21. The basic theorems of differential calculus

Local extrema of functions. Fermat's theorem

Local extrema of functions

18A/37:04 (12:31)

Definition.

Let f act from E to $\mathbb{R}$, let the point $x_0 \in E$ be the limit point of the set E. It is said that the point x_0 is the *local minimum point* of the function f if there exists a neighborhood U_{x_0} of the point x_0 such that for any $x \in U_{x_0} \cap E$ the inequality $f(x) \geq f(x_0)$ holds:

$$\exists\, U_{x_0} \quad \forall\, x \in U_{x_0} \cap E \quad f(x) \geq f(x_0).$$

The value of the function f at the point x_0 is called the *local minimum*.

The local maximum point is defined similarly. The point x_0 is called the *local maximum point* of the function f if

$$\exists\, U_{x_0} \quad \forall\, x \in U_{x_0} \cap E \quad f(x) \leq f(x_0).$$

The value of the function f at the local maximum point is called the *local maximum*.

We also introduce the concepts of strict local maximum and minimum.

Definition.

Let f act from E to $\mathbb{R}$, let the point $x_0 \in E$ be the limit point of the set E. The point x_0 is called the *point of a strict local minimum* of the function f if there exists a punctured neighborhood $\mathring{U}_{x_0}$ of the point x_0 such that for any $x \in \mathring{U}_{x_0} \cap E$ the inequality $f(x) > f(x_0)$ holds:

$~~f(x_0) > f(x)~~$

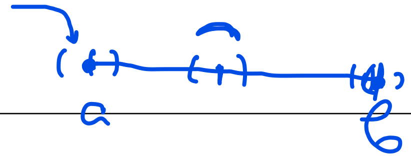
$x \neq x_0$ x_0

$$\exists\, \mathring{U}_{x_0} \quad \forall\, x \in \mathring{U}_{x_0} \cap E \quad f(x) > f(x_0).$$

The point of a strict local maximum is defined similarly. The point x_0 is called the *point of a strict local maximum* of the function f if

$$\exists\, \mathring{U}_{x_0} \quad \forall\, x \in \mathring{U}_{x_0} \cap E \quad f(x) < f(x_0).$$

It should be noted that when we give definitions of the points of strict local minimum and maximum, punctured neighborhoods of these points are

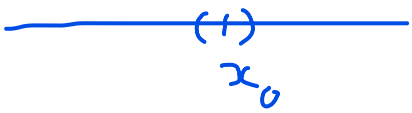


considered, since strict inequalities of the form $f(x) > f(x_0)$ or $f(x) < f(x_0)$ cannot be fulfilled for $x = x_0$.

The points of local minima and maxima are called the *points of local extrema*, and the values at these points are called the *local extrema* of this function. As for minima and maxima, strict and non-strict local extrema can be defined.

DEFINITION.

The point x_0 is called the *interior point* of the set E if the set E contains some neighborhood of this point:



$$\exists V_{x_0} \quad V_{x_0} \subset E.$$

DEFINITION.

Let f act from E to $\mathbb{R}$, $x_0 \in E$. It is said that the point x_0 is a *point of an interior local minimum, maximum* or *extremum* if it is a point of a local minimum, maximum or extremum and at the same time it is an interior point of the set E.

Fermat's theorem

`18A/49:35 (02:40), 18B/00:00 (13:02)`

THEOREM (FERMAT'S THEOREM ON INTERIOR LOCAL EXTREMA). interior point

1) Let the function f be defined in some neighborhood V_{x_0} of the point x_0 and 2) x_0 be the local extremum point of the function f. 3) Let the function f be differentiable at the point x_0. Then $f'(x_0) = 0$.

REMARK.

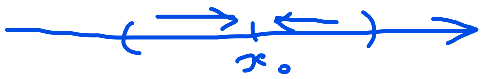
Under the conditions of the theorem, it is not assumed that the point x_0 is a point of ~~strict~~ local extremum, but it is required that it is a point of *interior* local extremum.

PROOF[1].

A=0 ?

