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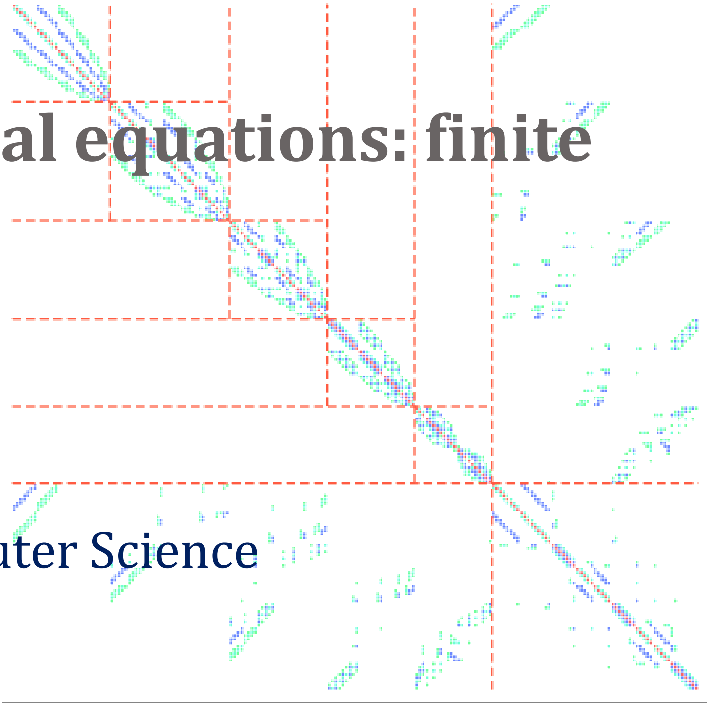
имени И.И. Воровича

Numerical Methods of Linear Algebra for Sparse Matrices

Lecture 9

Discretization of partial differential equations: finite difference method



Anna Nasedkina
Department of Mathematical Modeling
Institute of Mathematics, Mechanics and Computer Science
Southern Federal University

Discretization of PDEs

Discretization methods

Finite differences for 1D and 2D problems

Discretization of Poisson’s equation with finite difference method

Discretization of Partial Differential Equations

- Partial differential equations (PDEs) are the main source of linear systems with sparse matrices of large and extra-large size

- Discretization of PDE is the approximation of a boundary-value problem by a linear system of equations with *finite* number of unknowns

Discretization methods

- Finite difference method (FDM): the derivatives are approximated using the finite difference formulas based on low order Taylor series expansions
- Finite element method (FEM): unknown functions are approximated by piecewise-polynomial functions that are continuous on small elements of simple shape
- Finite volume method (FVM): volume integrals that contain a divergence term are converted to surface integrals, using the divergence theorem
- other methods

Poisson’s equation

- Poisson’s equation is a classic case to study discretization methods

Consider 2D Poisson's equation

$-\Delta u(\underline{x}) = f(\underline{x})$ for $\underline{x} = (x_1, x_2)$ in a bounded open domain $\Omega \subset \mathbb{R}^2$,

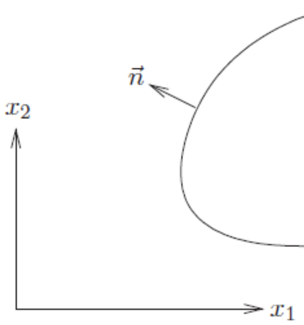
$\Gamma = \partial\Omega$ is the boundary of Ω, $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closed domain

$u(\underline{x}) = u(x_1, x_2)$ is a scalar **unknown function**, $f(\underline{x})$ is a known function

$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is Laplacian: $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$

$\underline{n}$ is external unit vector normal to Γ

(directed outwards)



Types of boundary conditions

- There are three common types of boundary conditions (BC) for PDEs
- Equation together with boundary conditions form a boundary-value problem

- Dirichlet (essential, BC of the 1st kind)

$$u(\underline{x}) = \varphi(\underline{x}) \text{ on } \Gamma_1$$

- Neumann (natural BC of the 2nd kind)

$$\frac{\partial u(\underline{x})}{\partial \underline{n}} = 0 \text{ on } \Gamma_2$$

- Robin (BC of the 3rd kind)

$$\alpha(\underline{x})u(\underline{x}) + \beta(\underline{x})\frac{\partial u(\underline{x})}{\partial n} = \gamma(\underline{x}) \quad \text{on } \Gamma_3$$

Poisson’s equation in 1D and 2D

1D Poisson's equation

$$-\frac{d^2u(x)}{dx^2} = f(x),\ x \in \Omega \subset \mathbb{R}$$

Consider domain as a segment on $\mathbb{R}$:

$$\Omega = (0,l),\ \overline{\Omega} = [0,l],\ \Gamma = \partial\Omega$$

Consider homogeneous boundary conditions: $u(x)\big|_\Gamma = 0$ or $u(0) = u(l) = 0$

2D Poisson's equation

$$-\Delta u(x_1,x_2) = f(x_1,x_2) \text{ or}$$

$$-\frac{\partial^2 u(x_1,x_2)}{\partial x_1^2} - \frac{\partial^2 u(x_1,x_2)}{\partial x_2^2} = f(x_1,x_2),\ \underline{x} = (x_1,x_2) \in \Omega \subset \mathbb{R}^2$$

