

Numerical Methods of Linear Algebra for Sparse Matrices

Lecture 11

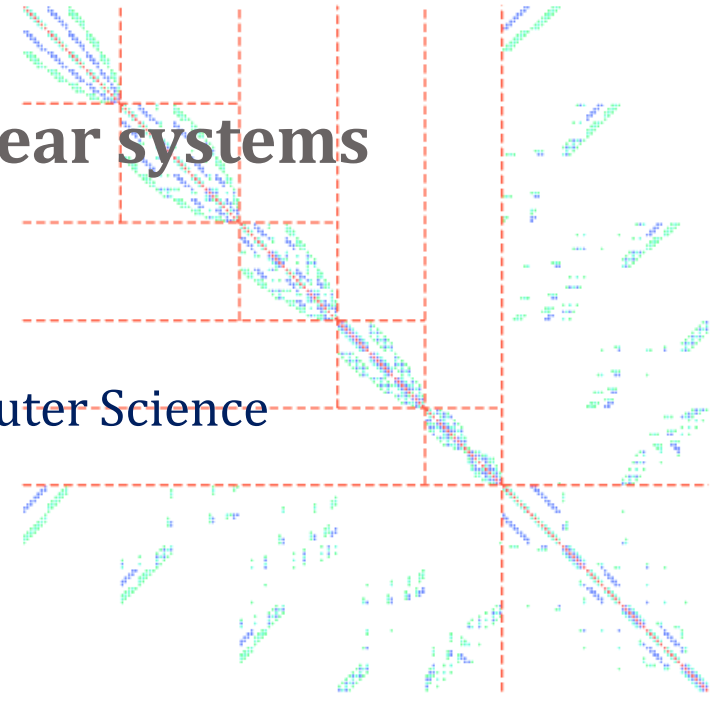
Classical iterative methods for linear systems

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Classical iterative methods for linear systems

Simple iteration method

Jacobi method

Gauss-Seidel method and its variations

SOR, SSOR, USOR methods

Matrix splitting methods and preconditioners

Formulation of classical iterative method

Idea of an iterative method to solve the system $Ax = b$:

1. Take *initial guess* $x^{(0)}$
2. Apply iterative process until convergence

$$x^{(k+1)} = G_k x^{(k)} + g_k, \text{ where}$$

$x^{(k)}$ is *approximate solution* at k -th iteration

G_k is iteration (transition) matrix, g_k is iteration vector

For stationary process $G_k = G$, $g_k = g$: $x^{(k+1)} = Gx^{(k)} + g$

At every iteration $x^{(k)}$ is improved by the modification of one or several components of it, until convergence is reached.

This is called a *relaxation step* : $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$

Also the goal is to make the norm of the residual vector

$\|r_k\| = \|b - Ax^{(k)}\|$ smaller at every iteration.

The simple iteration (Richardson) method

Idea of the method: assume $x^{(k+1)} - x^{(k)} = r_k = b - Ax^{(k)}$

$$x^{(k+1)} = x^{(k)} + r_k = x^{(k)} + b - Ax^{(k)} = (I - A)x^{(k)} + b$$

$$x^{(k+1)} = (I - A)x^{(k)} + b$$

Hence $G = I - A$, $g = b$

Converges, when $M(A) < 2$,

where $M(A)$ is maximal eigenvalue of A

Jacobi method

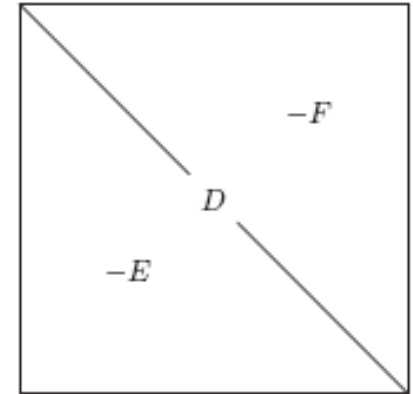
Consider splitting of matrix A :

