14. Alternating series and conditional convergence

Alternating series

Definition of conditional convergence, alternating series, and the Leibniz series

3.11B/00:00 (04:22)

Definition 1.

The series $\sum_{k=1}^{\infty} a_k$ is called *conditionally convergent* if it converges and the series $\sum_{k=1}^{\infty} |a_k|$ diverges. Thus, a convergent series is called conditionally convergent if it does not converge absolutely.

Such a situation is possible only when the terms of a series have different signs.

Definition 2.

A series of the form $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ is called an it alternating series, if all elements of the sequence $\{a_k\}$ have the same sign.

$a_k > 0$

Definition 3.

An alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ is called the *Leibniz series*, if the sequence $\{a_k\}$ monotonously approaches zero as $k \to \infty$.

Remarks.

1. When studying Leibniz series of the form $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$, we assume, for definiteness, that $a_k > 0$, $k \in \mathbb{N}$ (in this case, the sequence $\{a_k\}$ is a *non-increasing* sequence approaching zero).

$\sum_{k=1}^{\infty} \frac{1}{k}$ divers

2. The "Leibniz series" notion is also referred to the alternating series of a special form $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$, which was studied by G. W. Leibniz (he proved that the sum of this series is equal to $\frac{\pi}{4}$).

$\sum_{k=1}^{\infty} a_k$ conv $\Rightarrow$ $a_k \to 0$

$\not\Leftarrow$

Theorem on the convergence of the Leibniz series

3.11B/04:22 (11:11)

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ conv

Theorem (on the convergence of the Leibniz series).

The Leibniz series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

$a_k > 0$ $a_k \to 0$ monotonusly

$a_1 - a_2 + a_3 - a_4 + a_5 - \dots$

Proof.

Consider the partial sums of the Leibniz series with an even number of terms:

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n}. \tag{1}$$

We place parentheses on the right-hand side of equality (1) as follows:

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}).$$

Since the sequence $\{a_k\}$ is non-increasing, we obtain that each expression in parentheses is non-negative: $a_{2k-1} - a_{2k} \geq 0$, $k = 1, 2, \ldots$ Hence,

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}) \geq S_{2n}.$$

This estimate means that the sequence of partial sums $\{S_{2n}\}$ is non-decreasing.

Now we put parentheses in (1) in another way:

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}.$$

Since, as before, each expression in parentheses is non-negative, we obtain that the sum S_{2n} is estimated from above by the value a_1:

$$S_{2n} \leq a_1.$$

Thus, the sequence $\{S_{2n}\}$ is not only non-decreasing, but also bounded from above. Then, by virtue of the convergence theorem for monotone bounded sequences, the sequence $\{S_{2n}\}$ has a finite limit S:

$$\lim_{n\to\infty} S_{2n} = S.$$

Consider the partial sums of the Leibniz series with an odd number of terms: S_{2n+1}. For them, the following equality holds:

$$S_{2n+1} = S_{2n} + a_{2n+1}. \tag{2}$$

We have already proved that $S_{2n} \to S$ as $n \to \infty$. In addition, $a_{2n+1} \to 0$ as $n \to \infty$, since by condition $a_k \to 0$ as $k \to \infty$ and thus the subsequence $\{a_{2n+1}\}$ of the sequence $\{a_k\}$ must also converge to this limit by the theorem on the limit of subsequences of a converging sequence.

Therefore, the right-hand side of equality (2) has a limit S, so the left-hand side approaches the same limit.

So, we have proved that $S_{2n} \to S$ as $n \to \infty$ and $S_{2n+1} \to S$ as $n \to \infty$. This means that the entire sequence $\{S_n\}$ converges to the limit of S, since any neighborhood of the point S contains all elements of the sequence $\{S_n\}$

(with even and odd indices), with the possible exception of some finite number of its initial elements.

The convergence of the sequence of partial sums $\{S_n\}$ to a finite limit means that the corresponding series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges. $\square$

Remark.

The theorem on the convergence of the Leibniz series guarantees only its conditional convergence. For example, the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ is a Leibniz series, however, we previously established that the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$, consisting of absolute values of terms of the initial series, is divergent. In what follows, we will prove that the sum of the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ is equal to $\ln 2$.

