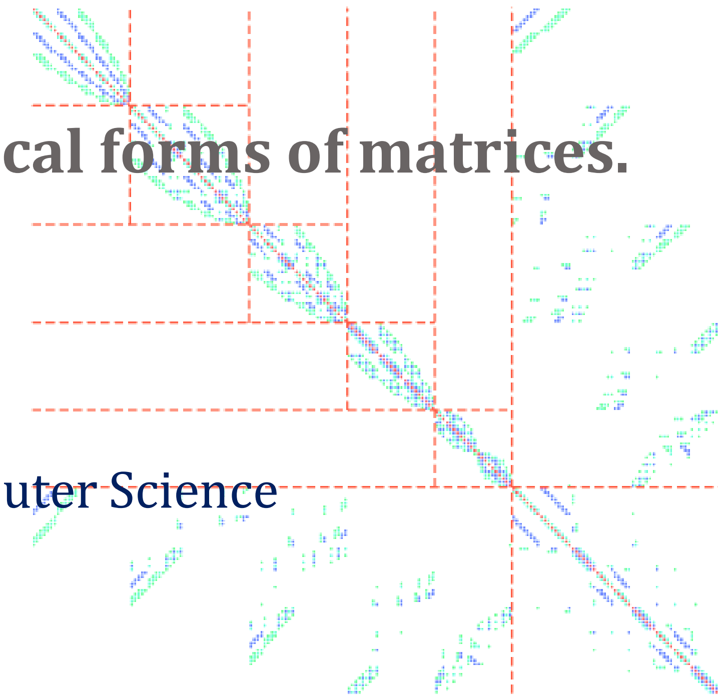
Numerical Methods of Linear Algebra for Sparse Matrices

Lecture 5

Multiplicities of eigenvalues. Canonical forms of matrices.



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Outline

- Multiplicities of eigenvalues
- Canonical forms by similarity transformation: diagonal form, Jordan form, Schur form
- Singular value decomposition (SVD factorization)
- Relation between Schur and SVD

Multiplicities of eigenvalues

Consider a matrix $A \in \mathbb{C}^{n\times n}$

- Algebraic multiplicity $\mu(\lambda)$ of eigenvalue $\lambda \in \sigma(A)$ is

a multiplicity of λ as a root of characteristic polynomial

$p_A(\lambda) = \det(A - \lambda I),$ where I is identity matrix, $\sigma(A)$ is the spectrum of A.

- Geometric multiplicity $\nu(\lambda)$ of eigenvalue $\lambda \in \sigma(A)$ is

the number of linear independent vectors in the eigenspace:

$\nu(\lambda) = \dim(Null(A - \lambda I))$

Note that for any λ geometric multiplicity cannot exceed

algebraic multiplicity: $\nu(\lambda) \le \mu(\lambda)$

- simple eigenvalue: $\mu(\lambda) = 1$
- multiple eigenvalue: $\mu(\lambda) > 1$
- semisimple eigenvalue: $\nu(\lambda) = \mu(\lambda)$
- defective eigenvalue: $\nu(\lambda) < \mu(\lambda)$
- derogatory eigenvalue: $\nu(\lambda) > 1$

Example. If $\mu(\lambda) = 3,\ \nu(\lambda) = 2,$
then λ is defective and derogatory

Simple, derogatory and defective matrices

Consider a matrix $A \in \mathbb{C}^{n \times n}$

- simple matrix: $\forall \lambda \in \sigma(A)$ is simple $(\mu(\lambda) = 1)$
- semisimple matrix: $\forall \lambda \in \sigma(A)$ is semisimple $\nu(\lambda) = \mu(\lambda)$
- defective matrix: $\exists$ defective λ $(\nu(\lambda) < \mu(\lambda))$
- derogatory matrix: $\exists$ derogatory λ $(\nu(\lambda) > 1)$

Theorem 1. For a simple matrix all eigenvalues λ_i are distinct, hence all eigenvectors u_i are linear independent.

