

Numerical Methods of Linear Algebra for Sparse Matrices

Lecture 1

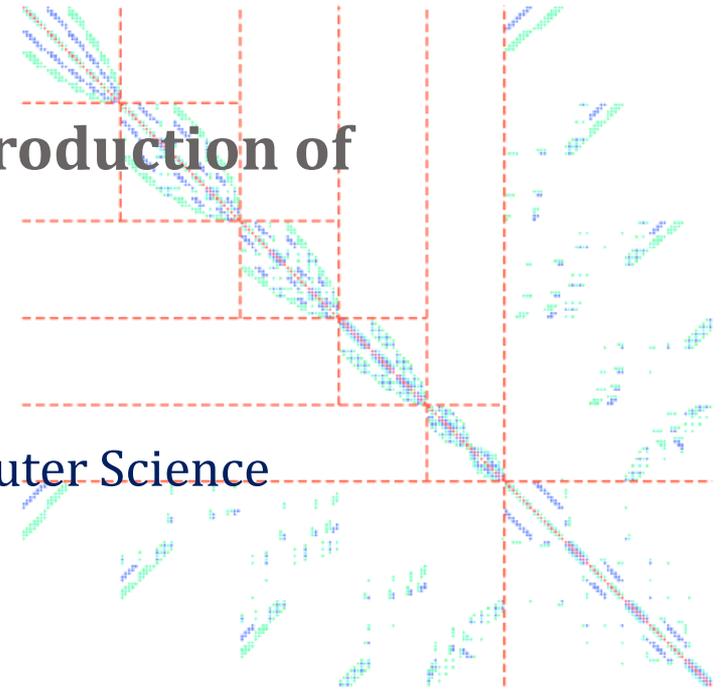
Fundamentals of Linear Algebra. introduction of elementary notation.

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Outline

- Matrices and matrix algorithms in modern world
- Linear systems and sparse linear systems
- Fundamentals of Linear Algebra (review of basic matrix theory, introduction of elementary notation)

Matrices and matrix algorithms in modern world

- Matrix algorithms pervade most areas of science and engineering
- Matrix algorithms are used by most disciplines requiring numerical computing
- Computations involving matrices is the topic becoming increasingly important in Computer Science
- Matrix algorithms in computer science are used for data mining, information retrieval, search engines, pattern recognition, graphics, ...

Example of a linear system

- **Problem** (from Yosef Saad lecture). A set of 12 coins containing nickels (5c each), dimes (10c each) and quarters (25c each) totals to \$1.45. In addition, the total without the nickels amounts to \$1.25. How many of each coin are there?
- **Solution.** Let x_n be the number of nickels, x_d be the number of dimes, x_q be the number of quarters

- **Equation form**

1st equation: $x_n + x_d + x_q = 12$

2nd equation: $5x_n + 10x_d + 25x_q = 145$

3rd equation: $0x_n + 10x_d + 25x_q = 125$

What's the answer?



- **Matrix form**

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 10 & 25 \\ 0 & 10 & 25 \end{pmatrix} \begin{pmatrix} x_n \\ x_d \\ x_q \end{pmatrix} = \begin{pmatrix} 12 \\ 145 \\ 125 \end{pmatrix}$$

$$\text{Matrix } A = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 10 & 25 \\ 0 & 10 & 25 \end{pmatrix}$$

$$\text{Right-hand side vector } b = \begin{pmatrix} 12 \\ 145 \\ 125 \end{pmatrix}$$

$$\text{Vector of unknowns } x = \begin{pmatrix} x_n \\ x_d \\ x_q \end{pmatrix}$$

$$\text{Matrix equation: } Ax = b$$

Solve in Matlab

- Matrix form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 10 & 25 \\ 0 & 10 & 25 \end{pmatrix} \begin{pmatrix} x_n \\ x_d \\ x_q \end{pmatrix} = \begin{pmatrix} 12 \\ 145 \\ 125 \end{pmatrix}$$

- **Answer:** 4 nickels, 5 dimes and 3 quarters
- Question: is the order of equations important or not?
- Will the solution change?

```
A=[1 1 1; 5 10 25; 0 10 25]
```

```
A = 3x3
```

```
    1    1    1  
    5   10   25  
    0   10   25
```

```
b=[12; 145; 125]
```

```
b = 3x1
```

```
    12  
   145  
   125
```

```
x=A\b
```

```
x = 3x1
```

```
    4  
    5  
    3
```