Consider rectangular domain on $\mathbb{R}^2$:

$$\Omega = (0,l_1)\times(0,l_2),\ \overline{\Omega} = [0,l_1]\times[0,l_2],\ \Gamma = \partial\Omega$$

Consider homogeneous boundary conditions: $u(x_1,x_2)\big|_\Gamma = 0$

1D domain for rectangular grid

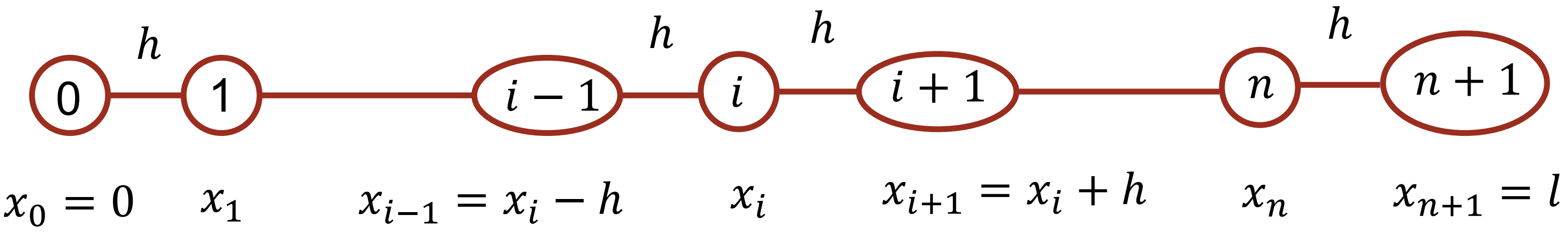
1D domain $\Omega = (0, l)$

Take n internal nodes with the step $h = \frac{l}{n+1}$

Points: $x_0, x_1, x_2, \dots, x_n, x_{n+1}$

$x_i = ih,\ i = 0, 1, 2, \dots, n, n+1$

Internal points: $x_1, x_2, \dots, x_n$



2D domain for rectangular grids

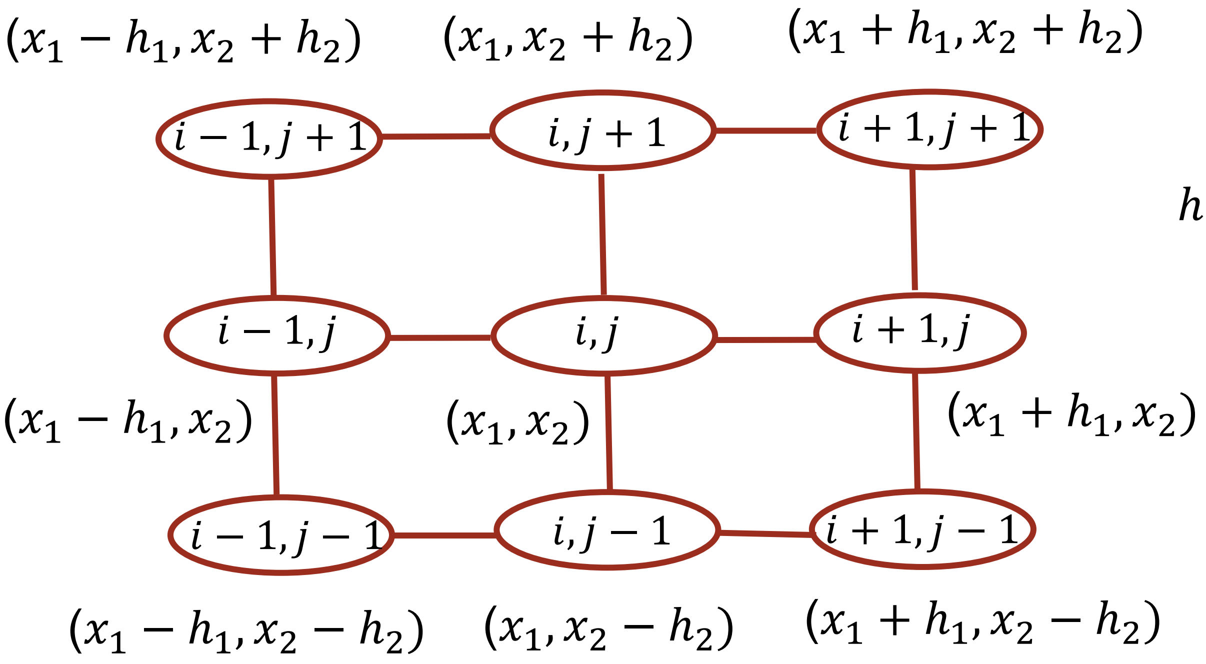
2D domain $\Omega = [0, l_1] \times [0, l_2]$