Denote $f'(x_0) = A$. By definition of the derivative, we have:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = A.$$



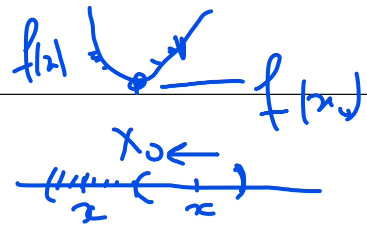
Since the specified limit exists, we obtain, by virtue of the criterion for the existence of the limit in terms of one-sided limits, that the left-hand and right-hand limits also exist and are equal to A:

$$\lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0 + 0} \frac{f(x) - f(x_0)}{x - x_0} = A.$$

Let the point x_0 be a local minimum point. This means that there exists a neighborhood $U_{x_0} \subset V_{x_0}$ for which the following condition holds:

[1] In video lectures, a more complicated method of proof is given.

$x - x_0 < 0$



$x < x_0$

$x > x_0$

$x - x_0 > 0$

$$\forall x \in U_{x_0} \quad f(x) \geq f(x_0).$$

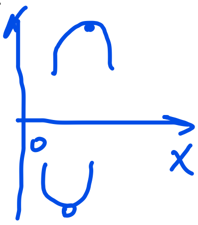
Then for the left-hand neighborhood $U_{x_0}^-$, we have $f(x) - f(x_0) \geq 0$, $x - x_0 < 0$, whence $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$. Passing to the limit, as $x \to x_0 - 0$, and using the first theorem on passing to the limit in inequalities for functions (its version for one-sided limits), we obtain $\lim_{x \to x_0 - 0} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$, i. e., $\boxed{A \leq 0}$.

For the right-hand neighborhood $U_{x_0}^+$, we have $f(x)-f(x_0) \geq 0$, $x-x_0 > 0$, whence $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$. Passing to the limit, as $x \to x_0 + 0$, and using the same theorem, we obtain $\lim_{x \to x_0 + 0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$, i. e., $\boxed{A \geq 0}$.

$A = 0$

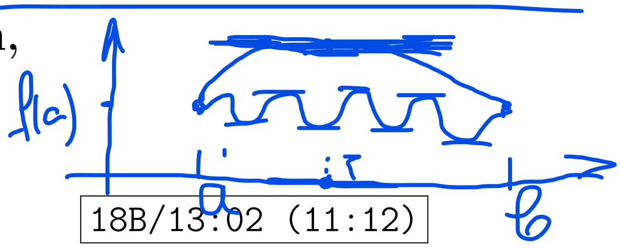
It follows from the inequalities $A \leq 0$ and $A \geq 0$ that $A = 0$, i. e., $f'(x_0) = 0$. We have proved the statement of the theorem for the case of a local minimum.

$0 = (-f)' = (-1) \cdot f' = (-1) \cdot f' = 0$



If x_0 is the local maximum point of the function f, then this point is the local minimum point of the function $-f$. Using the result already proved for a local minimum, we obtain $(-f)'(x_0) = 0$, whence $f'(x_0) = 0$. □

Rolle's theorem, Lagrange's theorem, and Cauchy's mean value theorem



ξ xi

Rolle's theorem

18B/13:02 (11:12)

Theorem (Rolle's theorem). 1)

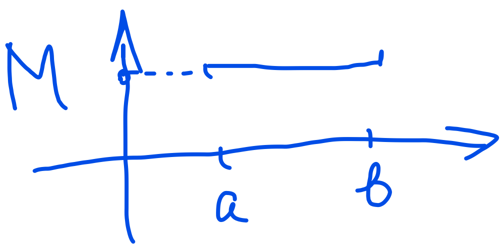
Let the function f be defined and continuous on the segment $[a, b]$ and 2) differentiable on the interval (a, b). 3) Let the function take the same values at the endpoints of the segment: $f(a) = f(b)$. Then there exists a point $\xi \in (a, b)$ for which $f'(\xi) = 0$.

Proof.

Since the function f is continuous on the segment, it attains its maximum M and minimum m values on this segment, by virtue of the second Weierstrass theorem:

$$\exists\, x_1 \in [a, b] \quad f(x_1) = \max_{x \in [a,b]} f(x) = M,$$

$$\exists\, x_2 \in [a, b] \quad f(x_2) = \min_{x \in [a,b]} f(x) = m.$$



Two cases are possible.

Case 1: $M = m$. This means that the function f is constant: $f(x) = M$ for all $x \in [a, b]$. Since the derivative of the constant function is 0, we can choose any point in the interval (a, b) as ξ: $f'(\xi) = 0$ for all $\xi \in (a, b)$.