Experiment in Matlab

```
A=[1 1 1; 5 10 25; 0 10 25]
b=[12; 145; 125]
x=A\b
```

```
p=[2 1 3]
A=A(p,:)
b=b(p,:)
x=A\b
```

- **Question:** is the order of equations important or not? Will the solution change?
- **Answer:** No!

```
x = 3x1
     4
     5
     3
```

```
p = 1x3
     2     1     3
```

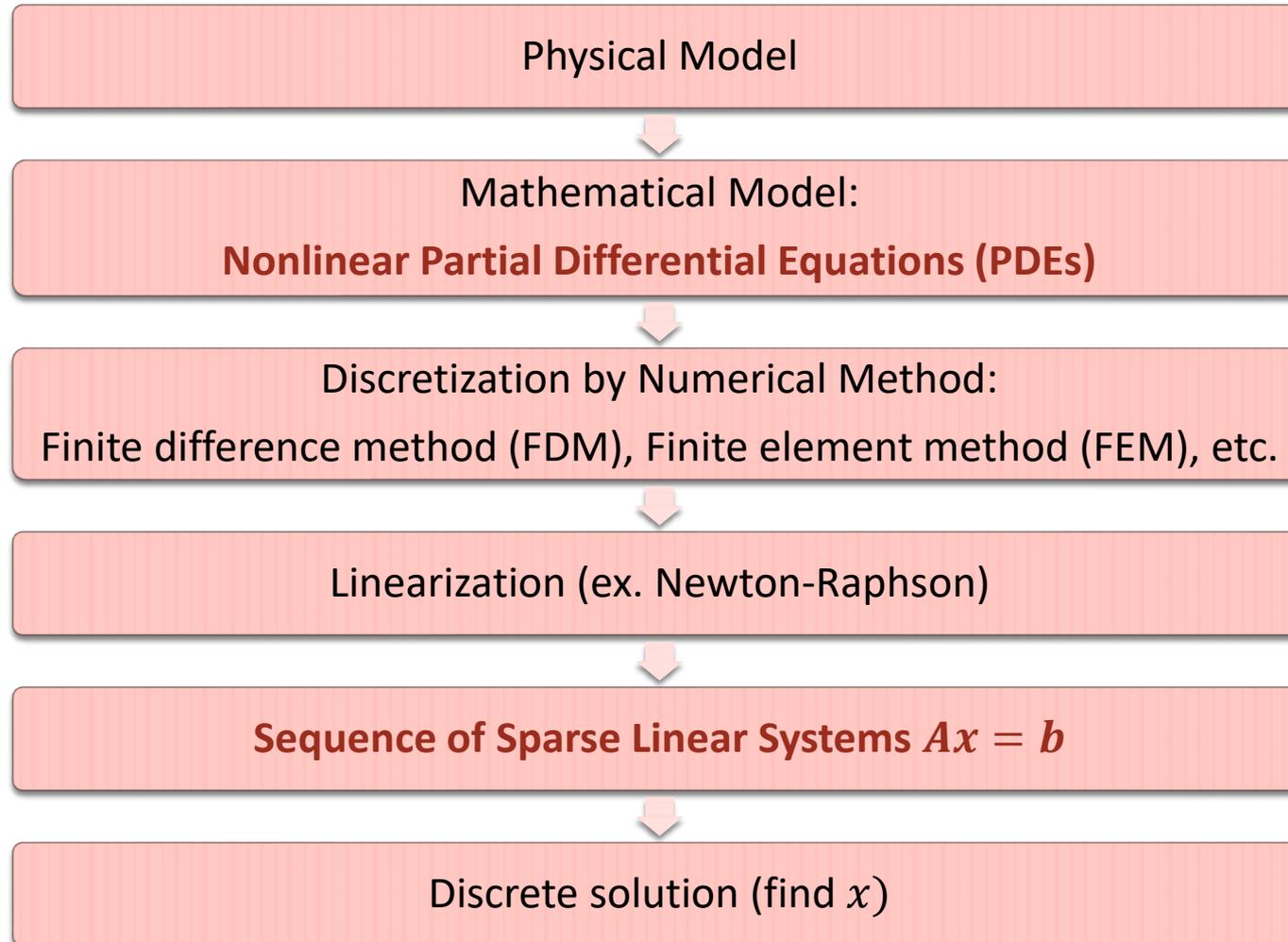
```
A = 3x3
     5    10    25
     1     1     1
     0    10    25
```

```
b = 3x1
    145
     12
    125
```

```
x = 3x1
     4
     5
     3
```

Source of large linear systems

- Typical Large-scale problem (ex. Fluid flow)



What is a PDE?

