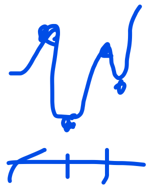
24. Application of differential calculus to the study of functions

Local extrema of functions

A necessary condition for the existence of a local extremum

20B/38:53 (06:18)

The necessary condition for the existence of an interior local extremum at the point x_0 for the function f can be obtained using previously proved Fermat's theorem.



x_0

THEOREM (A NECESSARY CONDITION FOR THE EXISTENCE OF A LOCAL EXTREMUM).

Let the function f be defined and continuous in some neighborhood U_{x_0} of the point x_0 (this, in particular, means that the point x_0 is the interior point of the domain of the function f). Let the point x_0 be the point of the interior local extremum of the function f. Then either the function f is not differentiable at the point x_0, or the function f is differentiable at the given point and $f'(x_0) = 0$.

x_0 - local extr $\Rightarrow f'(x_0)=0$

PROOF.

The theorem is a reformulation of Fermat's theorem. □

It follows from this theorem that if the function f is differentiable at the point x_0, but its derivative at this point does not vanish, then this point cannot be a point of the interior local extremum of the function f.

$f'(x_0)=0$

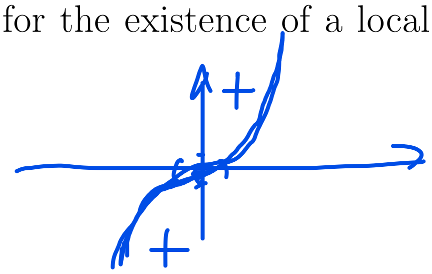
Interior points at which the function is non-differentiable or the derivative vanishes are called *critical points*, or *points suspected for a local extremum*. However, a critical point may not be an extremum point.

$\cancel{f'(x_0)\neq 0}$

For example, for the function $f(x) = x^3$, the point 0 is a critical point, since $f'(0) = 2x|_{x=0} = 0$, but it is not a local extremum point, since the inequality $f(x) < f(0)$ holds for all $x < 0$, and the inequality $f(x) > f(0)$ holds for all $x > 0$.

Thus, the formulated necessary condition for the existence of a local extremum is not a sufficient condition.

$f'(x)=3x^2\geq 0$ $f'(0)=0$



The first sufficient condition for the existence of a local extremum

`21A/17:34 (07:51)`

In the sufficient condition considered in this section, the differential properties of the function are analyzed not at the point of a local extremum, but in its neighborhood.

THEOREM (FIRST SUFFICIENT CONDITION FOR THE EXISTENCE OF A LOCAL EXTREMUM).

Let the function f be differentiable in some punctured neighborhood $\overset{\circ}{U}_{x_0}$ of the point x_0. Then the presence or absence of a local extremum at the point x_0 is determined by the signs of the derivative $f'(x)$ in the left-hand and the right-hand neighborhoods ($U^-_{x_0}$ and $U^+_{x_0}$) of the point x_0 as shown in Table 3.

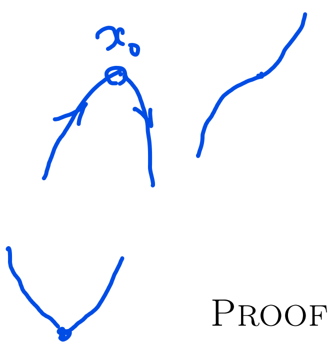
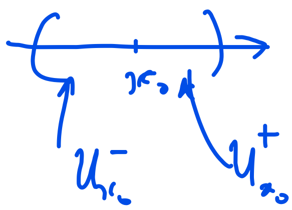


Table 3

Signs of a derivative and local extrema

$U^-_{x_0}$	$U^+_{x_0}$	Local extremum
$f'(x) > 0$	$f'(x) > 0$	no extremum
$f'(x) > 0$	$f'(x) < 0$	strict local maximum
$f'(x) < 0$	$f'(x) > 0$	strict local minimum
$f'(x) < 0$	$f'(x) < 0$	no extremum



PROOF.