Take n_1 internal nodes in horizontal direction with the step

$$h_1 = \frac{l_1}{n_1+1},\ x_{1,i} = ih_1,\ i = 0, 1, 2, \dots, n_1, n_1+1$$

Take n_2 internal nodes in vertical direction with the step

$$h_2 = \frac{l_2}{n_2+1},\ x_{2,j} = jh_2,\ j = 0, 1, 2, \dots, n_2, n_2+1$$



$(x_1 - h_1, x_2 + h_2)$ $(x_1, x_2 + h_2)$ $(x_1 + h_1, x_2 + h_2)$

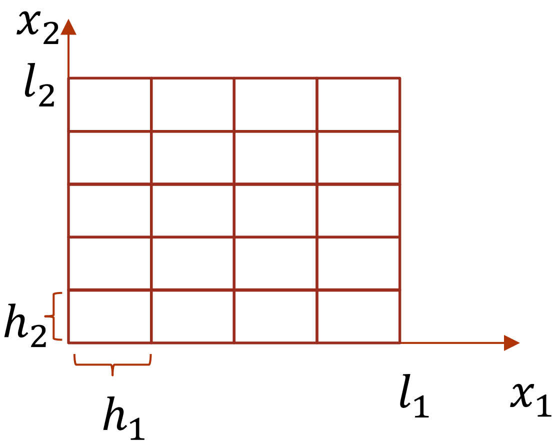
$i-1, j+1$ $i, j+1$ $i+1, j+1$

$i-1, j$ i, j $i+1, j$

$(x_1 - h_1, x_2)$ (x_1, x_2) $(x_1 + h_1, x_2)$

$i-1, j-1$ $i, j-1$ $i+1, j-1$

$(x_1 - h_1, x_2 - h_2)$ $(x_1, x_2 - h_2)$ $(x_1 + h_1, x_2 - h_2)$



Basic approximations by finite differences

Idea of FDM is to use expansion in Taylor series in the vicinity of points $x+h$ and $x-h$

$$u(x+h)=\sum_{k=0}^{\infty}\frac{h^k}{k!}\frac{d^k u(x)}{dx^k}$$

$$u(x-h)=\sum_{k=0}^{\infty}\frac{(-1)^k h^k}{k!}\frac{d^k u(x)}{dx^k}$$

$$u(x+h)=u(x)+h\frac{du(x)}{dx}+\frac{h^2}{2!}\frac{d^2u(x)}{dx^2}+\frac{h^3}{3!}\frac{d^3u(x)}{dx^3}+\ldots$$

$$u(x-h)=u(x)-h\frac{du(x)}{dx}+\frac{h^2}{2!}\frac{d^2u(x)}{dx^2}-\frac{h^3}{3!}\frac{d^3u(x)}{dx^3}+\ldots$$

$$u(x+h)=u(x)+h\frac{du(x)}{dx}+O(h)$$

$$u(x-h)=u(x)-h\frac{du(x)}{dx}+O(h)$$

Approximations for the 1st derivative

$$u(x+h) = u(x) + h\frac{du(x)}{dx} + O(h) \qquad (1)$$

$$u(x-h) = u(x) - h\frac{du(x)}{dx} + O(h) \qquad (2)$$

From these two formulas we can get

Approximations for the first derivative

Forward difference:

$$\frac{du(x)}{dx} \approx \frac{u(x+h)-u(x)}{h} \text{ from (1)} \Rightarrow u' \approx \frac{u_{i+1}-u_i}{h}$$

Backward difference:

$$\frac{du(x)}{dx} \approx \frac{u(x)-u(x-h)}{h} \text{ from (2)} \Rightarrow u' \approx \frac{u_i-u_{i-1}}{h}$$

Centered difference:

$$\frac{du(x)}{dx} \approx \frac{u(x+h)-u(x-h)}{2h} \text{ from (1)} - \text{(2)} \Rightarrow u' \approx \frac{u_{i+1}-u_{i-1}}{2h}$$

Approximations for the 2nd derivative

$$u(x+h) = u(x) + h\frac{du(x)}{dx} + \frac{h^2}{2!}\frac{d^2u(x)}{dx^2} + O(h^2) \qquad (1)$$

$$u(x-h) = u(x) - h\frac{du(x)}{dx} + \frac{h^2}{2!}\frac{d^2u(x)}{dx^2} + O(h^2) \qquad (2)$$

From these two formulas we can get

Approximations for the second derivative

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \approx \frac{d^2u(x)}{dx^2} \quad \text{from (1)+(2)}$$

$$u'' = \frac{d^2u}{dx^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

Finite differences: derivatives for univariate functions

- Forward difference $F'(x) = \dfrac{F(x+h) - F(x)}{h} + O(h)$
- Backward difference $F'(x) = \dfrac{F(x) - F(x-h)}{h} + O(h)$
- Centered difference $F'(x) = \dfrac{F(x+h) - F(x-h)}{2h} + O(h^2)$
- Centered difference for 2nd derivative

$$F''(x) = \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} + O(h^2)$$

Finite differences: derivatives for bivariate functions

$$F_x(x,y) = \frac{\partial F(x,y)}{\partial x} \approx \frac{F(x+h_1,y) - F(x-h_1,y)}{2h_1}$$

$$F_y(x,y) = \frac{\partial F(x,y)}{\partial y} \approx \frac{F(x,y+h_2) - F(x,y-h_2)}{2h_2}$$

$$F_{xx}(x,y) = \frac{\partial^2 F(x,y)}{\partial x^2} \approx \frac{F(x+h_1,y) - 2F(x,y) + F(x-h_1,y)}{{h_1}^2}$$

$$F_{yy}(x,y) = \frac{\partial^2 F(x,y)}{\partial y^2} \approx \frac{F(x,y+h_2) - 2F(x,y) + F(x,y-h_2)}{{h_2}^2}$$

$$F_{xy}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} \approx$$

$$\approx \frac{F(x+h_1,y+h_2) - F(x+h_1,y-h_2) - F(x-h_1,y+h_2) + F(x-h_1,y-h_2)}{4h_1h_2}$$