- A **Partial Differential Equation (PDE)** is a relationship between an unknown function of several variables and its partial derivatives.
- $u = u(x_1, x_2, x_3, t)$ is unknown function (field variable), x_1, x_2, x_3 (space coordinates) and t (time) are independent variables

Order of a PDE

the highest derivative in the equation

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \quad \text{is a first-order PDE.}$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0 \quad \text{is a second-order PDE.}$$

$$\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^2 u}{\partial x_2^2} - u = 0 \quad \text{is a fourth-order PDE.}$$

$$\left(\frac{\partial u}{\partial x_1}\right)^3 + \frac{\partial u}{\partial x_2} + u^4 = 0 \quad \text{is a first-order PDE.}$$

Linear and nonlinear PDEs

A *linear* PDE is of the first degree both in its field variables and partial derivatives

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0 \quad \text{is linear .}$$

$$\frac{\partial u}{\partial x_1} + \left(\frac{\partial u}{\partial x_2}\right)^2 = 0 \quad \text{is nonlinear .}$$

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u^2 = 0 \quad \text{is nonlinear .}$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = x_1 \quad \text{is linear .}$$

$$\frac{\partial^2 u}{\partial x_1^2} + u \frac{\partial^2 u}{\partial x_2^2} = 0 \quad \text{is quasilinear .}$$

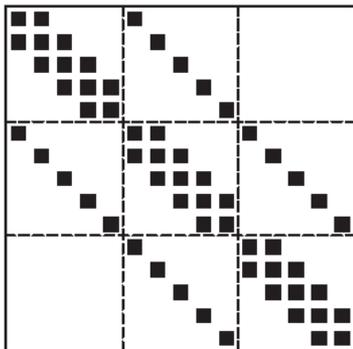
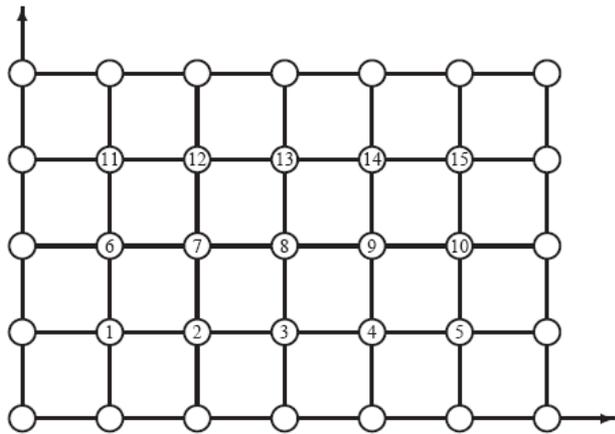
What is a sparse matrix?

- A **sparse matrix** is a matrix which has very few nonzero elements.
- **Sparsity** is the fraction of zero elements in a matrix
- **Density** is the fraction of non-zero elements in a matrix
- **Example of sparse matrix:** 64 elements, 52 zero elements and 12 nonzero elements (82% sparsity, 18% density)

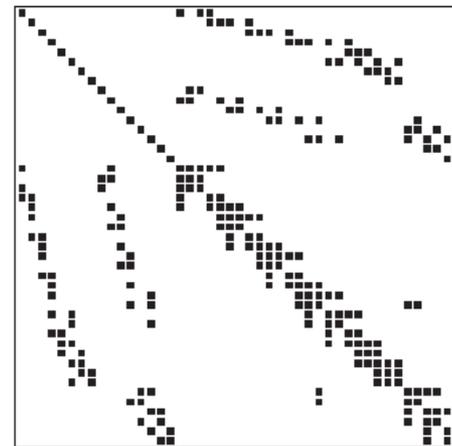
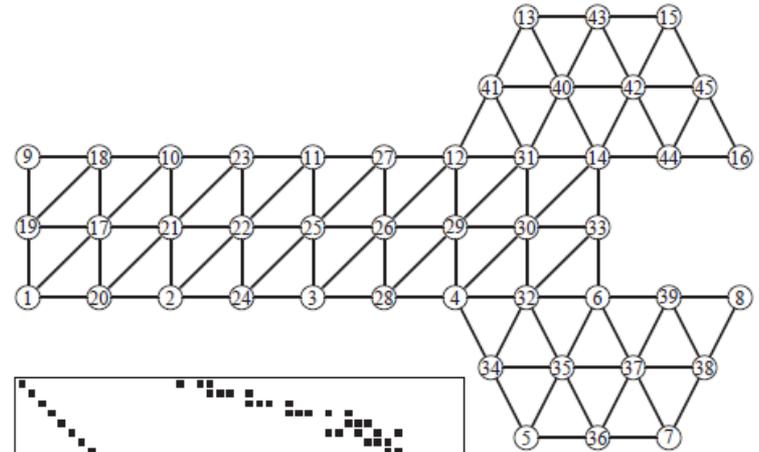
$$\begin{pmatrix} 1.0 & 0 & 5.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.0 & 0 & 0 & 0 & 0 & 11.0 & 0 \\ 0 & 0 & 0 & 0 & 9.0 & 0 & 0 & 0 \\ 0 & 0 & 6.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.0 & 0 & 0 & 0 & 0 \\ 2.0 & 0 & 0 & 0 & 0 & 10.0 & 0 & 0 \\ 0 & 0 & 0 & 8.0 & 0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 & 0 & 0 & 0 & 12.0 \end{pmatrix}$$

