Логарифмические неравенства

1) $\log_a f(x) > b$, $\quad b \equiv const$

$\log_a a = 1$

$\log_a a^b = b \cdot \log_a a$

$\log_a f(x) > \log_a a^b$

$a > 1: \quad \begin{cases} f(x) > 0 \\ f(x) > a^b \end{cases}$

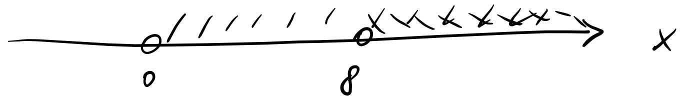
$0 < a < 1: \quad \begin{cases} f(x) > 0 \\ f(x) < a^b \end{cases}$

Пример 1. $\log_2 x > 3$

$\log_2 x > \log_2 2^3$

$\log_2 x > \log_2 8$

т.к. $2 > 1$, то $\begin{cases} x > 0 \\ x > 8 \end{cases} \Rightarrow \quad x > 8$



Ответ: $(8; +\infty)$

2) $\log_a f(x) > \log_a g(x)$

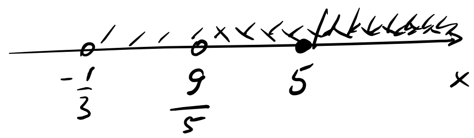
$a > 1: \quad \begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x) > g(x) \end{cases}$

$0 < a < 1: \quad \begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x) < g(x) \end{cases}$

Пример 2. $\log_{\pi/4}(5x-9) \le \log_{\pi/4}(3x+1)$

$\frac{\pi}{4} \approx \frac{3{,}14}{4} < 1$

$$\begin{cases} 5x-9>0 \\ 3x+1>0 \\ 5x-9 \geq 3x+1 \end{cases} \Rightarrow \begin{cases} x>\frac{9}{5} \\ x>-\frac{1}{3} \\ x \geq 5 \end{cases}$$



$5x-9 \geq 3x+1$

$5x-3x \geq 1+9$

$2x \geq 10$

$x \geq 5$

Ответ: $[5; +\infty)$

Примеры для самостоятельного решения:

1) $\log_{1/4} x > 2$ $\quad (0; \frac{1}{16})$

2) $\log_3(3x-1) < 1$ $\quad (\frac{1}{3}; \frac{4}{3})$

3) $\log_2(3x+1) \leq \log_2(x+2)$ $\quad (-\frac{1}{3}; \frac{1}{2}]$

4) $\log_{0,3}(x^2+22) < \log_{0,3}(13x)$ $\quad (0;2) \cup (11; +\infty)$

Пример 3. $-3\log_{0,25} x \leq 6 \quad |:(-3)$

$\log_{0,25} x \geq -2$

$\log_{0,25} x \geq \log_{0,25} 0{,}25^{-2}$

т.к. $0{,}25 < 1$, то $\begin{cases} x>0 \\ x \leq \left(\frac{1}{4}\right)^{-2} \end{cases} \Leftrightarrow \begin{cases} x>0 \\ x \leq 16 \end{cases}$