Theorem 2. Similar matrices have the same spectrum, hence for the corresponding eigenvalues the multiplicities are the same.

Similarity transformation

If $A \in \mathbb{C}^{n\times n}$ is a square matrix, sometimes we can reduce it to a simpler form (diagonal, bidiagonal, tridiagonal, etc.)

Reduction means a *similarity transformation*, which preserves eigenvalues.

- similarity transformation is a mapping $B \to A$ of matrix B to A, such that $A = XBX^{-1}$, where X is nonsingular.
- Matrices A and B are called similar

Eigenvalues of similar matrices are the same: $\sigma(A) = \sigma(B)$

If u_B is the eigenvector of B, then $u_A = Xu_B$ is the eigenvector of A.

If u_A is the eigenvector of A, then $u_B = X^{-1}u_A$ is the eigenvector of B.

Reduction to diagonal form

- Diagonal form of matrix $A \in \mathbb{C}^{n \times n}$ is the matrix D, such that $A = XDX^{-1}$, where X is nonsingular, $D = diag(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \sigma(A)$

- Diagonal form D is similar to initial matrix A: $D \sim A$

- Matrix $A \in \mathbb{C}^{n \times n}$ is called diagonalizable, if it admits reduction to diagonal form by similarity transformation: $\exists X : \ A = XDX^{-1}$, where X is nonsingular, $D = diag(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \sigma(A)$

When the reduction to a diagonal form is possible?

Existence and uniqueness of diagonal form

Theorem. $A \in \mathbb{C}^{n\times n}$ has a diagonal form: $A = XDX^{-1}$, $D = diag(\lambda_1, \ldots, \lambda_n)$ $\Leftrightarrow$ A has n linear independent eigenvectors $\Leftrightarrow$ A is semisimple

$\triangle$ $A = XDX^{-1} \ \Big|\cdot X$

$AX = XD$

Represent D, X as $X = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \ldots & x_n \\ | & | & & | \end{pmatrix}$, $D = \begin{pmatrix} d_1 & & 0 \\ & \ldots & \\ 0 & & d_n \end{pmatrix}$

$Ax_i = d_i x_i \Rightarrow x_i$ are eigenvectors of A, d_i are eigenvalues of A $\Rightarrow$ $\nu(d_i) = \mu(d_i)$, i.e. A is semisimple $\square$

Proposition. Diagonal form is not unique, since eigenvalues in D can be placed in any order.

Reduction to Jordan form: existence and uniqueness

Reduction to a Jordan form is possible for any $A \in \mathbb{C}^{n\times n}$:

$\exists\, X:\; A = XJX^{-1}$, where J is block-diagonal matrix, consisting of Jordan boxes:

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ldots & \\ 0 & & & J_p \end{pmatrix},\; J \in \mathbb{C}^{n\times n};\; J_i = \begin{pmatrix} J_{i_1} & & 0 \\ & \ldots & \\ 0 & & J_{s_i} \end{pmatrix},\; J_i \in \mathbb{C}^{m_i\times m_i};$$

$m_i = \mu(\lambda_i)$ is algebraic multiplicity; $s_i = \nu(\lambda_i)$ is geometric multiplicity;

$l_i \le m_i$, l_i is the index of λ_i

$$J_{i_k} = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ldots & \\ & & \ldots & 1 \\ 0 & & & \lambda_i \end{pmatrix},\quad J_{i_k} \in \mathbb{C}^{l_i\times l_i}$$ is a Jordan cell.

Matlab command:

- **jordan(A)**

Proposition. Jordan form is not unique, since Jordan cells can be placed in any order.

Jordan form and perturbations

Each Jordan cell J_{i_k} corresponds to a different eigenvector, associated with λ_i

Example. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$

Jordan form is not used in numerical analysis, as it is sensitive to perturbations.

