem**</u>. This algorithm builds minimum layered drawing.

This theorem is a corollary from the following theorem.

Layered graph drawing

Definition. A ***critical path*** in a DAG is a path with the maximum possible number of arcs.

Theorem. The depth of a DAG is equal to the number of vertices in the critical path of the DAG.

Proof

1. The number of layers cannot be less than the number of vertices in the critical path.

2. The algorithm builds the layered drawing with number of layers = number of vertices in the critical path.

Layered graph drawing

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