$$A = D - E - F$$

$-E$ is a strict lower part of A , $E = \{-a_{ij}\}, j < i$

$-F$ is a strict upper part of A , $F = \{-a_{ij}\}, j > i$

D is a diagonal part of A , $D = \{a_{ii}\}, i = 1, \dots, n$



$$Ax = b, A \in \mathbb{C}^{n \times n}, b \in \mathbb{C}^n$$

$$(D - E - F)x = b$$

$$Dx = (E + F)x + b$$

Iterative process:

$$Dx^{(k+1)} = (E + F)x^{(k)} + b$$

$$x^{(k+1)} = D^{-1}(E + F)x^{(k)} + D^{-1}b$$

$$G_J = D^{-1}(E + F) = I - D^{-1}A, \quad g_J = D^{-1}b$$

Component-wise form:

$$x_i^{(k+1)} = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j^{(k)} + \frac{b_i}{a_{ii}}$$

When we compute $x_i^{(k+1)}$, we already know previous entries

$$x_{i-1}^{(k+1)}, x_{i-2}^{(k+1)}, \dots, x_1^{(k+1)}$$

Gauss-Seidel method

Consider splitting of matrix A :

$$A = D - E - F$$

$-E$ is a strict lower part of A , $E = \{-a_{ij}\}, j < i$

$-F$ is a strict upper part of A , $F = \{-a_{ij}\}, j > i$

D is a diagonal part of A , $D = \{a_{ii}\}, i = 1, \dots, n$

$$Ax = b, A \in \mathbb{C}^{n \times n}, b \in \mathbb{C}^n \Rightarrow (D - E - F)x = b$$

$$(D - E)x = Fx + b$$

Iterative process:

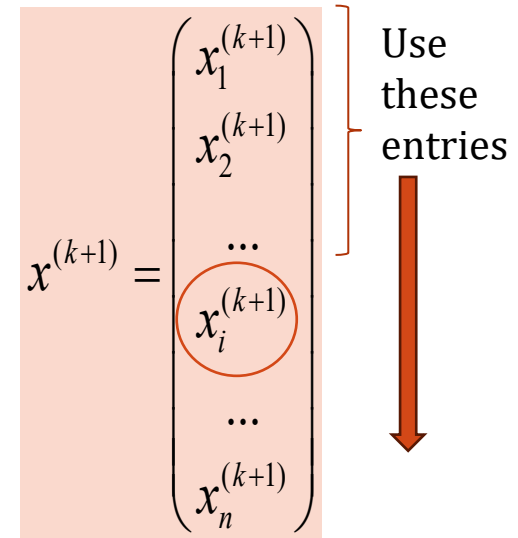
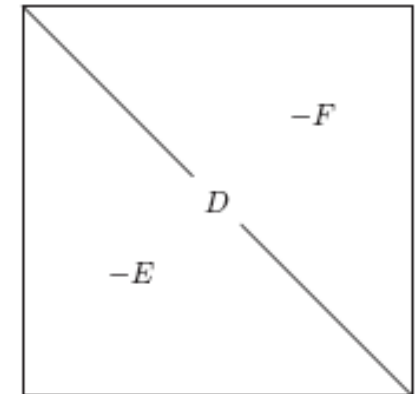
$$(D - E)x^{(k+1)} = Fx^{(k)} + b$$

$$x^{(k+1)} = (D - E)^{-1} Fx^{(k)} + (D - E)^{-1} b$$

$$G_{GS} = (D - E)^{-1} F = I - (D - E)^{-1} A, \quad g_{GS} = (D - E)^{-1} b$$

Component-wise form:

$$x_i^{(k+1)} = - \underbrace{\frac{1}{a_{ii}} \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)}}_{\text{Use these entries}} - \frac{1}{a_{ii}} \sum_{j=i+1}^n a_{ij} x_j^{(k)} + \frac{b_i}{a_{ii}}$$