Example. $A = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix} \Rightarrow J = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$

If $J = \begin{pmatrix} z & 1 \\ \varepsilon & z \end{pmatrix}$, where ε is very small: $\varepsilon = 1e\text{-}10 \Rightarrow J = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$

Reduction to Schur form: existence and uniqueness

Reduction to a Schur form is possible for any $A \in \mathbb{C}^{n \times n}$:

$\exists\, Q:\ A = QRQ^{-1}$, where Q is unitary ($Q^{-1} = Q^H$), R is upper triangular,

$diag(R) = diag(\lambda_1, \ldots, \lambda_n)$

Since Q is unitary, reduction to a Schur form is a unitary transformation:

$A = QRQ^H$

Matrix R is called Schur form of A, $\sigma(R) = \sigma(A)$

Proposition. Schur form is not unique, since eigenvalues in R can be placed in any order.

Quasi-Schur form (real Schur form)

When matrix is real $A \in \mathbb{R}^{n\times n}$, but its spectrum is complex: $\sigma(A) \in \mathbb{C}$, there is a real Schur form:

$$R = \begin{pmatrix} \bigcirc & \times & \times & \times & \times & \times \\ & \bigcirc & \times & \times & \times & \times \\ & & \square & \square & \times & \times \\ & & \square & \square & \times & \times \\ & & & & \square & \square \\ & & & & \square & \square \end{pmatrix}, \text{ where } \bigcirc \text{ is } 1\times 1 \text{ block for real eigenvalue,}$$

$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$ is 2×2 block for complex conjugate pair of eigenvalues

$\lambda_1 = x + iy,\ \lambda_2 = x - iy$

$$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} x & z_1 \\ z_2 & x \end{pmatrix},\ x = \mathrm{Re}(\lambda)$$

Matlab commands:

- **schur(A)**
- **schur(A, 'complex')**

Schur form in Matlab

```
A=rand(4)
```

```
A = 4x4
    0.7577    0.1712    0.0462    0.3171
    0.7431    0.7060    0.0971    0.9502
    0.3922    0.0318    0.8235    0.0344
    0.6555    0.2769    0.6948    0.4387
```

```
eig(A)
```

```
ans = 4x1 complex
   1.6423 + 0.0000i
   0.5196 + 0.0674i
   0.5196 - 0.0674i
   0.0445 + 0.0000i
```

```
schur(A)
```

```
ans = 4x4
    1.6423    0.0562    0.9304   -0.1392
         0    0.5196    0.2864    0.6072
         0   -0.0159    0.5196    0.3270
         0         0         0    0.0445
```

```
schur(A,'complex')
```

```
ans = 4x4 complex
   1.6423 + 0.0000i  -0.2131 + 0.0547i   0.0129 - 0.9056i  -0.1392 + 0.0000i
   0.0000 + 0.0000i   0.5196 + 0.0674i  -0.2705 - 0.0000i  -0.0749 - 0.5910i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.5196 - 0.0674i   0.1391 + 0.3183i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0445 + 0.0000i
```

Singular value decomposition (SVD): existence and uniqueness

- singular value of a matrix

σ_i is called a singular value of $A \in \mathbb{C}^{n\times n}$, if $\sigma_i{}^2$ is an eigenvalue of AA^H

$\sigma_i = \sqrt{\lambda_i}$, $\lambda_i \in \sigma(AA^H)$. Singular values are always non-negative.

- singular value decomposition of matrix is its representation as a product

of three matrices: $A = U\Sigma V^H$

Singular value decomposition is possible for any $A \in \mathbb{C}^{n\times m}$:

$\exists$ unitary matrices $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{m\times m}$ and diagonal matrix $\Sigma \in \mathbb{C}^{n\times m}$,

$\Sigma = diag(\sigma_1,\ldots,\sigma_k)$, $k = \min\{n,m\}$: $A = U\Sigma V^H$

Proposition. Singular value decomposition is not unique, since singular values in R can be placed in any order.

