Numerical Methods of Linear Algebra for Sparse Matrices

Discretization of partial differential equations

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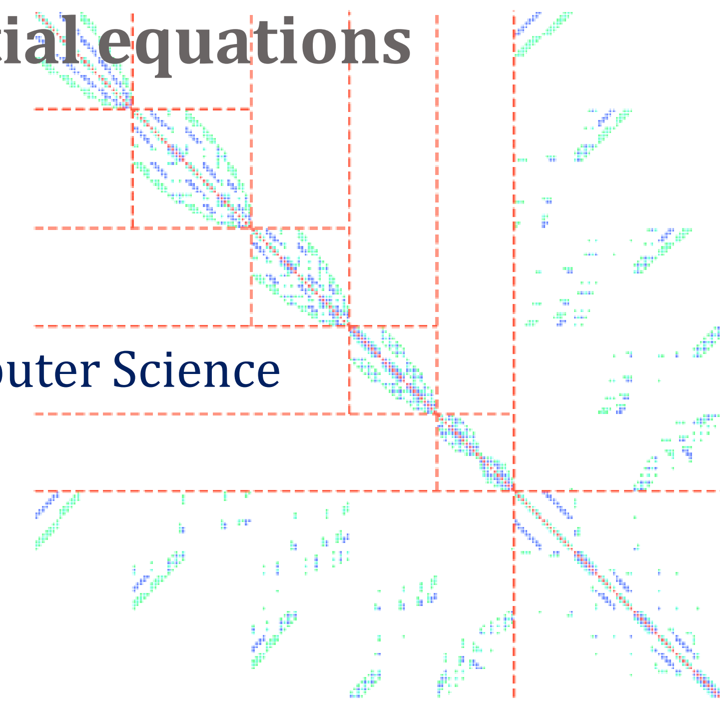


Discretization of PDEs

Discretization of partial differential equations (summary).

Finite differences for 1D and 2D problems

Finite element method: assembly process

Discretization of Partial Differential Equations

Methods

- Finite differences
- Finite elements
- Finite volumes

Finite differences: derivatives for univariate functions

- Forward difference $$F'(x) = \frac{F(x+h) - F(x)}{h} + O(h)$$
- Backward difference $$F'(x) = \frac{F(x) - F(x-h)}{h} + O(h)$$
- Centered difference $$F'(x) = \frac{F(x+h) - F(x-h)}{2h} + O(h^2)$$
- Centered difference for 2nd derivative

$$F''(x) = \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} + O(h^2)$$

Finite differences: derivatives for bivariate functions

$$F_x(x,y) \approx \frac{F(x+h,y)-F(x-h,y)}{2h}$$

$$F_y(x,y) \approx \frac{F(x,y+k)-F(x,y-k)}{2k}$$

$$F_{xx}(x,y) \approx \frac{F(x+h,y)-2F(x,y)+F(x-h,y)}{h^2}$$

$$F_{yy}(x,y) \approx \frac{F(x,y+k)-2F(x,y)+F(x,y-k)}{k^2}$$

$$F_{xy}(x,y) \approx \frac{F(x+h,y+k)-F(x+h,y-k)-F(x-h,y+k)+F(x-h,y-k)}{4hk}$$

Finite differences for 1D Poisson's equation

$$\begin{aligned} -u''(x) &= f(x) \text{ for } x \in (0,1) \\ u(0) = u(1) &= 0. \end{aligned}$$

$$x_i = i \times h,\ i = 0, \ldots, n+1$$

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i$$

- Matrix of the system $Ax = f$

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

Finite differences for 2D Poisson's equation

$$
\begin{aligned}
-\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) &= f \quad \text{in}\,\Omega = (0, l_1) \times (0, l_2) \\
u &= 0 \quad \text{on}\,\Gamma
\end{aligned}
$$

$$x_{1,i} = i \times h_1, i = 0, \ldots, n_1 + 1 \quad x_{2,j} = j \times h_2, j = 0, \ldots, n_2 + 1$$