Estimation of the Leibniz series in terms of its partial sums

`3.11B/15:33 (14:19)`

Theorem (on the estimation of the Leibniz series in terms of its partial sums).

Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = S$ be the Leibniz series and $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ be its partial sums. Then, for any $k \in \mathbb{N}$, the following estimate holds:

$$|S - S_k| \le a_{k+1}. \tag{3}$$

Proof.

In the proof of the previous theorem, we established that the sequence $\{S_{2n}\}$ is non-decreasing and has a limit S. This means that the following equality holds for all $n \in \mathbb{N}$:

$$S_{2n} \le S. \tag{4}$$

On the other hand, the sequence $\{S_{2n+1}\}$ is non-increasing since

$$S_{2n+1} = a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1}) - (a_{2n} - a_{2n+1}) \ge$$
$$\ge a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1}) = S_{2n-1}.$$

In addition, its limit is also equal to S. Therefore, the equality holds for all $n \in \mathbb{N}$:

$$S \le S_{2n+1}. \tag{5}$$

Let us subtract S_{2n} from both sides of inequality (5):

$$S - S_{2n} \le S_{2n+1} - S_{2n} = a_{2n+1}. \tag{6}$$

It follows from inequality (4) that $S - S_{2n} \ge 0$. Therefore, inequality (6) can be rewritten in the form

$|S - S_{2n}| \le a_{2n+1}.$

We have obtained estimate (3) for the case of even k.

Now we turn to inequality (4) and subtract S_{2n-1} from both its parts:

$S_{2n} - S_{2n-1} \le S - S_{2n-1}.$

Since $S_{2n} - S_{2n-1} = -a_{2n}$, this inequality can be transformed as follows:

$$S_{2n-1} - S \le a_{2n}. \tag{7}$$

It follows from inequality (5) that $S_{2n-1} - S \ge 0$. Therefore, inequality (7) can be rewritten in the form

$|S_{2n-1} - S| \le a_{2n}.$

We have obtained estimate (3) for the case of odd k.

Thus, estimate (3) is proved for all positive integers k. □

Dirichlet's test and Abel's test for conditional convergence of a numerical series

Dirichlet's test for conditional convergence of a numerical series `3.11B/29:52 (04:29), 3.12A/00:00 (03:18)`

Theorem (Dirichlet's test for conditional convergence of a numerical series).

Let the following conditions be satisfied for the series $\sum_{k=1}^{\infty} a_k b_k$:

1) $\exists M \quad \forall n \in \mathbb{N} \quad \left|\sum_{k=1}^{n} a_k\right| \le M$;

2) $b_k \to 0$ as $k \to \infty$, $\{b_k\}$ is monotone.

Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges (generally speaking, conditionally).

Proof[1].

Let us show that, for the series $\sum_{k=1}^{\infty} a_k b_k$, the condition for the Cauchy criterion for the convergence of a numerical series is fulfilled. For this, we will obtain an estimate for the sum $\left|\sum_{k=m+1}^{m+p} a_k b_k\right|$ when $m, p \in \mathbb{N}$.

First, let us transform the sum $\sum_{k=m+1}^{m+p} a_k b_k$ using the auxiliary notation $A_n = \sum_{k=1}^{n} a_k$:

$$\sum_{k=m+1}^{m+p} a_k b_k = \sum_{k=m+1}^{m+p} (A_k - A_{k-1}) b_k = \sum_{k=m+1}^{m+p} A_k b_k - \sum_{k=m+1}^{m+p} A_{k-1} b_k =$$

$$= \sum_{k=m+2}^{m+p+1} A_{k-1} b_{k-1} - \sum_{k=m+1}^{m+p} A_{k-1} b_k =$$

[1] There is no proof of this theorem in video lectures.