Matlab command:

- **svd(A)**

Property. $\|A\|_2 = \|\Sigma\|_2$

Form of diagonal matrix Σ in SVD

Consider a matrix $A \in \mathbb{C}^{n\times m}$

σ_i can be placed in descending order: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq 0$

$$\text{For } n > m \quad \left.\underbrace{\begin{pmatrix} \left.\begin{matrix} \sigma_1 & & \\ & \ldots & \\ & & \sigma_m \end{matrix}\right\} n \\ \begin{matrix} 0 & \ldots & 0 \\ \\ 0 & \ldots & 0 \end{matrix} \end{pmatrix}}_{m}\right\} m$$

$$\text{For } n < m \quad \underbrace{\left.\begin{pmatrix} \underbrace{\begin{matrix} \sigma_1 & & \\ & \ldots & \\ & & \sigma_n \end{matrix}}_{n} & \begin{matrix} 0 & \ldots & 0 \\ \\ 0 & \ldots & 0 \end{matrix} \end{pmatrix}\right\} n}_{m}$$

SVD in Matlab

SVD for rectangular matrix

```
A=rand(3,5)
```

```
A = 3x5
    0.8308    0.9172    0.7537    0.0759    0.7792
    0.5853    0.2858    0.3804    0.0540    0.9340
    0.5497    0.7572    0.5678    0.5308    0.1299
```

```
[U,S,V] = svd(A)
```

```
U = 3x3
   -0.7309   -0.0556    0.6802
   -0.4836   -0.6611   -0.5737
   -0.4816    0.7482   -0.4563

S = 3x5
    2.2395         0         0         0         0
         0    0.7548         0         0         0
         0         0    0.2610         0         0

V = 5x5
   -0.5157   -0.0289   -0.0820    0.2002   -0.8285
   -0.5239    0.4326    0.4384    0.4519    0.3768
   -0.4502    0.1741    0.1355   -0.8636    0.0521
   -0.1506    0.4733   -0.8488    0.0511    0.1736
   -0.4839   -0.7468   -0.2493    0.0851    0.3726
```

```
A-U*S*V'
```

```
ans = 3x5
10^-15 x
    0.1110         0         0    0.1110    0.1110
         0   -0.2220         0    0.1943    0.1110
   -0.6661   -0.2220   -0.2220   -0.2220   -0.1665
```

```
norm(A-U*S*V')
```

```
ans = 8.0167e-16
```

```
AA=A*A'
```

```
AA = 3x3
    2.7125    1.7670    1.7207
    1.7670    1.4443    0.9042
    1.7207    0.9042    1.4966
```

```
lambdas=eig(AA)
```

```
lambdas = 3x1
    0.0681
    0.5697
    5.0155
```

```
sigmas=sqrt(lambdas)
```

```
sigmas = 3x1
    0.2610
    0.7548
    2.2395
```

Singular values are positive

A*A' is Hermitian positive definite, its eigenvalues are real and positive (theorem will be given later)