Discretization of PDEs

- Example: grid in finite difference method and corresponding sparse matrix
- Result: *structured* sparse matrix (nonzero entries form a regular pattern). Here we see blocks of diagonals



- Example: final element mesh and corresponding sparse matrix
- Result: *unstructured* sparse matrix (nonzero entries are located irregularly). Although, we can spot a symmetric pattern here!



Definition of a matrix

A **complex matrix** $A = a_{ij}$, $A \in \mathbb{C}^{n \times m}$ is an array $n \times m$ of **complex** numbers,

A **real matrix** $A = a_{ij}$, $A \in \mathbb{R}^{n \times m}$ is an array $n \times m$ of **real** numbers,

where n is the number of rows, m is the number of columns

i is the row index, $i = 1, 2, \dots, n$

j is the column index, $j = 1, 2, \dots, m$

a_{ij} are the entries of the matrix

\in stands for "belongs to"

Row-vector of a matrix

$a_{i*} = (a_{i1} \ a_{i2} \ \dots \ a_{in})$ is the i -th row

Column-vector of a matrix

$a_{*j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$ is the j -th column

The diagram shows a matrix A with two rows and three columns. The matrix is represented as $A = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{pmatrix}$. A bracket on the right side of the matrix is labeled "rows" and encompasses the two rows. A bracket below the matrix is labeled "columns" and encompasses the three columns. The entries are labeled with their row and column indices: the first row has entries 10, 20, 30 and is labeled "1"; the second row has entries 40, 50, 60 and is labeled "2". The columns are labeled "1", "2", and "3" below the matrix.

Matrix A has

$n = 2$ rows, $m = 3$ columns

Matrix coefficients (entries) are

$a_{11}=10$, $a_{12}=20$, $a_{13}=30$,

$a_{21}=40$, $a_{22}=50$, $a_{23}=60$

Matrices in Matlab

```
% , or space separates columns
```

```
% ; separates rows
```

```
% Real matrices
```

```
A=[10,20,30;40,50,60]
```

```
B=[1 2;3 4]
```

```
a12=A(1,2) % matrix element
```

```
A_row2=A(2,:) % 2nd row
```

```
A_col3=A(:,3) % 3rd column
```

```
B_row_first=B(1,:) % 1st row
```

```
B_col_last=B(:,end) % last column
```

```
%complex matrix
```

```
A=[1+2i 3-1i; 4+5i -1i]
```

```
A = 2x3
      10    20    30
      40    50    60
```

```
B = 2x2
      1    2
      3    4
```

```
a12 = 20
A_row2 = 1x3
      40    50    60
```

```
A_col3 = 2x1
      30
      60
```

```
B_row_first = 1x2
      1    2
```

```
B_col_last = 2x1
      2
      4
```

```
A = 2x2 complex
      1.0000 + 2.0000i    3.0000 - 1.0000i
      4.0000 + 5.0000i    0.0000 - 1.0000i
```

Main operations with matrices

Addition

$$C = A + B, \quad A, B, C \in \mathbb{C}^{n \times m}$$

$$c_{ij} = a_{ij} + b_{ij}, \quad i = \overline{1, n}, \quad j = \overline{1, m}$$

Multiplication by a scalar

$$C = \alpha A, \quad A, C \in \mathbb{C}^{n \times m}, \quad \alpha \in \mathbb{C}$$

$$c_{ij} = \alpha a_{ij}, \quad i = \overline{1, n}, \quad j = \overline{1, m}$$

Multiplication by a matrix

$$C = AB, \quad A \in \mathbb{C}^{n \times m}, \quad B \in \mathbb{C}^{m \times p}, \quad C \in \mathbb{C}^{n \times p}$$

$$c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}, \quad i = \overline{1, n}, \quad j = \overline{1, p}$$

Matrix multiplication

Multiplication by a matrix

$$C = AB, \quad A \in \mathbb{C}^{n \times m}, \quad B \in \mathbb{C}^{m \times p}, \quad C \in \mathbb{C}^{n \times p}$$

$$c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}, \quad i = \overline{1, n}, \quad j = \overline{1, p}$$

True or false?