$$= A_{m+p}b_{m+p} + \sum_{k=m+2}^{m+p} A_{k-1}b_{k-1} - \sum_{k=m+2}^{m+p} A_{k-1}b_k - A_m b_{m+1} =$$

$$= A_{m+p}b_{m+p} + \sum_{k=m+2}^{m+p} A_{k-1}(b_{k-1} - b_k) - A_m b_{m+1}.$$

Let us estimate the value $\left|\sum_{k=m+1}^{m+p} a_k b_k\right|$ using condition 1 of the theorem, from which it follows that $|A_k| \le M$ for $k \in \mathbb{N}$:

$$\left|\sum_{k=m+1}^{m+p} a_k b_k\right| = \left|A_{m+p}b_{m+p} + \sum_{k=m+2}^{m+p} A_{k-1}(b_{k-1} - b_k) - A_m b_{m+1}\right| \le$$

$$\le M|b_{m+p}| + M\sum_{k=m+2}^{m+p} |b_{k-1} - b_k| + M|b_{m+1}|. \tag{8}$$

Since, by condition 2 of the theorem, the sequence $\{b_k\}$ monotonously approaches 0, we obtain that all the differences $b_{k-1} - b_k$ have the same sign. Therefore, in the sum $\sum_{k=m+2}^{m+p} |b_{k-1} - b_k|$, the absolute value sign can be moved outside the sum sign:

$$\sum_{k=m+2}^{m+p} |b_{k-1} - b_k| = \left|\sum_{k=m+2}^{m+p} (b_{k-1} - b_k)\right| =$$

$$= |(b_{m+1} - b_{m+2}) + (b_{m+2} - b_{m+3}) + \cdots + (b_{m+p-1} - b_{m+p})| =$$

$$= |b_{m+1} - b_{m+p}| \le |b_{m+1}| + |b_{m+p}|.$$

Now we substitute the estimate for $\sum_{k=m+2}^{m+p} |b_{k-1} - b_k|$ into inequality (8):

$$\left|\sum_{k=m+1}^{m+p} a_k b_k\right| \le M|b_{m+p}| + M(|b_{m+1}| + |b_{m+p}|) + M|b_{m+1}| =$$

$$= 2M(|b_{m+1}| + |b_{m+p}|).$$

It remains to use the condition $b_k \to 0$ as $k \to \infty$, which can be written as follows:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N \quad \forall p \in \mathbb{N} \quad |b_{m+p}| < \frac{\varepsilon}{4M}.$$

For $\left|\sum_{k=m+1}^{m+p} a_k b_k\right|$, we finally get

$$\left|\sum_{k=m+1}^{m+p} a_k b_k\right| \le 2M(|b_{m+1}| + |b_{m+p}|) < 2M\left(\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\right) = \varepsilon.$$

We have proved that the Cauchy criterion condition is satisfied for the initial series:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N \quad \forall p \in \mathbb{N} \quad \left| \sum_{k=m+1}^{m+p} a_k b_k \right| < \varepsilon.$$

Therefore, the series $\sum_{k=1}^{\infty} a_k b_k$ converges. $\square$

Examples of applying Dirichlet's test `3.12A/03:18 (11:41)`

1. Once again, let us turn to the Leibniz series and write it in the following form: $\sum_{k=1}^{\infty} (-1)^{k+1} b_k$. By the definition of the Leibniz series, two conditions are satisfied for the sequence $\{b_k\}$: $b_k \to 0$ as $k \to \infty$, $\{b_k\}$ is monotone. Thus, the condition 2 of Dirichlet's test is satisfied for $\{b_k\}$. Also we can take the sequence $\{(-1)^{k+1}\}$ as the sequence $\{a_k\}$. Obviously, this sequence satisfies condition 1 of Dirichlet's test:

$1 - 1 + 1 - 1 \; . - .$ $\qquad$ 0 or 1

$$\forall n \in \mathbb{N} \quad \left| \sum_{k=1}^{n} a_k \right| = |1 - 1 + 1 - 1 + \ldots| \le 1.$$

Thus, the convergence of the Leibniz series follows directly from the Dirichlet's test.

2. Consider the following series: $\sum_{k=1}^{\infty} \frac{\sin kx}{k^\alpha}$, $x \in \mathbb{R}$, $\alpha > 0$. If $\alpha > 1$, then this series converges absolutely for any $x \in \mathbb{R}$, since, in this case, the absolute value of its common term can be estimated as follows:

$$\left| \frac{\sin kx}{k^\alpha} \right| \le \frac{1}{k^\alpha}.$$

$\alpha > 1$ converges

Earlier, when discussing the integral convergence test, we established that the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges for $\alpha > 1$. Therefore, using the comparison test, we obtain that the series $\sum_{k=1}^{\infty} \left| \frac{\sin kx}{k^\alpha} \right|$ also converges, which means that the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^\alpha}$ converges absolutely.