SVD in Matlab

SVD for square matrix

```
A=rand(3)+1i*rand(3)
```

```
A = 3x3 complex
    0.5688 + 0.6020i   0.3371 + 0.6892i   0.3112 + 0.0838i
    0.4694 + 0.2630i   0.1622 + 0.7482i   0.5285 + 0.2290i
    0.0119 + 0.6541i   0.7943 + 0.4505i   0.1656 + 0.9133i
```

```
[U,S,V] = svd(A)
```

```
U = 3x3 complex
   -0.3312 - 0.4453i   0.4226 - 0.1242i   0.2118 + 0.6731i
   -0.2947 - 0.4047i   0.4977 - 0.0653i  -0.1610 - 0.6866i
   -0.2656 - 0.6090i  -0.7442 - 0.0119i  -0.0386 - 0.0563i

S = 3x3
    1.9489         0         0
         0    0.8978         0
         0         0    0.3239

V = 3x3 complex
   -0.5658 + 0.0000i   0.4070 + 0.0000i   0.7171 + 0.0000i
   -0.6437 - 0.0673i  -0.5655 - 0.4347i  -0.1869 + 0.1936i
   -0.5075 - 0.0593i   0.2619 + 0.5070i  -0.5490 - 0.3346i
```

```
A-U*S*V'
```

```
ans = 3x3 complex
10^-15 x
    0.2220 - 0.1110i   0.3886 - 0.1110i   0.1110 + 0.1388i
    0.0555 + 0.0555i   0.0555 + 0.5551i   0.2220 + 0.0833i
   -0.0399 - 0.2220i   0.3331 + 0.5551i   0.1110 + 0.2220i
```

```
norm(A-U*S*V')
```

```
ans = 9.9158e-16
```

```
AA=A*A'
```

```
AA = 3x3 complex
    1.3785 + 0.0000i   1.1793 - 0.0344i   1.1069 - 0.2397i
    1.1793 + 0.0344i   1.2073 + 0.0000i   0.9402 - 0.2275i
    1.1069 + 0.2397i   0.9402 + 0.2275i   2.1235 + 0.0000i
```

```
lambdas=eig(AA)
```

```
lambdas = 3x1
    0.1049
    0.8061
    3.7983
```

```
sigmas=sqrt(lambdas)
```

```
sigmas = 3x1
    0.3239
    0.8978
    1.9489
```

Relation between Schur form and SVD

Proposition 1. If $A = U\Sigma V^H$, then diagonal matrix $D = \Sigma\Sigma^H$ is Schur form of AA^H :

$AA^H = U\Sigma V^H (U\Sigma V^H)^H = U\Sigma V^H V\Sigma^H U^H = U\Sigma I\Sigma^H U^H = U\Sigma\Sigma^H U^H = UDU^H$,

as $D = \Sigma\Sigma^H$ is diagonal matrix.

Note that AA^H is Hermitian *positive semidefinite* (detailed definition later) :

1) Hermitian: $(AA^H)^H = (A^H)^H A^H = AA^H$

2) positive semidefinite : $AA^H \geq 0$ or $(Au, u) \geq 0 \ \forall u \in \mathbb{C}^n$

Proposition 2. Any Hermitian matrix is unitarily similar to a real diagonal matrix: $A = A^H \ \Rightarrow \ A = QDQ^H$, where D is real.

Relation between Schur form and SVD

Theorem. If $A \in \mathbb{C}^{n \times n}$ is *Hermitian positive definite*, then its Schur form is diagonal matrix: $A = QRQ^H$, where $R = diag(\lambda_1, \ldots, \lambda_n) = diag(\sigma_1, \ldots, \sigma_n)$, λ_i are eigenvalues of A and σ_i are singular values of A

For *Hermitian positive definite* matrix Schur form R and matrix Σ in singular value decomposition are the *same* up to the order of singular values on the diagonal: $R \simeq \Sigma$

Check what happens to Schur and SVD, when the matrix is Hermitian (symmetric), but not positive definite.

Schur and svd in Matlab

Schur and SVD for arbitratry real matrix

```
A=rand(3)
```

```
A = 3x3
    0.4253    0.1788    0.5985
    0.3127    0.4229    0.4709
    0.1615    0.0942    0.6959
```

```
lambdas=eig(A)
```

```
lambdas = 3x1
    1.0510
    0.1701
    0.3230
```

```
sigmas=svd(A)
```

```
sigmas = 3x1
    1.2137
    0.3011
    0.1580
```

```
sigmas=sqrt(eig(A*A'))
```

```
sigmas = 3x1
    0.1580
    0.3011
    1.2137
```

```
[U,S,V] = svd(A)
```

```
U = 3x3
   -0.6141    0.0051    0.7892
   -0.5512    0.7129   -0.4335
   -0.5648   -0.7013   -0.4349

S = 3x3
    1.2137         0         0
         0    0.3011         0
         0         0    0.1580

V = 3x3
   -0.4323    0.3716    0.8216
   -0.3264    0.7849   -0.5267
   -0.8406   -0.4959   -0.2181
```

```
norm(A-U*S*V') % check svd
```

```
ans = 3.0866e-16
```

```
[U,T]=schur(A)
```

```
U = 3x3
   -0.6162   -0.7435   -0.2598
   -0.6453    0.6658   -0.3746
   -0.4515    0.0632    0.8900

T = 3x3
    1.0510    0.0534   -0.5795
         0    0.1701   -0.1104
         0         0    0.3230
```

```
norm(A-U*T*U')  % check schur
```

```
ans = 2.8347e-16
```