$0 < x \leq 16$

Ответ: $(0; 16]$

Пример 4. $\log_2 x + 6\log_4 x - 3\log_8 x \le 6$

$\log_{a^p} x = \frac{1}{p}\cdot\log_a x$

$4 = 2^2, \quad 8 = 2^3$

$\log_2 x + 6\cdot\frac{1}{2}\log_2 x - 3\cdot\frac{1}{3}\log_2 x \le 6$

$\cancel{\log_2 x} + 3\log_2 x \cancel{- \log_2 x} \le 6$

$3\log_2 x \le 6 \quad | : 3$

$\log_2 x \le 2$

$\log_2 x \le \log_2 2^2$

$\begin{cases} x > 0 \\ x \le 4 \end{cases} \Rightarrow \quad 0 < x \le 4$

Ответ: $(0; 4]$

Пример 5. $\log_2(\log_5 x) < 1$

Замена: $\log_5 x = t > 0$

$\log_2 t < 1 = \log_2 2^1$

$\begin{cases} t > 0 \\ t < 2 \end{cases} \quad 0 < t < 2$

$0 < \log_5 x < 2$

$\log_a 1 = 0$

$\log_5 1 < \log_5 x < \log_5 5^2$

$$\begin{cases} x > 0 \\ 1 < x < 25 \end{cases} \qquad 1 < x < 25$$

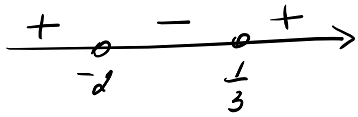
Ответ: $(1; 25)$

Пример 6. $$\log_{1/3} \frac{3x-1}{x+2} < 1$$

$$\log_{1/3} \frac{3x-1}{x+2} < \log_{1/3} \frac{1}{3}$$

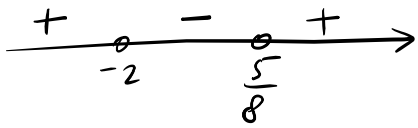
$$\begin{cases} \dfrac{3x-1}{x+2} > 0 \\ \dfrac{3x-1}{x+2} > \dfrac{1}{3} \end{cases} \qquad \begin{cases} x < -2,\ x > \dfrac{1}{3} \\ x < -2,\ x > \dfrac{5}{8} \end{cases}$$

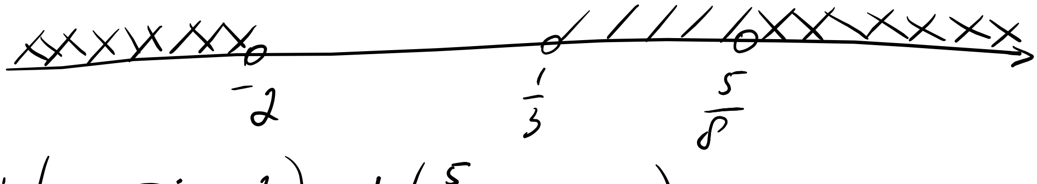
$$\frac{3x-1}{x+2} > 0 \qquad \begin{aligned} 3x-1 &= 0 \\ x+2 &= 0 \end{aligned} \qquad \begin{aligned} x &= \tfrac{1}{3} \\ x &= -2 \end{aligned}$$



$$\frac{3x-1}{x+2} - \frac{1}{3} > 0 \Rightarrow \frac{3(3x-1)-(x+2)}{3(x+2)} > 0 \Rightarrow \frac{8x-5}{3(x+2)} > 0$$

$$\begin{aligned} 8x-5 &= 0 \\ 3(x+2) &= 0 \end{aligned} \qquad \begin{aligned} x &= \tfrac{5}{8} \\ x &= -2 \end{aligned}$$





Ответ: $(-\infty; -2) \cup \left(\frac{5}{8}; +\infty\right)$

Примеры для самостоятельного решения:

5) $3\log_{0,2} x > -6$ $(0; 25)$

6) $\log_{\sqrt{3}} x + \log_{\sqrt[3]{3}} x - \log_{\sqrt[6]{3}} x \geq -2$ $(0; 9]$

7) $\log_{0,5}\left(\log_2(2x-1)\right) \geq -1$ $(1; 2,5]$

8) $\log_{7/9} \dfrac{x+1}{x-1} > 1$ $(-8; -1)$

6) $\sqrt{3} = 3^{\frac{1}{2}} \Rightarrow \log_{\sqrt{3}} x = \log_{3^{\frac{1}{2}}} x = \frac{1}{1/2} \log_3 x = 2\log_3 x$

$\sqrt[3]{3} = 3^{\frac{1}{3}} \Rightarrow \log_{\sqrt[3]{3}} x = 3\log_3 x$

$\sqrt[6]{3} = 3^{\frac{1}{6}} \Rightarrow \log_{\sqrt[6]{3}} x = 6\log_3 x$

$2\log_3 x + 3\log_3 x - 6\log_3 x \geq -2$

$-\log_3 x \geq -2 \quad | : (-1)$

$\log_3 x \leq 2 \Rightarrow \log_3 x \leq \log_3 3^2$

$\begin{cases} x > 0 \\ x \leq 9 \end{cases}$ $0 < x \leq 9$

7) $t = \log_2(2x-1)$

$\log_{0,5} t \geq -1 \Rightarrow \log_{0,5} t \geq \log_{0,5} 0,5^{-1}$

$0,5^{-1} = \left(\frac{1}{2}\right)^{-1}$

$\begin{cases} t > 0 \\ t \leq 2 \end{cases} \Rightarrow 0 < t \leq 2$

$0 < \log_2(2x-1) \leq 2$

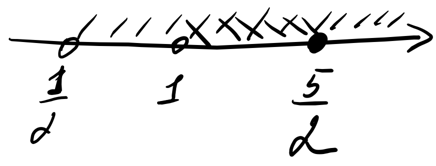
$\log_2 1 < \log_2(2x-1) \leq \log_2 2^2$

$$\begin{cases} 2x-1>0 \\ 1<2x-1\le 4 \end{cases} \qquad \begin{cases} x>\frac{1}{2} \\ 1<x\le\frac{5}{2} \end{cases}$$

$$\underbrace{1}<\underbrace{2x-1}\le\underbrace{4} \quad \Big| +1$$

$$2<2x\le 5 \quad \Big| :2$$

$$1<x\le\frac{5}{2}$$



$$1<x\le\frac{5}{2}$$

$$\log_{1/3} x+\log_{1/3}(4-x)>-1$$

Пример 7.