Variations of Gauss-Seidel method: backward and symmetric

Backward GS ($E \leftrightarrow F$):

$$x^{(k+1)} = (D - F)^{-1} E x^{(k)} + (D - F)^{-1} b$$

Component-wise form:

$$x_i^{(k+1)} = -\frac{1}{a_{ii}} \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \underbrace{\frac{1}{a_{ii}} \sum_{j=i+1}^n a_{ij} x_j^{(k+1)}} + \frac{b_i}{a_{ii}}$$

Symmetric GS: combine one iteration

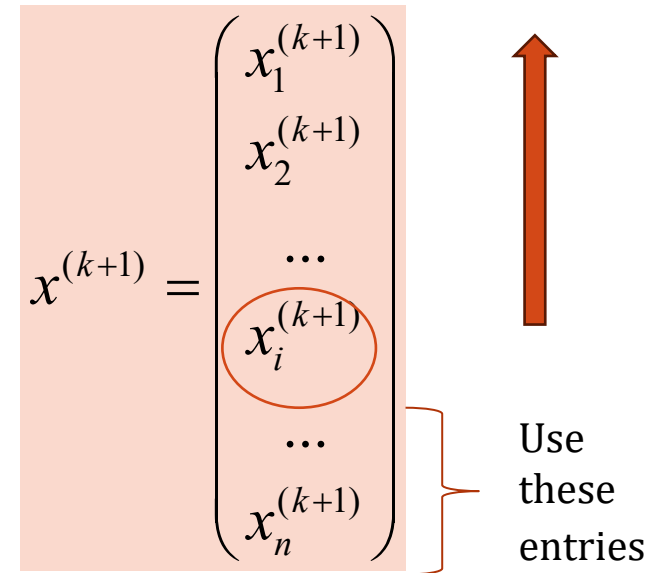
half-step of GS with one iteration half-step of backward GS

$$x^{(k+\frac{1}{2})} = (D - E)^{-1} F x^{(k)} + (D - E)^{-1} b$$

$$x^{(k+1)} = (D - F)^{-1} E x^{(k+\frac{1}{2})} + (D - F)^{-1} b$$

By eliminating $x^{(k+\frac{1}{2})}$, we get $x^{(k+1)} = G_{SGS} x^{(k)} + g_{SGS}$, where

$$G_{SGS} = (D - F)^{-1} E (D - E)^{-1} F, \quad g_{SGS} = (D - F)^{-1} D (D - E)^{-1} b$$



Diagonal dominance

From component-wise formulas of Jacobi and GS methods it can be seen, that convergence is related to diagonal elements.

Row diagonal dominance of matrix $A \in \mathbb{C}^{n \times n}$

- weak diagonal dominance

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|, \quad i = \overline{1, n}$$

- strict diagonal dominance

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|, \quad i = \overline{1, n}$$

- irreducible diagonal dominance

$$\forall i = \overline{1, n} \quad |a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \quad \text{and} \quad \exists i : \quad |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|,$$

where A is irreducible, i.e. cannot be reduced to block-diagonal form:

$$\text{there is no } P: PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

Proposition. For any norm $|\lambda_i| \leq \|A\|$, where λ_i is eigenvalue of A

Gershgorin theorem

Diagonal dominance is related to Gershgorin circles and location of eigenvalues.

Gershgorin Theorem

Any eigenvalue λ of matrix $A \in \mathbb{C}^{n \times n}$ is located in one of the closed circles of the complex plane with the center in a_{ii}

and the radius $R_i = \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|$, $i = \overline{1, n}$

$$\forall \lambda \in \sigma(A) \quad \exists i: \quad |\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|$$

Convergence of Jacobi and GS methods

Sufficient convergence condition

Theorem 1. If matrix A has strict diagonal dominance, then Jacobi and GS methods converge for any initial guess and

$$\|G_{GS}\|_{\infty} \leq \|G_J\|_{\infty} < 1$$

(Recall that ∞ denotes the maximal row norm)

Theorem 2. If matrix A has irreducible diagonal dominance or strict diagonal dominance, then Jacobi and GS methods converge for any initial guess and $\rho(G_{GS}) < \rho(G_J) < 1$