$a_k = \sin kx, \; b_k = \frac{1}{k^\alpha}$

Consider the case $\alpha \in (0, 1]$ and show that in this case all conditions of Dirichlet's test are satisfied for the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^\alpha}$.

First, we discard the case of $x = 2\pi m$, $m \in \mathbb{Z}$, since in this case all terms of the series turn to 0 and therefore the sum of the series is also 0.

We take $\frac{1}{k^\alpha}$ as b_k, since it is obvious that the sequence $\left\{\frac{1}{k^\alpha}\right\}$ is monotone (decreasing) and approaches zero as $k \to \infty$. We take $\sin kx$ as a_k and show that condition 1 of Dirichlet's test is satisfied for partial sum $\sum_{k=1}^{n} \sin kx$. To do this, we transform this partial sum by multiplying and dividing the common term by $2 \sin \frac{x}{2}$ (this factor is not equal to 0, since we assume that $x \ne 2\pi m$, $m \in \mathbb{Z}$):

$$\sum_{k=1}^{n} \frac{2 \sin kx \sin \frac{x}{2}}{2 \sin \frac{x}{2}} = \frac{1}{2 \sin \frac{x}{2}} \sum_{k=1}^{n} 2 \sin kx \sin \frac{x}{2}. \tag{9}$$

$\left| \int_a^b \sin x \, dx \right| \le 2$

Let us transform the product of the sines $\sin kx \sin \frac{x}{2}$ according to the formula $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$:

$$\sum_{k=1}^{n} 2\sin kx \sin\frac{x}{2} = \sum_{k=1}^{n}\Big(\cos\Big(kx-\frac{x}{2}\Big) - \cos\Big(kx+\frac{x}{2}\Big)\Big) =$$

$$= \cos\frac{x}{2} - \cos\frac{3x}{2} + \cos\frac{3x}{2} - \cos\frac{5x}{2} + \cdots +$$

$$+ \cos\frac{(2n-1)x}{2} - \cos\frac{(2n+1)x}{2} = \cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}.$$

Now let us transform the last difference using the formula $\cos\alpha - \cos\beta = 2\sin\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}$:

$$\cos\frac{x}{2} - \cos\frac{(2n+1)x}{2} = 2\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}.$$

Substituting the resulting expression into the right-hand side of (9), we finally obtain

$$\sum_{k=1}^{n}\sin kx = \frac{1}{2\sin\frac{x}{2}}\cdot 2\sin\frac{(n+1)x}{2}\sin\frac{nx}{2} = \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

This implies the following estimate for partial sum $\sum_{k=1}^{n}\sin kx$, $n\in\mathbb{N}$:

$$\Big|\sum_{k=1}^{n}\sin kx\Big| \le \frac{1}{|\sin\frac{x}{2}|}.$$

Thus, condition 1 of Dirichlet's test is also satisfied, and the series $\sum_{k=1}^{\infty}\frac{\sin kx}{k^\alpha}$ is convergent for $\alpha\in(0,1]$. However, for these values of α, convergence is conditional.

The proof of the absence of absolute convergence

3.12A/14:59 (06:03)

The fact that the series $\sum_{k=1}^{\infty}\frac{\sin kx}{k^\alpha}$ is not absolutely convergent for $\alpha\in(0,1]$ is proved in the same way as a similar fact for the improper integral $\int_1^{+\infty}\frac{\sin x}{x}\,dx$. First of all, recall the estimate for the function $\frac{\sin kx}{k^\alpha}$; this estimate is valid for all k and x:

$$\Big|\frac{\sin kx}{k^\alpha}\Big| \ge \frac{\sin^2 kx}{k^\alpha}. \tag{10}$$

Let us prove that the series $\sum_{k=1}^{\infty}\frac{\sin^2 kx}{k^\alpha}$ diverges. To do this, consider its partial sum and transform it as follows:

$$\sum_{k=1}^{n}\frac{\sin^2 kx}{k^\alpha} = \sum_{k=1}^{n}\frac{1-\cos 2kx}{2k^\alpha} = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k^\alpha} - \frac{1}{2}\sum_{k=1}^{n}\frac{\cos 2kx}{k^\alpha}. \tag{11}$$

The second term on the right-hand side of (11) has a finite limit as $n \to \infty$, since the series $\sum_{k=1}^{\infty} \frac{\cos 2kx}{k^\alpha}$ converges (this fact can be proved in the same way as the convergence of the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^\alpha}$). The first term on the right-hand side of (11) approaches infinity as $n \to \infty$, since the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ diverges for $\alpha \in (0, 1]$.