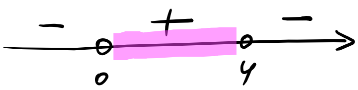
$$\log_a x+\log_a y=\log_a(x\cdot y)$$

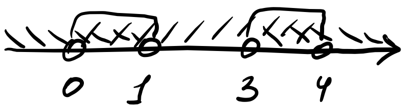
$$\log_a x-\log_a y=\log_a\left(\frac{x}{y}\right)$$

$$\log_{\frac{1}{3}}\big(x(4-x)\big)>\log_{\frac{1}{3}}\left(\frac{1}{3}\right)^{-1}$$

$$\begin{cases} x(4-x)>0 \\ x(4-x)<3 \end{cases} \qquad \begin{cases} 0<x<4 \\ x<1,\ x>3 \end{cases} \qquad \left[\begin{array}{l} 0<x<1 \\ 3<x<4 \end{array}\right.$$

$x(4-x)>0$

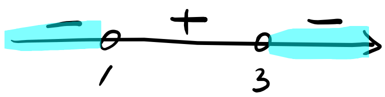




$x=0, \quad x=4$

$x(4-x)-3<0$

$-x^2+4x-3<0$



$x^2-4x+3=0$

$+\,\Big|\ 4\ \Big|\ x_1=1$

$\cdot\,\Big|\ 3\ \Big|\ x_2=3$

Ответ: $(0;1)\cup(3;4)$

$$\lg^2 x-\lg x-2>0$$

Пример 8.

Замена: $t=\lg x$

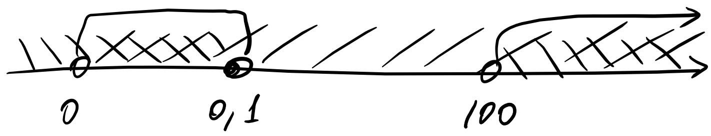
$t^2 - t - 2 > 0$

$t^2 - t - 2 = 0$

$\begin{array}{c|c|} + & 1 \\ \cdot & -2 \end{array}$ $\quad t_1 = -1$, $t_2 = 2$

$\left[\begin{array}{l} t < -1 \\ t > 2 \end{array} \right. \Rightarrow \left[\begin{array}{l} \lg x < -1 \\ \lg x > 2 \end{array} \right. \Rightarrow \left[\begin{array}{l} \lg x < \lg 10^{-1} \\ \lg x > \lg 10^2 \end{array} \right.$

$\begin{cases} x > 0 \\ \left[\begin{array}{l} x < 0{,}1 \\ x > 100 \end{array} \right. \end{cases}$



Ответ: $(0; 0{,}1) \cup (100; +\infty)$

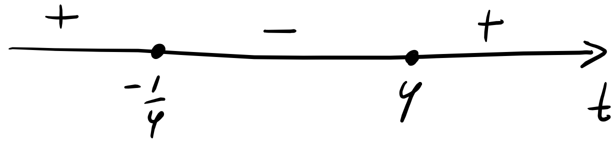
$$\log_2^2 x^2 - 15 \cdot \log_2 x - 4 \le 0$$

Пример 9.

$\log_2^2 x^2 = (\log_2 x^2)^2 = (2 \cdot \log_2 x)^2 = 4 \cdot \log_2^2 x$

Замена: $t = \log_2 x$

$4t^2 - 15t - 4 \le 0$



$4t^2 - 15t - 4 = 0$

$D = (-15)^2 - 4 \cdot 4 \cdot (-4) = 225 + 64 = 289 = 17^2$

$t_{1,2} = \dfrac{15 \pm 17}{2 \cdot 4} \Rightarrow t_1 = \dfrac{32}{2 \cdot 4} = 4$, $\quad t_2 = \dfrac{-2}{2 \cdot 4} = -\dfrac{1}{4}$

$-\dfrac{1}{4} \le t \le 4$

$$-\frac{1}{4} \le \log_2 x \le 4$$

$$\log_2 2^{-\frac{1}{4}} \le \log_2 x \le \log_2 2^4$$

$$\begin{cases} x > 0 \\ \frac{1}{\sqrt[4]{2}} \le x \le 16 \end{cases} \qquad \frac{1}{\sqrt[4]{2}} \le x \le 16$$