For this matrix eigenvalues are all real

Schur form contains eigenvalues on the diagonal

SVD contains singular values on the diagonal

Schur and svd in Matlab

Schur and SVD for arbitratry complex matrix

```
A=rand(3)+1i*rand(3)
```

```
A = 3x3 complex
   0.7184 + 0.2665i   0.3251 + 0.4401i   0.7788 + 0.8754i
   0.9686 + 0.1537i   0.1056 + 0.5271i   0.4235 + 0.5181i
   0.5313 + 0.2810i   0.6110 + 0.4574i   0.0908 + 0.9436i
```

```
lambdas=eig(A)
```

```
lambdas = 3x1 complex
   1.5126 + 1.4299i
  -0.1979 - 0.1820i
  -0.4000 + 0.4893i
```

```
sigmas=svd(A)
```

```
sigmas = 3x1
    2.3099
    0.6381
    0.2400
```

```
sigmas=sqrt(eig(A*A'))
```

```
sigmas = 3x1
    0.2400
    0.6381
    2.3099
```

Schur form contains eigenvalues on the diagonal
SVD contains singular values on the diagonal

```
[U,S,V] = svd(A)
```

```
U = 3x3 complex
  -0.6256 - 0.1410i   0.0274 - 0.2970i   0.7006 + 0.0948i
  -0.5117 - 0.1394i  -0.6686 + 0.0064i  -0.4827 + 0.1964i
  -0.4830 - 0.2724i   0.4924 + 0.4706i  -0.2508 - 0.4071i

S = 3x3
    2.3099         0         0
         0    0.6381         0
         0         0    0.2400

V = 3x3 complex
  -0.5789 + 0.0000i  -0.4894 + 0.0000i  -0.6522 + 0.0000i
  -0.3518 + 0.2333i   0.5126 + 0.4809i  -0.0723 - 0.5679i
  -0.5197 + 0.4653i  -0.0464 - 0.5142i   0.4961 - 0.0273i
```

```
norm(A-U*S*V') % check svd
```

```
ans = 3.1151e-15
```

```
[U,T]=schur(A)
```

```
U = 3x3 complex
  -0.5810 + 0.2939i   0.2977 - 0.5536i   0.0381 + 0.4237i
  -0.4566 + 0.2871i   0.2378 + 0.4280i   0.4916 - 0.4772i
  -0.5134 + 0.1471i  -0.5937 + 0.1125i  -0.5594 - 0.1916i

T = 3x3 complex
   1.5126 + 1.4299i  -0.1828 + 0.8748i  -0.4143 + 0.1047i
   0.0000 + 0.0000i  -0.1979 - 0.1820i  -0.0767 + 0.0937i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.4000 + 0.4893i
```

```
norm(A-U*T*U')  % check schur
```

```
ans = 1.9536e-15
```