$$h_1 = \frac{l_1}{n_1 + 1} \quad h_2 = \frac{l_2}{n_2 + 1}$$

when $h_1 = h_2 = h$

- Discretized equation

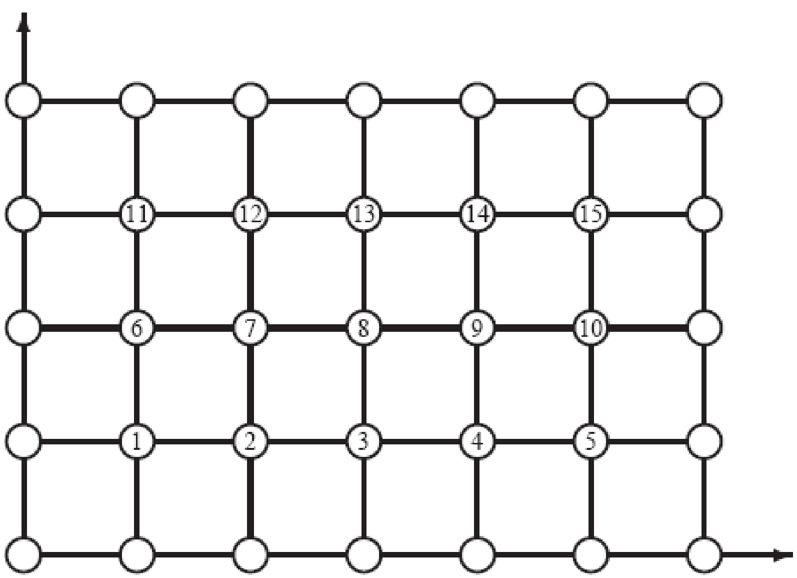
$$-u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} = h^2 f_{i,j} + h^2 \tau_{i,j}$$

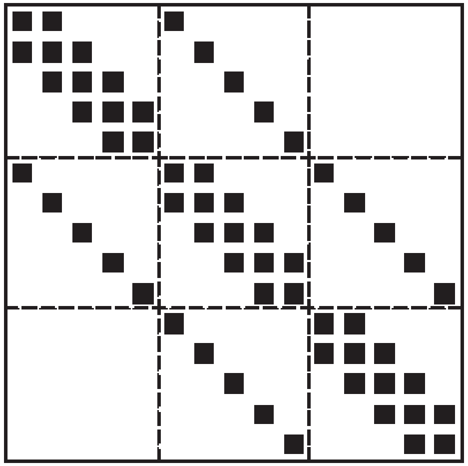
Finite differences for 2D Poisson's equation

- Matrix of the system $Ax = f$

$$A = \frac{1}{h^2}\begin{pmatrix} B & -I & \\ -I & B & -I \\ & -I & B \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix}$$