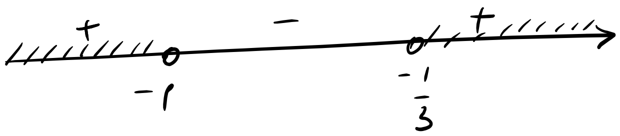
Ответ: $\left[\sqrt[4]{\frac{1}{2}}\ ;\ 16\right]$

Пример 10.

$$2\log_9\left(4x^2+1\right) \ge \log_3\left(3x^2+4x+1\right)$$

$$2 \cdot \frac{1}{2} \cdot \log_3(\underbrace{4x^2+1}_{>0}) \ge \log_3(3x^2+4x+1)$$

$$\begin{cases} 3x^2+4x+1 > 0 \\ 4x^2+1 \ge 3x^2+4x+1 \end{cases} \qquad \begin{cases} x < -1,\ x > -\frac{1}{3} \\ x \le 0,\ x \ge 4 \end{cases}$$



$3x^2+4x+1 > 0$

$3x^2+4x+1 = 0$

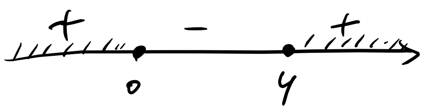
$D = 4^2 - 4\cdot 3\cdot 1 = 16-12 = 4 \Rightarrow x_{1,2} = \frac{-4\pm 2}{2\cdot 3} = \frac{-2\pm 1}{3} \Rightarrow \left[\begin{array}{l} x_1 = -1 \\ x_2 = -\frac{1}{3} \end{array}\right.$

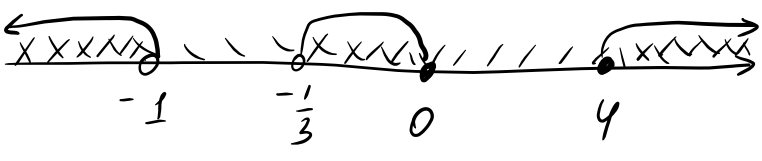
$4x^2+1 \ge 3x^2+4x+1$

$x^2-4x \ge 0$

$x(x-4) \ge 0$

$\left[\begin{array}{l} x_1 = 0 \\ x_2 = 4 \end{array}\right.$





Ответ: $(-\infty; -1) \cup \left(-\frac{1}{3}; 0\right] \cup [4; +\infty)$

Примеры для самостоятельного решения:

9) $\lg(7-x)+\lg x>1$ $\quad (2;5)$

10) $\log_2^2 x+2\log_2 x-3\le 0$ $\quad \left[\frac{1}{8};2\right]$

11) $\log_{1/5}^2 x^2-31\log_{1/5} x-8<0$ $\quad \left(\frac{1}{5^8};\sqrt[4]{5}\right)$

12) $2\log_4\left(3x^2+2\right)\ge\log_2\left(2x^2+5x+2\right)$

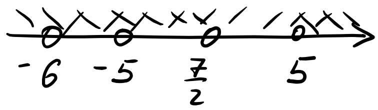
$(-\infty;-2)\cup\left(-\frac{1}{2};0\right]\cup[5;+\infty)$

Пример 11 (ЕГЭ-2024).

Решите неравенство $\dfrac{\log_2(2x^2-17x+35)-1}{\log_7(x+6)}\leqslant 0.$

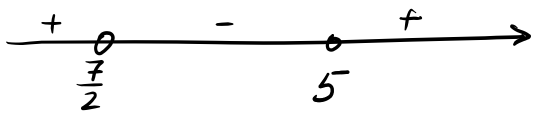
Решение. ОДЗ: $\quad x\in(-6;-5)\cup\left(-5;\frac{7}{2}\right)\cup(5;\infty)$

$$\begin{cases}2x^2-17x+35>0\\ x+6>0\\ x+6\ne 1\end{cases}\qquad \begin{cases}x<\frac{7}{2},\ x>5\\ x>-6\\ x\ne -5\end{cases}$$



$2x^2-17x+35>0$

$2x^2-17x+35=0$



$D=(-17)^2-4\cdot 2\cdot 35=289-280=9$

$x_{1,2}=\frac{17\pm 3}{2\cdot 2}\Rightarrow\left[\begin{array}{l}x_1=\frac{20}{4}=5\\ x_2=\frac{14}{2\cdot 2}=\frac{7}{2}\end{array}\right.$

Найдём нули числителя:

$\log_2(2x^2-17x+35)-1=0$

$\log_2(2x^2-17x+35)=1$

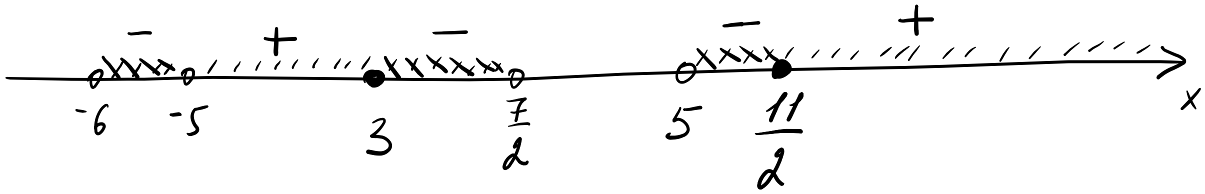
$\log_2(2x^2-17x+35)=\log_2 2$

$2x^2-17x+35=2$

$2x^2-17x+33=0$

$D=(-17)^2-4\cdot 2\cdot 33=289-264=25$

$x_{1,2}=\dfrac{17\pm 5}{2\cdot 2} \Rightarrow \left[\begin{array}{l} x_1=\dfrac{17-5}{2\cdot 2}=\dfrac{12}{4}=3 \\ x_2=\dfrac{17+5}{2\cdot 2}=\dfrac{22}{2\cdot 2}=\dfrac{11}{2} \end{array}\right.$



Ответ: $(-6;-5)\cup\left[3;\frac{7}{2}\right)\cup\left(5;\frac{11}{2}\right]$

Пример 12 (ЕГЭ-2022).

Решите неравенство

$$1+\frac{6}{\log_3 x-3}+\frac{5}{\log_3^2 x-\log_3(27x^6)+12}\geqslant 0.$$

Решение. Замена $t=\log_3 x \Rightarrow$

$\log_3(27x^6)=\log_3 27+\log_3 x^6=\log_3 3^3+\log_3 x^6=$

$=3\log_3 3+6\log_3 x=3+6\log_3 x=3+6t;$

$\log_3^2 x=t^2$

$1+\dfrac{6}{t-3}+\dfrac{5}{t^2-3-6t+12}\geqslant 0$

$$1+\frac{6}{t-3}+\frac{5}{t^2-6t+9}\geqslant 0$$

$$1+\frac{6}{t-3}+\frac{5}{(t-3)^2}\geqslant 0$$

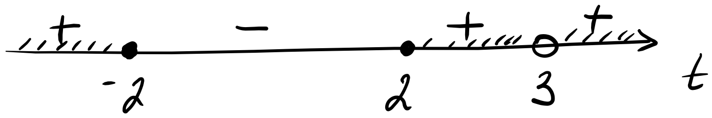
$$\frac{(t-3)^2+(t-3)\cdot 6+5}{(t-3)^2}\geqslant 0$$

$$\frac{t^2-6t+9+6t-18+5}{(t-3)^2}\geqslant 0$$

$$\frac{t^2-4}{(t-3)^2}\geqslant 0$$

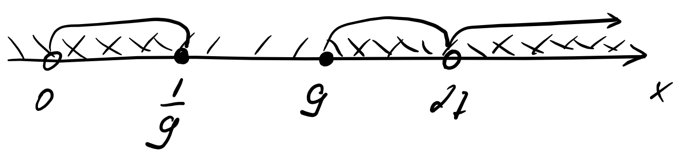
$$\frac{(t-2)(t+2)}{(t-3)^2}\geqslant 0$$

М. интервалов:



$$\left[\begin{array}{l} t\leq -2 \\ 2\leq t<3 \\ t>3 \end{array}\right. \Rightarrow \left[\begin{array}{l} \log_3 x\leq -2 \\ 2\leq \log_3 x<3 \\ \log_3 x>3 \end{array}\right.$$

$$\begin{cases} x>0 \\ \left[\begin{array}{l} x\leq 3^{-2} \\ 3^2\leq x<3^3 \\ x>3^3 \end{array}\right. \end{cases}$$



Ответ: $\left(0;\frac{1}{9}\right]\cup[9;27)\cup(27;+\infty)$

Домашнее задание:

1) ЕГЭ-2023

Решите неравенство $\log_{0,1}(x^3-5x^2-25x+125)\leqslant\log_{0,01}(x-5)^4$.

2) ЕГЭ-2018

Решите неравенство $\log_{x^2+1}\dfrac{2\cdot 4^x-15\cdot 2^x+23}{4^x-9\cdot 2^x+14}\geqslant 0$.