Finite differences for 1D Poisson's equation

$$-u''(x) = f(x) \text{ for } x \in (0,l)$$

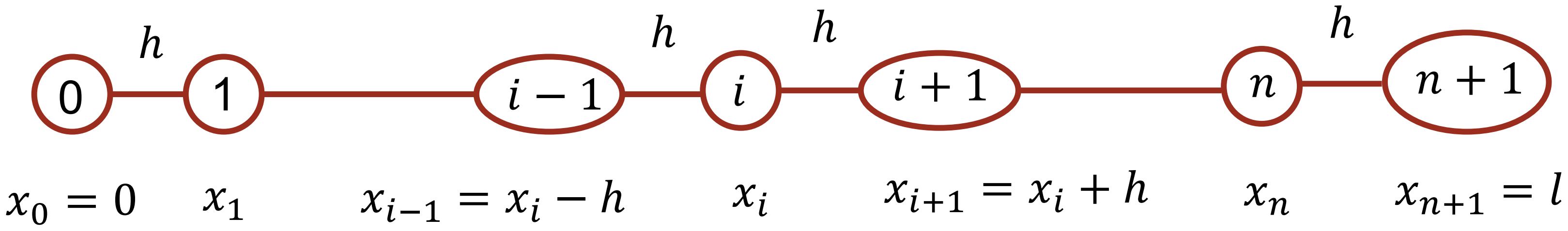
$$u(0) = u(l) = 0$$

$$x_i = i \times h,\ i = 0,1,\ldots,n+1$$

Discretized equation

$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i + h^2 \tau_i$, τ_i is the residual from Taylor series

$$\begin{cases} \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i + \tau_i \\ u_0 = u_{n+1} = 0 \end{cases}$$



Obtain the matrix for 1D Poisson's equation

$$-u_{i-1}+2u_i-u_{i+1}=h^2f_i+h^2\tau_i,\ i=1,2,....,n$$

We get a system of n equations

$$i=1:\ \ -u_0+2u_1-u_2=h^2f_1+h^2\tau_1,\ u_0=0 \text{ from BC}$$

$$i=2:\ \ -u_1+2u_2-u_3=h^2f_2+h^2\tau_2$$

$$i=3:\ \ -u_2+2u_3-u_4=h^2f_3+h^2\tau_3$$

...

$$i=n:\ \ -u_{n-1}+2u_n-u_{n+1}=h^2f_n+h^2\tau_n,\ u_{n+1}=0 \text{ from BC}$$

Write n equations one below another for corresponding variables

$$2u_1\quad -u_2 \qquad\qquad\qquad \approx h^2f_1$$

$$-u_1+2u_2\quad -u_3 \qquad\qquad \approx h^2f_2$$

$$-u_2+2u_3-u_4 \qquad \approx h^2f_3$$

...

$$-u_{n-1}+2u_n\approx h^2f_n$$

Obtain the matrix for 1D Poisson's equation

Write n equations in matrix form:

$$\begin{pmatrix} 2 & -1 & 0 & & \dots & & 0 \\ -1 & 2 & -1 & & \dots & & 0 \\ & & & \dots & & & \\ 0 & & \dots & & -1 & 2 & -1 \\ 0 & & \dots & & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-1} \\ u_n \end{pmatrix} \approx h^2 \begin{pmatrix} f_1 \\ f_2 \\ \dots \\ f_{n-1} \\ f_n \end{pmatrix}$$