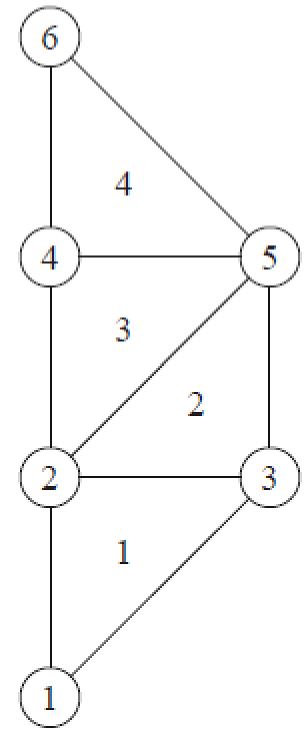




Finite element method and assembly process

Element stiffness matrix for triangle

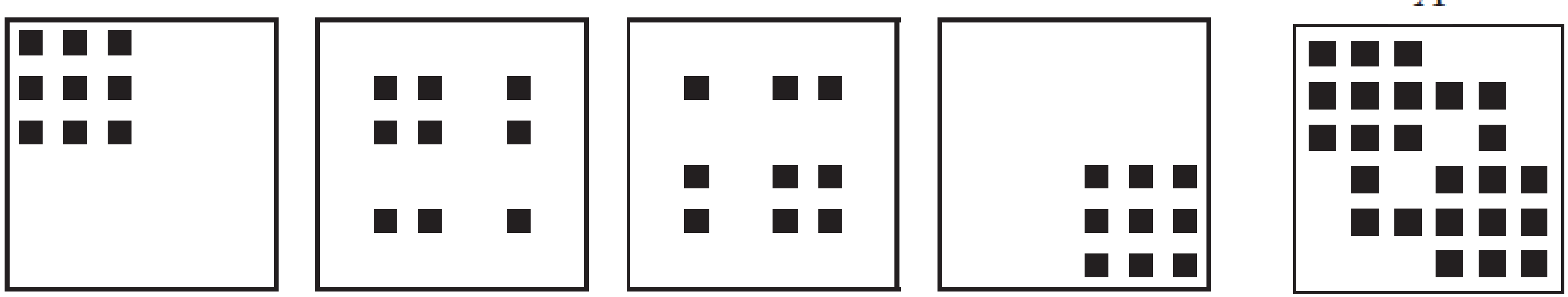
$$A_K = \begin{pmatrix} a_K(\phi_i,\phi_i) & a_K(\phi_i,\phi_j) & a_K(\phi_i,\phi_k) \\ a_K(\phi_j,\phi_i) & a_K(\phi_j,\phi_j) & a_K(\phi_j,\phi_k) \\ a_K(\phi_k,\phi_i) & a_K(\phi_k,\phi_j) & a_K(\phi_k,\phi_k) \end{pmatrix}$$



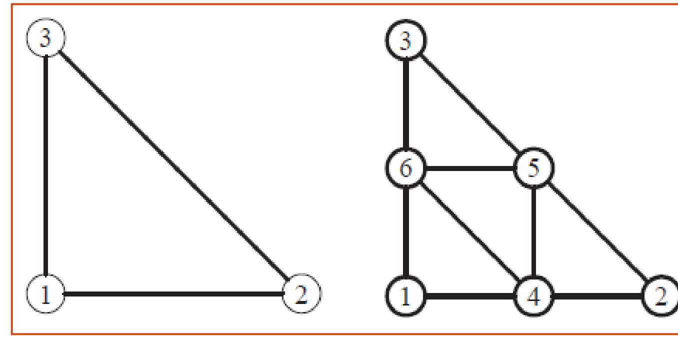
Assembly process $$A = \sum_{e=1}^{nel} A^{[e]}$$

$$A^{[e]} = P_e A_{K_e} P_e^T$$

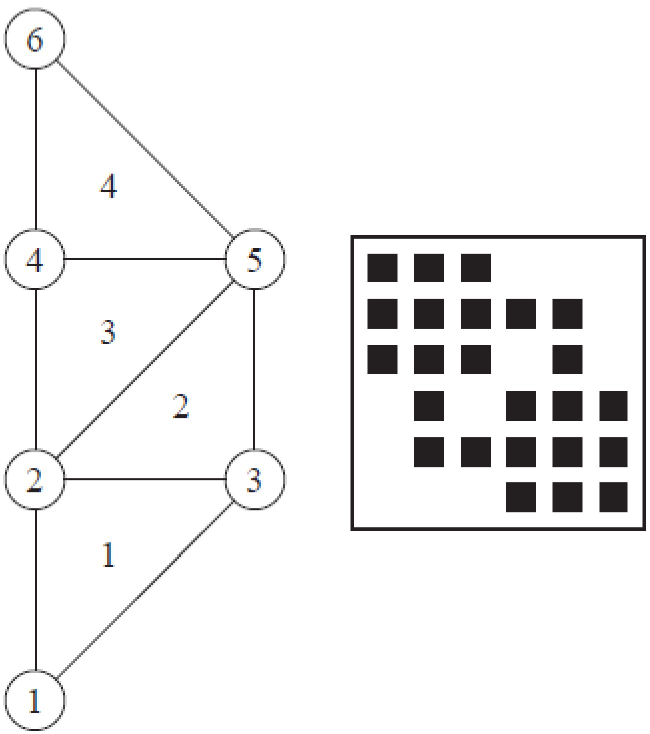
$A^{[1]}$ $A^{[2]}$ $A^{[3]}$ $A^{[4]}$ A



Mesh refinement in finite element method



Original mesh and assembled matrix



Refined mesh and assembled matrix