Successive over relaxation (SOR) method

Consider splitting of matrix A : $A = D - E - F$

$$Ax = b, \quad A \in \mathbb{C}^{n \times n}, \quad b \in \mathbb{C}^n$$

$$\omega Ax = \omega b$$

$$\begin{aligned} \omega A &= \omega D - \omega E - \omega F + D - D = (D - \omega E) - (D - \omega D) - \omega F = \\ &= (D - \omega E) - (\omega F + (1 - \omega)D) \end{aligned}$$

$$\left((D - \omega E) - (\omega F + (1 - \omega)D) \right) x = \omega b$$

$$(D - \omega E)x = (\omega F + (1 - \omega)D)x + \omega b$$

Iterative process:

$$(D - \omega E)x^{(k+1)} = (\omega F + (1 - \omega)D)x^{(k)} + \omega b$$

$$x^{(k+1)} = (D - \omega E)^{-1}[\omega F + (1 - \omega)D]x^{(k)} + \omega(D - \omega E)^{-1}b$$

$$G_{SOR} = (D - \omega E)^{-1}[\omega F + (1 - \omega)D], \quad g_{SOR} = \omega(D - \omega E)^{-1}b$$

Successive over relaxation (SOR) method

Component-wise form:

$$x_i^{(k+1)} = \omega \left(-\frac{1}{a_{ii}} \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \frac{1}{a_{ii}} \sum_{j=i+1}^n a_{ij} x_j^{(k)} + \frac{b_i}{a_{ii}} \right) + (1 - \omega) x_i^{(k)}$$

$$x_i^{(k+1)} = \omega x_i^{GS} + (1 - \omega) x_i^{(k)}$$

ω is relaxation parameter, $\omega \in [0, 2]$

For $\omega = 1$ we get Gauss-Seidel method.

Variations of SOR method: backward SOR

Backward SOR is derived in a similar way: $E \leftrightarrow F$

$$\omega A = (D - \omega F) - (\omega E + (1 - \omega)D)$$

$$((D - \omega F) - (\omega E + (1 - \omega)D))x = \omega b$$

$$(D - \omega F)x = (\omega E + (1 - \omega)D)x + \omega b$$

Iterative process:

$$(D - \omega F)x^{(k+1)} = (\omega E + (1 - \omega)D)x^{(k)} + \omega b$$

$$x^{(k+1)} = (D - \omega F)^{-1}[\omega E + (1 - \omega)D]x^{(k)} + \omega(D - \omega F)^{-1}b$$

$$G_{SOR} = (D - \omega F)^{-1}[\omega E + (1 - \omega)D], \quad g_{SOR} = \omega(D - \omega F)^{-1}b$$

Symmetric successive over relaxation (SSOR) method

Consider splitting of matrix A : $A = D - E - F$

Idea of symmetric SOR (SSOR) is to combine one iteration half-step of SOR with one iteration half-step of backward SOR

$$x^{(k+\frac{1}{2})} = (D - \omega E)^{-1}[\omega F + (1 - \omega)D]x^{(k)} + \omega(D - \omega E)^{-1}b$$

$$x^{(k+1)} = (D - \omega F)^{-1}[\omega E + (1 - \omega)D]x^{(k+\frac{1}{2})} + \omega(D - \omega F)^{-1}b$$

$$G_{SSOR} = (D - \omega F)^{-1}[\omega E + (1 - \omega)D](D - \omega E)^{-1}[\omega F + (1 - \omega)D],$$

$$g_{SOR} = \omega(2 - \omega)(D - \omega F)^{-1}D(D - \omega E)^{-1}b$$

Component-wise form:

$$x_i^{(k+\frac{1}{2})} = \omega \left(-\frac{1}{a_{ii}} \sum_{j=1}^{i-1} a_{ij} x_j^{(k+\frac{1}{2})} - \frac{1}{a_{ii}} \sum_{j=i+1}^n a_{ij} x_j^{(k)} + \frac{b_i}{a_{ii}} \right) + (1 - \omega)x_j^{(k)}, \quad i = 1, 2, \dots, n;$$

$$x_i^{(k+1)} = \omega \left(-\frac{1}{a_{ii}} \sum_{j=1}^{i-1} a_{ij} x_j^{(k+\frac{1}{2})} - \frac{1}{a_{ii}} \sum_{j=i+1}^n a_{ij} x_j^{(k+1)} + \frac{b_i}{a_{ii}} \right) + (1 - \omega)x_j^{(k+\frac{1}{2})}, \quad i = n, n-1, \dots, 1.$$