- $AB = BA$
- $A(BC) = (AB)C$

Is it possible to multiply A by B in these cases?

$$1) \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 6 & 7.5 \\ 0 & 3.1 & 2 \end{pmatrix}$$

$$2) \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix}$$

$$\begin{array}{c}
 \begin{matrix} \text{1st Row} \\ \text{Times} \\ \text{1st Column} \end{matrix} \left[\begin{array}{cc|cc} \hline \textcircled{1} & \textcircled{0} & \textcircled{-1} & 4 \\ -3 & 2 & 3 & 5 \\ \hline \end{array} \right] \cdot \begin{matrix} \text{1st Row} \\ \text{Times} \\ \text{2nd Column} \end{matrix} \left[\begin{array}{cc|cc} \hline \textcircled{1} & \textcircled{0} & \textcircled{-1} & \textcircled{4} \\ -3 & 2 & 3 & 5 \\ \hline \end{array} \right] \\
 \begin{matrix} \text{1st Row} \\ \text{Times} \\ \text{1st Column} \end{matrix} \left[\begin{array}{cc|cc} \hline (1)(-1) + (0)(3) & (1)(4) + (0)(5) \\ -1 + 0 & 4 + 0 \\ -1 & 4 \\ \hline \end{array} \right] \\
 \begin{matrix} \text{2nd Row} \\ \text{Times} \\ \text{1st Column} \end{matrix} \left[\begin{array}{cc|cc} \hline (-3)(-1) + (2)(3) & (-3)(4) + (2)(5) \\ 3 + 6 & -12 + 10 \\ 9 & -2 \\ \hline \end{array} \right] \\
 \begin{matrix} \text{2nd Row} \\ \text{Times} \\ \text{2nd Column} \end{matrix} \left[\begin{array}{cc|cc} \hline (-3)(-1) + (2)(3) & (-3)(4) + (2)(5) \\ 3 + 6 & -12 + 10 \\ 9 & -2 \\ \hline \end{array} \right] \\
 \begin{matrix} \text{2nd Row} \\ \text{Times} \\ \text{2nd Column} \end{matrix} \left[\begin{array}{cc|cc} \hline \textcircled{1} & \textcircled{0} & \textcircled{-1} & 4 \\ -3 & 2 & 3 & 5 \\ \hline \end{array} \right] \cdot \begin{matrix} \text{1st Row} \\ \text{Times} \\ \text{2nd Column} \end{matrix} \left[\begin{array}{cc|cc} \hline \textcircled{1} & \textcircled{0} & \textcircled{-1} & \textcircled{4} \\ -3 & 2 & 3 & 5 \\ \hline \end{array} \right]
 \end{array}$$

Final Answer: $\begin{bmatrix} -1 & 4 \\ 9 & -2 \end{bmatrix}$

<http://www.gradeamathhelp.com/matrix-multiplication.html>

Matrix operations in Matlab

```
A=[10,20,30; 40,50,60]
B=[1 2 3; 4 5 6]
```

```
% addition
```

```
C_ad=A+B
```

```
% multiplication by a scalar
```

```
C_ms=3.6*A
```

```
% random matrices
```

```
A1=rand(2,3)
```

```
B1=rand(3,2)
```

```
% matrix multiplication
```

```
C1=A1*B1
```

```
C=A*B
```



```
A = 2x3
```

```
    10    20    30
    40    50    60
```

```
B = 2x3
```

```
    1    2    3
    4    5    6
```

```
C_ad = 2x3
```

```
    11    22    33
    44    55    66
```

```
C_ms = 2x3
```

```
    36    72   108
   144   180   216
```

```
A1 = 2x3
```

```
    0.7513    0.5060    0.8909
    0.2551    0.6991    0.9593
```

```
B1 = 3x2
```

```
    0.5472    0.2575
    0.1386    0.8407
    0.1493    0.2543
```

```
C1 = 2x2
```

```
    0.6142    0.8454
    0.3797    0.8973
```

```
Error using *
Incorrect dimensions for matrix multiplication.
```

Matrix-matrix and matrix-vector product

- **Column-wise product** (Matlab notation)

$$C_{:,j} = \sum_{k=1}^n b_{kj} A_{:,k}$$

- **Row-wise product**

$$C_{i,:} = \sum_{k=1}^n a_{ik} B_{k,:}$$

- **Sum of 'outer-product' matrices**

$$C = \sum_{k=1}^n A_{:,k} B_{k,:}$$

 What happens to these 3 different approaches when B has one column ($p = 1$)?