$f'(x) = 0$

Case 2: $M > m$. Then, due to the condition $f(a) = f(b)$, at least one of the points x_1 or x_2 belongs to the interval (a,b). Indeed, if points x_1 and x_2 coincide with the endpoints of the segment $[a,b]$, then $M = f(x_1) = f(x_2) = m$, which contradicts our assumption $M > m$.

Let, for definiteness, such a point be x_1: $x_1 \in (a,b)$. Then the point x_1 is the point of the interior global maximum, therefore, it is also the point of the interior local maximum. In addition, by the condition of the theorem, the function f is differentiable at the point x_1. Thus, all the conditions of Fermat's theorem are satisfied for the point x_1. Therefore, $f'(x_1) = 0$. □

Lagrange's theorem

18B/24:14 (09:45)

Theorem (Lagrange's theorem).

Let the function f be defined and continuous on the segment $[a,b]$ and differentiable on the interval (a,b). Then there exists a point $\xi \in (a,b)$ for which the following relation holds:

$$\boxed{f(b) - f(a) = f'(\xi)(b-a).} \qquad (1)$$

$0 = f'(\xi)(b-a)$, $\neq 0$

$f'(\xi) = 0$

Remark 1.

Lagrange's theorem is a generalization of Rolle's theorem, since in the case $f(a) = f(b)$ the left-hand side of equality (1) turns to 0, which immediately implies that $f'(\xi) = 0$.

Remark 2 (geometric sense of Lagrange's theorem).

$k = \tan\alpha$

Equality (1) can be rewritten as follows:

$$\frac{f(b) - f(a)}{b-a} = f'(\xi). \qquad (2)$$

$y = kx + b$

In studying the geometric sense of the derivative, we obtained the equations of the secant line and the tangent line to the graph of the function, and at the same time established that the slope of the secant line passing through the points $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ is the number $\frac{f(x_2)-f(x_1)}{x_2-x_1}$, and the slope of the tangent line at the point $\big(x_0, f(x_0)\big)$ is $f'(x_0)$.

Therefore, equality (2) means that the slope of the secant line passing through the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ is equal to the slope of the tangent line at some point $\big(\xi, f(\xi)\big)$, where $\xi \in (a,b)$.

Taking into account that the slope of the straight line is equal to the tangent of its angle of inclination, we obtain that there exists a tangent line to the graph of the function on the interval (a,b), whose angle of inclination coincides with the angle of inclination of the secant line passing through the endpoints of the graph of the function on the segment $[a,b]$ (Fig. 10).

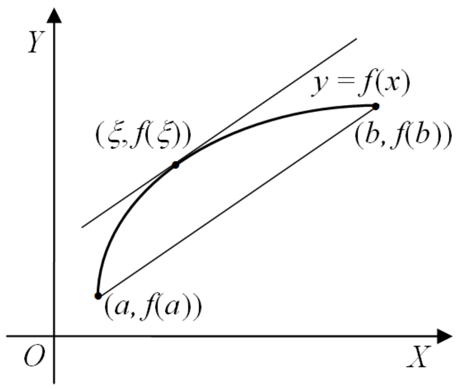


Fig. 10. Geometric sense of Lagrange's theorem

PROOF.

Introduce the auxiliary function

$(f(x) - g(x))' = f'(x) - g'(x)$

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

$g'(x) = (x - a)' = 1$

The function F is a linear combination of the function f and the linear function; therefore, it is continuous on the segment $[a, b]$ and differentiable on the interval (a, b). In addition, the function F takes the same values at the endpoints of the segment:

$F(a) = F(b)$

$$F(a) = f(a) - \frac{f(b) - f(a)}{b - a}\overbrace{(a - a)}^{=0} = f(a) - 0 = f(a),$$

$$F(b) = f(b) - \frac{f(b) - f(a)}{\cancel{b - a}}(\cancel{b - a}) = f(b) - \big(f(b) - f(a)\big) = f(a).$$

Thus, for the function F, all the conditions of Rolle's theorem are satisfied. By virtue of this theorem, we obtain that there exists a point $\xi \in (a, b)$ for which

$$F'(\xi) = 0. \tag{3}$$

Let us find the derivative of the function F:

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}. \tag{4}$$

From relations (3) and (4), we obtain:

$f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$

$$f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0.$$

If we move the second term of the last equality to the right-hand side and multiply both sides by $(b - a)$, we get (1). □

$f'(\xi)(b - a) = f(b) - f(a)$

Corollaries of Lagrange's theorem

19A/00:00 (18:01)

Corollary 1.