Variation of SSOR method: unsymmetric SOR (USOR)

In USOR two different relaxation parameters ω_1 and ω_2 are used:

$$x^{\left(k+\frac{1}{2}\right)} = (D - \omega_1 E)^{-1} [\omega_1 F + (1 - \omega_1) D] x^{(k)} + \omega_1 (D - \omega_1 E)^{-1} b$$

$$x^{(k+1)} = (D - \omega_2 F)^{-1} [\omega_2 E + (1 - \omega_2) D] x^{\left(k+\frac{1}{2}\right)} + \omega_2 (D - \omega_2 F)^{-1} b$$

Convergence of SOR method

Theorem 1. If matrix A is symmetric positive definite, then SOR method converges for any initial guess for any $\omega \in (0,2)$.

Theorem 2. If matrix A is symmetric and all its diagonal elements are positive $a_{ii} > 0$, then SOR method converges for any initial guess for any $\omega \in (0,2) \Leftrightarrow A$ is positive definite.

Matrix splitting

Consider matrix splitting: $A = M - N$,
where M is nonsingular, system $Mx = b$ may be easy to solve,
 M may be good approximation to A .

$$Ax = b$$

$$(M - N)x = b$$

$$Mx = Nx + b$$

Iterative process:

$$Mx^{(k+1)} = Nx^{(k)} + b$$

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b = Gx^{(k)} + g$$

Iteration matrix $G = M^{-1}N$, $g = M^{-1}b$

$$G = M^{-1}N = M^{-1}(M - A) = I - M^{-1}A$$

Preconditioners

The iteration $x^{(k+1)} = Gx^{(k)} + g$ can be viewed as technique for solving

$$(I - G)x = g$$

Since $G = I - M^{-1}A$, $g = M^{-1}b$

$$(I - I + M^{-1}A)x = M^{-1}b$$

$$M^{-1}Ax = M^{-1}b$$

System $M^{-1}Ax = M^{-1}b$ is a **preconditioned system**

for original system $Ax = b$. It has the same solution.

Matrix M is called a **preconditioning matrix** or **preconditioner**

The preconditioned system $M^{-1}Ax = M^{-1}b$ may be less sparse than initial system $Ax = b$. Preconditioner M may be sparse, but its inverse M^{-1} not.

Iteration matrix and preconditioning matrices for the main iterative methods

In general for the iterative method we have: $G = M^{-1}N = I - M^{-1}A$.

Jacobi method:

$$G_J = D^{-1}(E + F) = I - D^{-1}A$$

$$M_J = D, N_J = D - A$$

Gauss-Seidel method:

$$G_{GS} = (D - E)^{-1}F = I - (D - E)^{-1}A$$

$$M_{GS} = D - E, N_{GS} = F$$

SOR method:

$$G_{SOR} = (D - \omega E)^{-1}[\omega F + (1 - \omega)D]$$

$$M_{SOR} = \frac{1}{\omega}D - E, N_{SOR} = \frac{1}{\omega}[(1 - \omega)D + \omega F]$$

SSOR method:

$$G_{SSOR} = (D - \omega F)^{-1}[\omega E + (1 - \omega)D](D - \omega E)^{-1}[\omega F + (1 - \omega)D],$$

$$M_{SSOR} = \frac{1}{\omega(2 - \omega)}(D - \omega E)D^{-1}(D - \omega F), N_{SSOR} = \frac{1}{\omega(2 - \omega)}(D + \omega F)D^{-1}(D + \omega E)$$