The resulting system $Au = h^2 f$ can be written as $T_n u = h^2 f$,

where the matrix $A = T_n = tridiag(-1, 2 - 1) \in \mathbb{R}^{n \times n}$

Finite differences for 1D Poisson’s equation

$$-u''(x) = f(x) \text{ for } x \in (0,l)$$

$$u(0) = u(l) = 0$$

$$x_i = i \times h,\ i = 0,1,\ldots,n+1$$

Discretized equation

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} \approx f_i$$

Matrix of the system $Au = f$

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

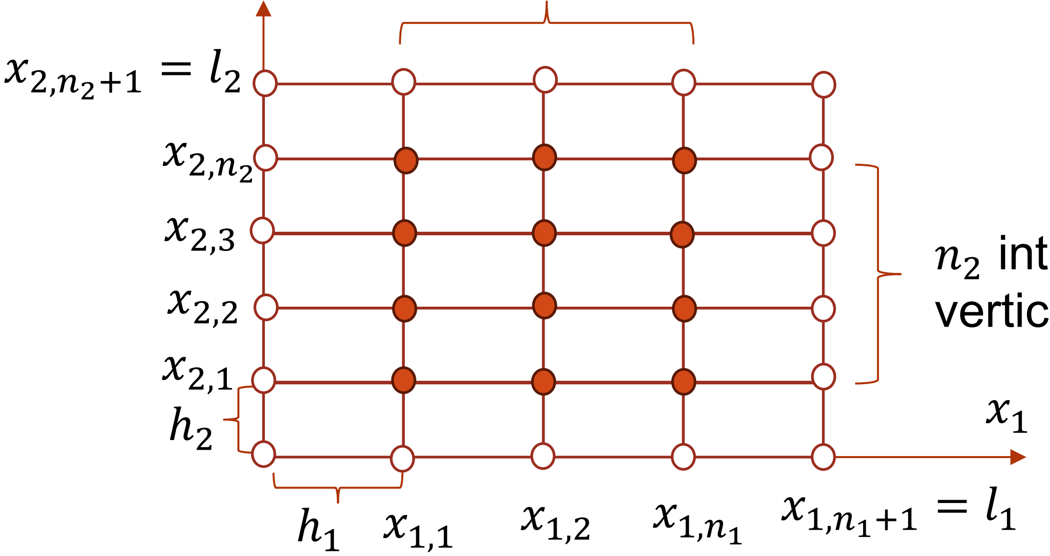
Finite differences for 2D Poisson's equation

$$
\begin{aligned}
-\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) &= f \quad \text{in}\, \Omega = (0, l_1) \times (0, l_2) \\
u &= 0 \quad \text{on}\, \Gamma
\end{aligned}
$$

$$x_{1,i} = i \times h_1, i = 0, \ldots, n_1 + 1 \quad x_{2,j} = j \times h_2, j = 0, \ldots, n_2 + 1$$

$$h_1 = \frac{l_1}{n_1 + 1} \quad h_2 = \frac{l_2}{n_2 + 1}$$

n_1 internal nodes in horizontal direction



n_2 internal nodes in vertical direction

Finite differences for 2D Poisson's equation

$u(x_{1,i};x_{2,j})=u_{ij}$

Obtain discretized equation for the case when

$l_1=l_2=l,\ h_1=h_2=h,\ n_1=n_2=n$

$$\begin{cases} -u_{i-1,j}+4u_{i,j}-u_{i+1,j}-u_{i,j-1}-u_{i,j+1}=h^2 f_{i,j}+h^2\tau_{i,j} \\ u_{0j}=u_{n+1,j}=u_{i,0}=u_{i,n+1}=0,\ i,j=1,2,\dots,n \end{cases}$$

We get the system with n^2 equations and n^2 unknowns.

When $n_1\neq n_2$ we get $n_1\cdot n_2$ unknowns.

For $n_1\neq n_2,\quad h_1\neq h_2$

$$-h_2^2u_{i-1,j}+2(h_1^2+h_2^2)u_{i,j}-h_2^2u_{i+1,j}-h_1^2u_{i,j-1}-h_1^2u_{i,j+1}=h_1^2h_2^2(f_{i,j}+\tau_{i,j})$$

Node renumeration for 2D Poisson's equation

To get the system of linear algebraic equations,

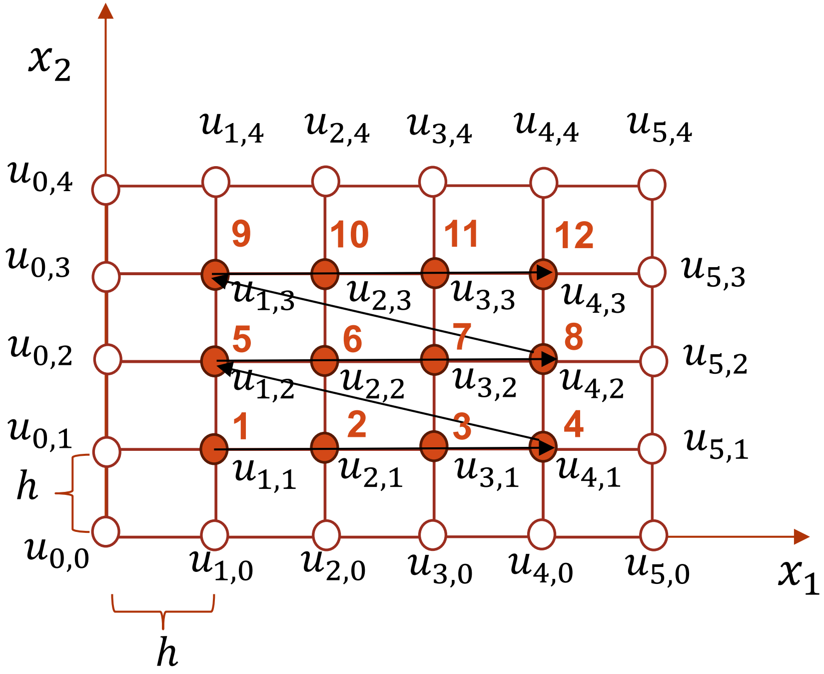
we need to renumerate the nodes to one index:

$u_{i,j} \to v_k$, where $k = 1, 2, \ldots, n_1 \cdot n_2$

There are many different algorithms of renumeration.