Schur and svd for symmetric pos. definite matrix

Schur and SVD for symmetric matrix

```
B=rand(3);
A=B*B' % symmetric
```

```
A = 3x3

    0.5944    0.1951    0.6358
    0.1951    0.0910    0.3206
    0.6358    0.3206    1.1826
```

```
lambdas=eig(A)
```

```
lambdas = 3x1
    0.0019
    0.1883
    1.6778
```

```
sigmas=svd(A)
```

```
sigmas = 3x1
    1.6778
    0.1883
    0.0019
```

```
sigmas=sqrt(eig(A*A'))
```

```
sigmas = 3x1
    0.0019
    0.1883
    1.6778
```

```
[U,S,V] = svd(A) % U=V
```

```
U = 3x3
   -0.5231    0.8478    0.0871
   -0.2301   -0.0421   -0.9722
   -0.8206   -0.5286    0.2171

S = 3x3
    1.6778         0         0
         0    0.1883         0
         0         0    0.0019

V = 3x3
   -0.5231    0.8478    0.0871
   -0.2301   -0.0421   -0.9722
   -0.8206   -0.5286    0.2171
```

```
norm(A-U*S*V')
```

```
ans = 7.2678e-16
```

```
norm(A-U*S*U')
```

```
ans = 6.7175e-16
```

```
[U,T]=schur(A)
```

```
U = 3x3
    0.0871    0.8478    0.5231
   -0.9722   -0.0421    0.2301
    0.2171   -0.5286    0.8206

T = 3x3
    0.0019         0         0
         0    0.1883         0
         0         0    1.6778
```

```
norm(A-U*T*U')
```

```
ans = 2.5458e-16
```

Eigenvalues and singular values are the same

Hence matrices S and T are the same up to the order of values on the diagonal

Schur and svd for Hermitian pos. definite matrix

Schur and SVD for Hermitian matrix

```
B=rand(3)+1i*rand(3);
A=B*B' % Hermitian
```

```
A = 3x3 complex
   1.4459 + 0.0000i   1.4923 + 0.0709i   1.4645 + 0.0448i
   1.4923 - 0.0709i   2.1318 + 0.0000i   1.5724 - 0.5092i
   1.4645 - 0.0448i   1.5724 + 0.5092i   2.0134 + 0.0000i
```

```
lambdas=eig(A)
```

```
lambdas = 3x1
   0.0358
   0.6008
   4.9545
```

```
sigmas=svd(A)
```

```
sigmas = 3x1
   4.9545
   0.6008
   0.0358
```

```
sigmas=sqrt(eig(A*A'))
```

```
sigmas = 3x1
   0.0358
   0.6008
   4.9545
```

```
[U,S,V] = svd(A)
```

```
U = 3x3 complex
  -0.5100 + 0.0000i   0.4810 + 0.0000i   0.7131 - 0.0000i
  -0.6106 + 0.0938i  -0.1453 + 0.6052i  -0.3387 - 0.3411i
  -0.5966 - 0.0478i  -0.1187 - 0.6060i  -0.3466 + 0.3745i

S = 3x3
   4.9545         0         0
        0    0.6008         0
        0         0    0.0358

V = 3x3 complex
  -0.5100 + 0.0000i   0.4810 + 0.0000i   0.7131 + 0.0000i
  -0.6106 + 0.0938i  -0.1453 + 0.6052i  -0.3387 - 0.3411i
  -0.5966 - 0.0478i  -0.1187 - 0.6060i  -0.3466 + 0.3745i
```

```
norm(A-U*S*V')
```

```
ans = 1.3139e-15
```

```
norm(A-U*S*U')
```

```
ans = 2.0239e-15
```

```
[U,T]=schur(A)
```

```
U = 3x3 complex
   0.4844 + 0.5234i  -0.0925 + 0.4720i   0.5084 - 0.0407i
   0.0203 - 0.4802i  -0.5659 - 0.2590i   0.6012 - 0.1423i
  -0.5103 + 0.0000i   0.6175 + 0.0000i   0.5986 + 0.0000i

T = 3x3
   0.0358         0         0
        0    0.6008         0
        0         0    4.9545
```

```
norm(A-U*T*U')
```

```
ans = 3.1015e-15
```