 Characterize the matrices AA^T and $A^T A$ when A is of dimension $n \times 1$.

Transpose of a matrix

Transposition: If $A \in \mathbb{R}^{m \times n}$ then its transpose is a matrix $C \in \mathbb{R}^{n \times m}$ with entries

$$c_{ij} = a_{ji}, i = 1, \dots, n, j = 1, \dots, m$$

Notation : A^T .

Transpose Conjugate: for complex matrices, the transpose conjugate matrix denoted by A^H is more relevant: $A^H = \bar{A}^T = \overline{A^T}$.

Properties

$$1) (AB)^T = B^T A^T; (AB)^H = B^H A^H$$

$$2) (\alpha A)^T = \alpha A^T; (\alpha A)^H = \bar{\alpha} A^H$$

$$3) (A^T)^T = A; (A^H)^H = A$$

$$4) (A+B)^T = A^T + B^T$$

$$(ABC)^T = ?$$

True or false?

- $AA^T = A^T A$

Given a complex number	Its conjugate
$a + bi$	$a - bi$
$a - bi$	$a + bi$

$$z = 2 + 3i \Rightarrow \bar{z} = 2 - 3i$$

Transpose in Matlab

```
% Random integer matrix
A=randi(10,2,3)

% Transposition for real matrix
A'

% Random complex matrix
A=rand(2)+1i*rand(2)

%Transpose
At=transpose(A)

%Complex conjugate transpose
Act=ctranspose(A)
Act=A'
```

```
A = 2x3
     4     8    10
     7     1     8
```

```
ans = 3x2
     4     7
     8     1
    10     8
```

```
A = 2x2 complex
     0.4868 + 0.5085i    0.4468 + 0.8176i
     0.4359 + 0.5108i    0.3063 + 0.7948i
```

```
At = 2x2 complex
     0.4868 + 0.5085i    0.4359 + 0.5108i
     0.4468 + 0.8176i    0.3063 + 0.7948i
```

```
Act = 2x2 complex
     0.4868 - 0.5085i    0.4359 - 0.5108i
     0.4468 - 0.8176i    0.3063 - 0.7948i
```

```
Act = 2x2 complex
     0.4868 - 0.5085i    0.4359 - 0.5108i
     0.4468 - 0.8176i    0.3063 - 0.7948i
```

Square matrix, identity matrix and inverse

- Square matrix: $n = m$ - so $A \in \mathbb{R}^{n \times n}$
- Identity matrix: square matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Notation: I .
- Property: $AI = IA = A$
- Inverse of A (when it exists) is a matrix C such that

$$AC = CA = I$$

Properties of inverse

- 1) $(AB)^{-1} = B^{-1}A^{-1}$
- 2) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$
- 3) $(A^T)^{-1} = (A^{-1})^T$

Notation: A^{-1}

Eigenvalues and eigenvectors

A complex scalar λ is an **eigenvalue** of the square matrix A if a nonzero vector u of \mathbb{C}^n exists such that

$$Au = \lambda u.$$

The vector u is an **eigenvector** of A associated with λ . The set of all the eigenvalues of A is the **spectrum** of A . Notation: $\lambda(A)$.

- λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$
- $p_A(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n in $\lambda =$ characteristic polynomial of A .
- $\lambda \in \lambda(A)$ if and only if λ is a root of the characteristic polynomial $p_A(\lambda)$.

Spectral radius and trace

- Spectral radius = The maximum modulus of the eigenvalues

$$\rho(A) = \max_{\lambda \in \lambda(A)} |\lambda|.$$

- Trace of A = sum of diagonal elements of A .

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}.$$

- $\text{tr}(A)$ = sum of all the eigenvalues of A counted with their multiplicities.
- Recall that $\det(A)$ = product of all the eigenvalues of A counted with their multiplicities.

- ☒ Trace, spectral radius, and determinant of

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}.$$