We'll use the **algorithm of natural ordering**.

Node renumeration for 2D Poisson's equation: natural ordering



- Example of 12 unknowns and 12 equations: $n_1 = 4, n_2 = 3, h_1 = h_2 = h$
- Natural ordering: renumerate the nodes from bottom up and from left to right
- $n_1 = 4$ internal nodes in horizontal direction, $i = 1,2,3,4$
- $n_2 = 3$ internal nodes in vertical direction, $j = 1,2,3$
- Total number of unknowns is $n_1 \cdot n_2 = 12$

- Renumerate unknowns u_{ij} to one index

$$u_{1,1} = v_1, \quad u_{2,1} = v_2, \quad u_{3,1} = v_3, \quad u_{4,1} = v_4,$$
$$u_{1,2} = v_5, \quad u_{2,2} = v_6, \quad u_{3,2} = v_7, \quad u_{4,2} = v_8,$$
$$u_{1,3} = v_9, \quad u_{2,3} = v_{10}, \quad u_{3,3} = v_{11}, \quad u_{4,3} = v_{12}$$

- The same for the values of known function f_{ij} and residual τ_{ij}

Obtain the matrix for 2D Poisson's equation

For the example $n_1 = 4,\; n_2 = 3,\; h_1 = h_2 = h$ discretized equation is

$$-u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} = h^2 f_{i,j} + h^2 \tau_{i,j},\; i = \overline{1,4},\;\; j = \overline{1,3}$$

We get a system of 12 equations with 12 unknowns $u_{i,j},\; i = \overline{1,4},\;\; j = \overline{1,3}$

Consider $i = \overline{1,4},\; j$ is fixed. We get $n_2 = 3$ blocks of $n_1 = 4$ equations

Block 1

1) $i = 1,\; j = 1:\;\; -\underset{0}{u_{0,1}} + 4\underset{v_1}{u_{1,1}} - \underset{v_2}{u_{2,1}} - \underset{0}{u_{1,0}} - \underset{v_5}{u_{1,2}} = h^2 \underset{f_1}{f_{1,1}} + h^2 \underset{\tau_1}{\tau_{1,1}},\;\; u_{0,1} = u_{1,0} = 0$ from BC

2) $i = 2,\; j = 1:\;\; -\underset{v_1}{u_{1,1}} + 4\underset{v_2}{u_{2,1}} - \underset{v_3}{u_{3,1}} - \underset{0}{u_{2,0}} - \underset{v_6}{u_{2,2}} = h^2 \underset{f_2}{f_{2,1}} + h^2 \underset{\tau_2}{\tau_{2,1}},\;\; u_{2,0} = 0$ from BC

3) $i = 3,\; j = 1:\;\; -\underset{v_2}{u_{2,1}} + 4\underset{v_3}{u_{3,1}} - \underset{v_4}{u_{4,1}} - \underset{0}{u_{3,0}} - \underset{v_7}{u_{3,2}} = h^2 \underset{f_3}{f_{3,1}} + h^2 \underset{\tau_3}{\tau_{3,1}},\;\; u_{3,0} = 0$ from BC

4) $i = 4,\; j = 1:\;\; -\underset{v_3}{u_{3,1}} + 4\underset{v_4}{u_{4,1}} - \underset{v_5}{u_{5,1}} - \underset{0}{u_{4,0}} - \underset{v_8}{u_{4,2}} = h^2 \underset{f_4}{f_{4,1}} + h^2 \underset{\tau_4}{\tau_{4,1}},\;\; u_{4,0} = 0$ from BC

Obtain the matrix for 2D Poisson's equation

Block 2

$$5)\ i=1,\ j=2:\ -\underset{0}{u_{0,2}}+4\underset{v_5}{u_{1,2}}-\underset{v_6}{u_{2,2}}-\underset{v_1}{u_{1,1}}-\underset{v_9}{u_{1,3}}=h^2\underset{f_5}{f_{1,2}}+h^2\underset{\tau_5}{\tau_{1,2}},\ u_{0,2}=0 \text{ from BC}$$

$$6)\ i=2,\ j=2:\ -\underset{v_5}{u_{1,2}}+4\underset{v_6}{u_{2,2}}-\underset{v_7}{u_{3,2}}-\underset{v_2}{u_{2,1}}-\underset{v_{10}}{u_{2,3}}=h^2\underset{f_6}{f_{2,2}}+h^2\underset{\tau_6}{\tau_{2,2}},$$

$$7)\ i=3,\ j=2:\ -\underset{v_6}{u_{2,2}}+4\underset{v_7}{u_{3,2}}-\underset{v_8}{u_{4,2}}-\underset{v_3}{u_{3,1}}-\underset{v_{11}}{u_{3,3}}=h^2\underset{f_7}{f_{3,2}}+h^2\underset{\tau_7}{\tau_{3,2}},$$

$$8)\ i=4,\ j=2:\ -\underset{v_7}{u_{3,2}}+4\underset{v_8}{u_{4,2}}-\underset{0}{u_{5,2}}-\underset{v_4}{u_{4,1}}-\underset{v_{12}}{u_{4,3}}=h^2\underset{f_8}{f_{4,2}}+h^2\underset{\tau_8}{\tau_{4,2}},\ u_{5,2}=0 \text{ from BC}$$

