

Lab 4. Linear Algebra and ODE

Linear algebra

1. Given a matrix: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & -2 & -6 \end{bmatrix}$. Swap the second and third rows, then multiply the last column by 10.

2. Given matrices $A = \begin{bmatrix} 5 & 7 & -3 & -4 \\ 7 & 6 & -4 & -5 \\ 6 & 4 & -3 & -2 \\ 8 & 5 & -6 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$. Find the products of matrices AB , BA and the determinants $\det A$, $\det B$. Calculate the inverse matrices A^{-1} , B^{-1} and the transpose matrices A^T , B^T . Determine whether the matrices are singular. Determine whether the matrices are symmetric. **Hint:** Use the **Equal** command to compare two matrices and check the symmetry.

3. Given a matrix $A = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$. Find the characteristic polynomial of the matrix (using the Maple command and the formula). Calculate the value of the matrix at the characteristic polynomial $P_A(\lambda)$ (substitute the matrix A for the variable λ in $P_A(\lambda)$). To simplify the result, you can use **evalm** or **simplify** commands). Find the spectrum of the matrix (i.e., all eigenvalues) using two commands: **Eigenvalues** and **Eigenvectors**. Check whether the matrix is positive definite.

4. Given a matrix $A = \begin{bmatrix} 3 & -4 & 0 & 2 \\ 4 & -5 & -2 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$. Find the Jordan normal form, eigenvectors, and eigenvalues, along with their multiplicities. Check that each eigenvector u corresponding to an eigenvalue λ satisfies the equation $Au = \lambda u$. To check, use the substitution command and the **Equal** command. You can also check vectors by norm (the norm of the vector $Au - \lambda u$ should be zero or very close to zero).

5. Solve the matrix equation: $AX = B$, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 & 0 \\ 10 & 2 & 7 \\ 10 & 7 & 8 \end{bmatrix}$.

6. Define a system of equations $\begin{cases} x + 2y + 3z = 8, \\ 3x + y + z = 6, \\ 2x + y + 2z = 6. \end{cases}$ and solve it using the **solve** command. Then convert

the system to matrix form using the appropriate Maple command (see in the lecture, which one), obtaining the system matrix A and the right-hand side vector b . Solve the equation $A \cdot x = b$ and find vector x .

7. Find the rank of a matrix and the size (number of rows and columns) of a matrix:

$$C = \begin{bmatrix} -6 & 4 & 8 & -1 & 6 \\ -5 & 2 & 4 & 1 & 3 \\ 7 & 2 & 4 & 1 & 3 \\ 2 & 4 & 8 & -7 & 6 \\ 3 & 2 & 4 & -5 & 3 \end{bmatrix}$$

using Maple commands. Obtain the LU decomposition of matrix C (matrices P, L, U). Check that $C = P \cdot L \cdot U$. Obtain the full (matrices Qf and Rf) and truncated (matrices Q and R) QR decompositions of matrix C . Determine the sizes of the obtained matrices. Check that $C = Q \cdot R, C = Qf \cdot Rf$. Is there a relationship between the rank of a matrix and the number of nonzero rows in upper triangular form, as well as between the rank and size of matrices in QR decomposition?

Solving ordinary differential equations

1. Find the general solution of the differential equation:

$$y'' - 2y' - 3y = xe^{4x} \sin x$$

2. Find a solution to the boundary value problem: $y'' - y = 2 \cdot x, y(0) = 0, y(1) = -1$. Check the obtained solution.

3. Find a solution to the system of differential equations:

$$\begin{cases} x'' + 5x' + 2y' + y = 0 \\ 3x'' + 5x + y' + 3y = 0 \end{cases}$$

under initial conditions $x(0)=1, x'(0)=0; y(0)=1$.

4. Plot a graph of the numerical solution of the Cauchy problem $y' = \sin(xy), y(0)=1$.

5. Solve the Cauchy problem numerically: $y'' = xy' - y^2, y(0) = 1, y'(0) = 2$. Find an approximate solution to this equation as a power series expansion. Plot the graphs of the obtained solutions on a single figure. Label the curves as a legend.

6. Plot a graph of the numerical solution of the Cauchy problem $y'' - xy' + xy = 0, y(0) = 1, y'(0) = -4$ on the interval $[-1.5; 3]$, using the **DEplot** command. Set the curve color to red.

7. Construct a phase portrait of a linear system of differential equations

$$\begin{cases} \dot{x} = 3x - 4y \\ \dot{y} = x - 2y \end{cases}$$

under several initial conditions, which should be chosen independently for the best visual clarity. Use three methods: **DEplot**, **phaseportrait** (phase portrait and direction field), and **dfieldplot** (direction field only). Set the curve color to blue and the arrow size to large. Determine the nature of the singular point by calculating the roots of the characteristic polynomial of the system matrix. Write down your answer in the text comment.