If the function f is defined and continuous on the segment $[a, b]$, differentiable on the interval (a, b), and $\forall\, x \in (a, b)\ f'(x) = 0$, then the function f is a constant on the segment $[a, b]$:

$$\forall\, x \in [a, b] \quad f(x) = c.$$

$f(x) = c \Rightarrow f'(x) = 0$

Remark.

The converse statement (that the derivative of the constant function vanishes) follows directly from the definition of the derivative; this statement was proved in the last section of Chapter 16.

Proof.

Let $x_1, x_2 \in [a, b]$ be arbitrarily chosen different points. For definiteness, suppose that $x_1 < x_2$.

Then the function f has the same properties on the segment $[x_1, x_2]$ as on the segment $[a, b]$, that is, it is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2) and $\forall\, x \in (x_1, x_2)\ f'(x) = 0$. Apply Lagrange's theorem for the segment $[x_1, x_2]$. By virtue of this theorem, there exists a point $\xi \in (x_1, x_2)$ for which the following relation holds:

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1). \quad = 0 \tag{5}$$

But by condition, $f'(\xi) = 0$, therefore equality (5) implies the equality $f(x_2) - f(x_1) = 0$, or $f(x_2) = f(x_1)$. $= C$

Since x_1 and x_2 were arbitrarily selected points from the segment $[a, b]$, we obtain that the function f is a constant function on this segment. □

Corollary 2.

If the function f is defined and continuous on the segment $[a, b]$, differentiable on the interval (a, b), and $\forall\, x \in (a, b)\ f'(x) \geq 0$, then the function f is non-decreasing on the segment $[a, b]$.

Proof.

As in the proof of corollary 1, we arbitrarily choose points $x_1, x_2 \in [a, b]$, $x_1 < x_2$, and apply Lagrange's theorem for the segment $[x_1, x_2]$:

$\forall x_1, x_2 \in [a, b]$

$$\exists\, \xi \in (x_1, x_2) \quad f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \geq 0 \tag{6}$$

By condition, $f'(\xi) \geq 0$, moreover, the estimate $x_2 - x_1 > 0$ holds for the difference $x_2 - x_1$, therefore the right-hand side of equality (6) is non-negative. Therefore, the left-hand side of equality (6) is also non-negative: $f(x_2) - f(x_1) \geq 0$, or $f(x_1) \leq f(x_2)$. Since the points x_1 and x_2 satisfying

$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ – non-decreasing property the inequality $x_1 < x_2$ were chosen arbitrarily from the segment $[a, b]$, we obtain that the function f is non-decreasing on this segment. □

Remark.

The following three statements can be proved in a similar way. It is assumed in each of these statements that the function f is defined and continuous on the segment $[a, b]$ and differentiable on the interval (a, b).

1. If $\forall\, x \in (a, b)$ $f'(x) \leq 0$, then the function f is non-increasing on the segment $[a, b]$.
2. If $\forall\, x \in (a, b)$ $f'(x) > 0$, then the function f is increasing on the segment $[a, b]$.
3. If $\forall\, x \in (a, b)$ $f'(x) < 0$, then the function f is decreasing on the segment $[a, b]$.

Corollary 3.

$\forall x \in (a, b)$ $f'(x) \neq 0$

Let the function f be defined and 1) continuous on the segment $[a, b]$, 2) differentiable on the interval (a, b), 3) its derivative does not vanish on the interval (a, b) and is continuous on this interval. Then the function f is strictly monotonous on the segment $[a, b]$ and has the inverse function f^{-1} acting from the segment $[c, d] = f([a, b])$ into the segment $[a, b]$. The inverse function is continuous on the segment $[c, d]$, differentiable on the interval (c, d), and has the same monotonicity type as the function f.

Proof.