Block 3

$$9)\ i=1,\ j=3:\ -\underset{0}{u_{0,3}}+4\underset{v_9}{u_{1,3}}-\underset{v_{10}}{u_{2,3}}-\underset{v_5}{u_{1,2}}-\underset{0}{u_{1,4}}=h^2\underset{f_9}{f_{1,3}}+h^2\underset{\tau_9}{\tau_{1,3}},\ u_{0,3}=u_{1,4}=0 \text{ from BC}$$

$$10)\ i=2,\ j=3:\ -\underset{v_9}{u_{1,3}}+4\underset{v_{10}}{u_{2,3}}-\underset{v_{11}}{u_{3,3}}-\underset{v_6}{u_{2,2}}-\underset{0}{u_{2,4}}=h^2\underset{f_{10}}{f_{2,3}}+h^2\underset{\tau_{10}}{\tau_{2,3}},\ u_{2,4}=0 \text{ from BC}$$

$$11)\ i=3,\ j=3:\ -\underset{v_{10}}{u_{2,3}}+4\underset{v_{11}}{u_{3,3}}-\underset{v_{12}}{u_{4,3}}-\underset{v_7}{u_{3,2}}-\underset{0}{u_{3,4}}=h^2\underset{f_{11}}{f_{3,3}}+h^2\underset{\tau_{11}}{\tau_{3,3}},\ u_{3,4}=0 \text{ from BC}$$

$$12)\ i=4,\ j=3:\ -\underset{v_{11}}{u_{3,3}}+4\underset{v_{12}}{u_{4,3}}-\underset{0}{u_{5,3}}-\underset{v_8}{u_{4,2}}-\underset{0}{u_{4,4}}=h^2\underset{f_{12}}{f_{4,3}}+h^2\underset{\tau_{12}}{\tau_{4,3}},\ u_{5,3}=u_{4,4}=0 \text{ from BC}$$

Obtain the matrix for 2D Poisson’s equation

Write 12 equations one below another for corresponding variables $v_1, v_2, \dots, v_{12}$

$$
\begin{array}{cccc|cccc|cccc|l}
4v_1 & -v_2 & +0 & +0 & -v_5 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & =h^2(f_1+\tau_1) \\
v_1 & +4v_2 & -v_3 & +0 & +0 & -v_6 & +0 & +0 & +0 & +0 & +0 & +0 & =h^2(f_2+\tau_2) \\
0 & +v_2 & +4v_3 & -v_4 & +0 & +0 & -v_7 & +0 & +0 & +0 & +0 & +0 & =h^2(f_3+\tau_3) \\
0 & +0 & -v_3 & +4v_4 & +0 & +0 & +0 & -v_8 & +0 & +0 & +0 & +0 & =h^2(f_4+\tau_4) \\
\cline{1-12}
-v_1 & +0 & +0 & +0 & +4v_5 & -v_6 & +0 & +0 & -v_9 & +0 & +0 & +0 & =h^2(f_5+\tau_5) \\
0 & -v_2 & +0 & +0 & -v_5 & +4v_6 & -v_7 & +0 & +0 & -v_{10} & +0 & +0 & =h^2(f_6+\tau_6) \\
0 & +0 & -v_3 & +0 & +0 & -v_6 & +4v_7 & -v_8 & +0 & +0 & -v_{11} & +0 & =h^2(f_7+\tau_7) \\
0 & +0 & +0 & -v_4 & +0 & +0 & -v_7 & +4v_8 & -v_9 & +0 & +0 & -v_{12} & =h^2(f_8+\tau_8) \\
\cline{1-12}
0 & +0 & +0 & +0 & -v_5 & +0 & +0 & +0 & +4v_9 & -v_{10} & +0 & +0 & =h^2(f_9+\tau_9) \\
0 & +0 & +0 & +0 & +0 & -v_6 & +0 & +0 & -v_9 & +4v_{10} & -v_{11} & +0 & =h^2(f_{10}+\tau_{10}) \\
0 & +0 & +0 & +0 & +0 & +0 & -v_7 & +0 & +0 & -v_{10} & +4v_{11} & -v_{12} & =h^2(f_{11}+\tau_{11}) \\
0 & +0 & +0 & +0 & +0 & +0 & +0 & -v_8 & +0 & +0 & -v_{11} & +4v_{12} & =h^2(f_{12}+\tau_{12})
\end{array}
$$

Obtain the matrix for 2D Poisson's equation

Write 12 equations in matrix form:

$$\left(\begin{array}{cccc|cccc|cccc}
4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4
\end{array}\right)
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}
\approx h^2
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \\ f_{12} \end{pmatrix}$$