Therefore, the right-hand side of equality (11) has an infinite limit as $n \to \infty$, this is also true for the left-hand side, so the series $\sum_{k=1}^{\infty} \frac{\sin^2 kx}{k^\alpha}$ diverges. Using the comparison test, we obtain from estimate (10) that the series $\sum_{k=1}^{\infty} \left|\frac{\sin kx}{k^\alpha}\right|$ also diverges. So, for $\alpha \in (0, 1]$, the initial series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^\alpha}$ converges conditionally.

Abel's test for conditional convergence of a numerical series

3.12A/21:02 (06:58)

Theorem (Abel's test for conditional convergence of a numerical series).

Let the following conditions be satisfied for a series $\sum_{k=1}^{\infty} a_k b_k$:

1) the series $\sum_{k=1}^{\infty} a_k$ converges;

2) the sequence $\{b_k\}$ is monotone and bounded.

Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges (generally speaking, conditionally).

Dirichlet's

1) $\left|\sum_{k=1}^{n} a_k\right| < M$

2) $b_k \to 0$

Remark.

If we compare Dirichlet's test and Abel' test, then it can be noted that in Abel's test, condition 1 is stronger (since the convergence of the corresponding series is required instead of uniformly boundedness of its partial sums) and condition 2 is weaker (since it is not necessary that the sequence $\{b_k\}$ had a zero limit).

Proof.

By virtue of the theorem on monotone and bounded sequences, the sequence $\{b_k\}$ has a finite limit: $b_k \to c$ as $k \to \infty$. $c \neq 0$

We transform the partial sum of the initial series as follows:

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k(b_k - c + c) = \underbrace{\sum_{k=1}^{n} a_k(b_k - c)}_{S'} + c\underbrace{\sum_{k=1}^{n} a_k}_{S''}. \qquad (12)$$

The second term on the right-hand side of (12) has a finite limit as $n \to \infty$, since, by condition 1, the series $\sum_{k=1}^{\infty} a_k$ converges.

The first term on the right-hand side of (12) is a partial sum of the series $\sum_{k=1}^{\infty} a_k(b_k - c)$, which converges according to Dirichlet's test. Indeed, condition 1 of Dirichlet's test follows from condition 1 of Abel's test, since if the series $\sum_{k=1}^{\infty} a_k$ converges, then its partial sums are uniformly bounded. Condition 2 of Dirichlet's test follows from condition 2 of Abel's test and the fact that $\lim_{k\to\infty} b_k = c$, since in this case the sequence $\{b_k - c\}$ monotonously approaches zero as $k \to \infty$. So, the first term on the right-hand side of (12) also has a finite limit.

Therefore, the partial sums $\sum_{k=1}^{n} a_k b_k$ also have a finite limit, and the initial series converges. □

Additional remarks on absolutely and conditionally convergent series

`3.12A/28:00 (07:07)`

The question arises: will the sum of the convergent series $\sum_{k=1}^{\infty} a_k$ change if the order of its terms is changed? For example, it is possible to organize the summation, for which, after each term a_k of the initial series with an odd index $(a_1, a_3, a_5, \dots)$, *several* terms with even indices will follow, and their amount will increase by 1 each time $(a_1+a_2+a_3+a_4+a_6+a_5+a_8+a_{10}+a_{12}+a_7+\dots)$ or it will double each time $(a_1+a_2+a_3+a_4+a_6+a_5+a_8+a_{10}+a_{12}+a_{14}+a_7+\dots)$.

It turns out that, for an absolutely convergent series, its sum does not change with any change in the order of its terms. However, for a conditionally convergent series, this statement is false.

Moreover, if the series conditionally converges, then, by rearranging its terms, it can be achieved that the resulting series converges to any pre-selected number $A \in \mathbb{R}$ or diverges. This fact is called the *Riemann theorem on conditionally convergent series* (its proof is given, for example, in [18, Ch. 8, Sec. 41.4]).