Since the function $f'(x)$ is continuous on (a, b) and for all $x \in (a, b)$ $f'(x) \neq 0$, we obtain, by virtue of the intermediate value theorem, that the function $f'(x)$ preserves the sign on the interval (a, b), that is, one of the following two situations is possible: either $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$. Indeed, if it turned out that at some points x_1 and x_2 from (a, b) the function f' takes values of different signs, then, by the intermediate value theorem, there would be a point $x_0 \in (x_1, x_2)$ such that $f'(x_0) = 0$, which contradicts the condition.

If any of two possible situations is fulfilled, we obtain, by virtue of the remark to corollary 2, that the function f is strictly monotonous on the segment $[a, b]$.

Therefore, if we assume that the function f acts from the segment $[a, b]$ into the image $f([a, b])$ of this segment, then it is one-to-one and, therefore, has the inverse function f^{-1}, and the function f^{-1} has the same monotonicity type as the function f (by the first part of the inverse function theorem).

Since, by condition, the function f is continuous on the segment $[a, b]$, we conclude, by virtue of the second part of the inverse function theorem, that the image $f([a,b])$ is the segment $[c,d]$ and the function f^{-1} is continuous on the segment $[c,d]$.

Since, by condition, the function f is differentiable on the interval (a,b) and $f'(x) \neq 0$ for any point $x \in (a,b)$, we conclude, by virtue of the theorem on the differentiation of an inverse function, that the function f^{-1} is differentiable on the interval (c,d). $\square$

$f(b) - f(a) = f'(\xi)(b-a)$

Cauchy's mean value theorem

19A/18:01 (13:01)

Theorem (Cauchy's mean value theorem).

Let two functions $x(t)$ and $y(t)$ be defined and continuous on the segment $[\alpha,\beta]$, differentiable on the interval (α,β), and, in addition, $x'(t) \neq 0$ for $x \in (\alpha,\beta)$. Then there exists a point $\tau \in (\alpha,\beta)$ for which the following relation holds:

$$\frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)} = \frac{y'(\tau)}{x'(\tau)}. \tag{7}$$

$y(t)=f(t)$

$x(t)=t$

$x'(t)=1$

$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(\tau)$

$\boxed{\frac{f(b)-f(a)}{b-a} = f'(\xi)}$

Remark.

This theorem is a generalization of Lagrange's theorem, since a formula similar to formula (1) for Lagrange's theorem can be obtained from formula (7) if we put $y(t) = f(t)$, $x(t) = t$ in formula (7).

Proof.

First of all, note that the denominator $x(\beta) - x(\alpha)$ in formula (7) does not turn into 0. This follows from Lagrange's theorem for the function x and the condition $x'(t) \neq 0$, which holds for all $t \in (\alpha,\beta)$, thus, for some value $\xi \in (\alpha,\beta)$, we obtain:

$$x(\beta) - x(\alpha) = x'(\xi)(\beta - \alpha) \neq 0.$$

Now, as in the proof of Lagrange's theorem, introduce the following auxiliary function:

$$F(t) = y(t) - \frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)}\big(x(t) - x(\alpha)\big).$$

This function is continuous on the segment $[\alpha,\beta]$ and differentiable on the interval (α,β). In addition, the function F takes the same values at the endpoints of the segment:

$$F(\alpha) = y(\alpha) - \frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)}\big(x(\alpha) - x(\alpha)\big) = y(\alpha) - 0 = y(\alpha),$$

$$F(\beta) = y(\beta) - \frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)}\big(x(\beta) - x(\alpha)\big) =$$

$$= y(\beta) - \big(y(\beta) - y(\alpha)\big) = y(\alpha).$$

Thus, all the conditions of Rolle's theorem are satisfied for the function F. By virtue of this theorem, we obtain that there exists a point $\tau \in (\alpha, \beta)$ for which

$$F'(\tau) = 0. \tag{8}$$

Let us find the derivative of the function F:

$$F'(t) = y'(t) - \frac{y(\beta) - y(\alpha)}{x(\beta) - x(\alpha)} x'(t). \tag{9}$$

From relations (8) and (9), we obtain:

$$y'(\tau) - \frac{y(\beta) - y(\alpha)}{x(\beta) - x(\alpha)} x'(\tau) = 0.$$

If we move the second term in the last equality to the right-hand side and divide both sides by $x'(\tau)$, we get (7). $\square$