Obtain the matrix for 2D Poisson's equation

The resulting system $Av = h^2 f$ can be written as $T_{n_1 \times n_2} v = h^2 f$,

where the matrix $A = T_{n_1 \times n_2} \in \mathbb{R}^{n_1 n_2 \times n_1 n_2}$ consists of $n_2 \times n_2 = 3 \times 3$ blocks,

each block has the size $n_1 \times n_1 = 4 \times 4$

$$T_{n_1 \times n_2} = \begin{pmatrix} T_{n_1} + 2I_{n_1} & -I_{n_1} & 0 \\ -I_{n_1} & T_{n_1} + 2I_{n_1} & -I_{n_1} \\ 0 & -I_{n_1} & T_{n_1} + 2I_{n_1} \end{pmatrix}$$

Here $T_{n_1} = tridiag(-1, 2 -1)$, the matrix from 1D case,

I_{n_1} is the identity matrix of the size $n_1 \times n_1$

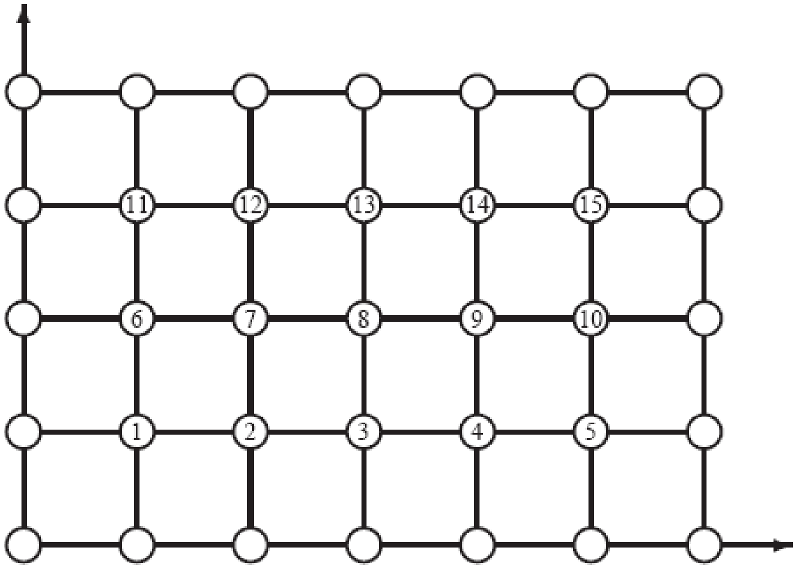
Finite differences for 2D Poisson's equation

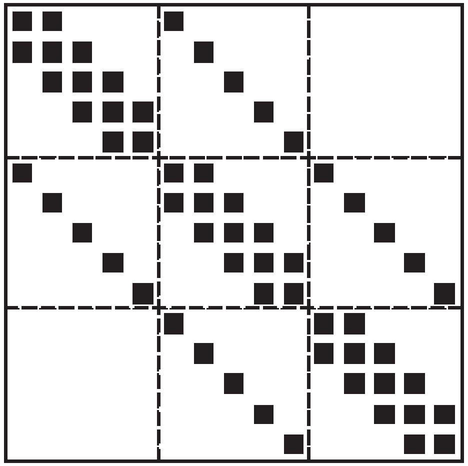
Matrix of the system $Av = f$

$$A = \frac{1}{h^2}\begin{pmatrix} B & -I & \\ -I & B & -I \\ & -I & B \end{pmatrix}$$

Grid and corresponding matrix for 5×3 internal nodes

$$B = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix}$$



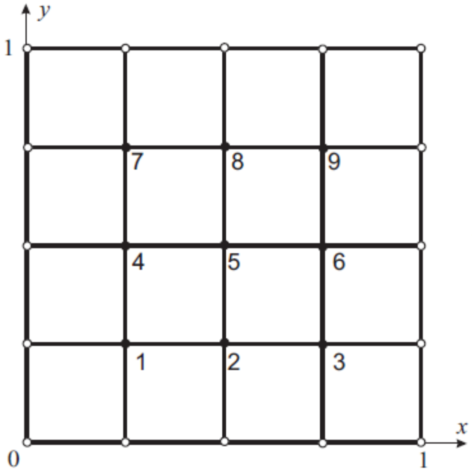


Renumeration algorithms and matrix patterns

Different algorithms for node enumeration reflect different permutations of matrix rows and columns

Natural ordering

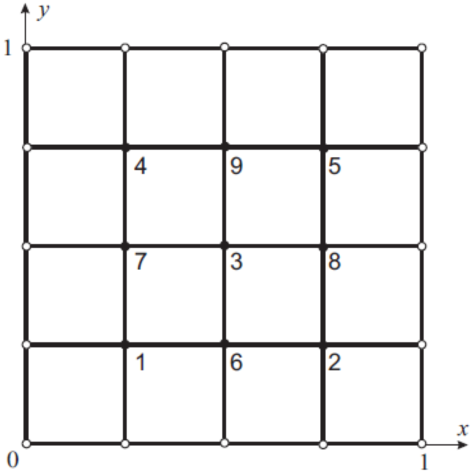
- From bottom up and from left to right



$$\mathbf{A} = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$$

Black-white ordering

- First enumerate the nodes with even sum of two indices, then enumerate the nodes with odd sum of two indices



$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 4 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 4 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 4 \end{pmatrix}$$