This statement follows from the second corollary of Lagrange's theorem: a function increases on an interval on which its derivative is positive, and decreases on an interval on which its derivative is negative. Thus, if the derivative changes sign when passing through the point x_0, then this point is a point of strict local extremum, and if the sign does not change, then there is no extremum. $\square$

The second sufficient condition for the existence of a local extremum

`21A/25:25 (15:50)`

In a sufficient condition considered in this section, as in the necessary condition, the differential properties of the function are analyzed at the local extremum point. However, unlike the necessary condition, it requires to analyze higher order derivatives.

THEOREM (SECOND SUFFICIENT CONDITION FOR THE EXISTENCE OF A LOCAL EXTREMUM).

Let the function f be continuously differentiable in some neighborhood of the point x_0 up to the derivative of order $n \in \mathbb{N}$, and let the first nonzero derivative at the point x_0 be the derivative of order n:

$f'(x_0) \neq 0$ $\quad f'''(x_0) \neq 0$ $\quad n$

$1,3,5,7,\ldots$

$$\underline{f'(x_0) = 0}, \quad f''(x_0) = 0, \quad \ldots, \quad f^{(n-1)}(x_0) = 0, \quad f^{(\boxed{n})}(x_0) \neq 0. \quad (1)$$

$2,4,6,8\ldots$

1) If n is an odd number, then there is no local extremum at the point x_0, and 2) if n is an even number, then in the case $f^{(n)}(x_0) > 0$, the point x_0 is a point of a strict local minimum, and in the case of $f^{(n)}(x_0) < 0$, the point x_0 is a point of a strict local maximum.

Proof.

We use Taylor's formula with the remainder term in the Peano form and expand the function f according to this formula at the point x_0 up to the term with the derivative of order n:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o\big((x - x_0)^n\big), \quad x \to x_0.$$

By virtue of conditions (1), all terms corresponding to $k = 1, 2, \ldots, n-1$ disappear in the sum, and only terms corresponding to the function itself and its nth derivative remain:

$$f(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o\big((x - x_0)^n\big), \quad x \to x_0.$$

Transfer the term $f(x_0)$ to the left-hand side of the equality and represent the expression $o\big((x - x_0)^n\big)$ in the form $\alpha(x)(x - x_0)^n$, where $\alpha(x) \to 0$ as $x \to x_0$:

$x - x_0 < 0$

$x - x_0 > 0$

$x \quad x$

x_0

$x > x_0$

<1

no extr

$$f(x) - f(x_0) = \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \alpha(x)(x - x_0)^n =$$

$$= \left(\frac{f^{(n)}(x_0)}{n!} + \alpha(x)\right)(x - x_0)^n. \quad (2)$$

0 $\quad 1,3,5$ $\quad \to 0$ $\quad +2,4,6\ldots$

$f(x) - f(x_0) > 0$ $\quad f(x) > f(x_0)$

min

$f(x) - f(x_0) < 0$

$f(x) < f(x_0)$

Since $\frac{f^{(n)}(x_0)}{n!} \neq 0$ and $\alpha(x) \to 0$, we can choose a neighborhood U_{x_0} of the point x_0 in which the absolute value of function $\alpha(x)$ will be less than the absolute value of $\frac{f^{(n)}(x_0)}{n!}$. This means that the behavior of the function $\alpha(x)$ in the neighborhood U_{x_0} will not affect the sign of the factor $\left(\frac{f^{(n)}(x_0)}{n!} + \alpha(x)\right)$ on the right-hand side of (2); this sign will be determined by the sign of the derivative $f^{(n)}(x_0)$. We can also say that the factor $\left(\frac{f^{(n)}(x_0)}{n!} + \alpha(x)\right)$ preserves the sign in the neighborhood U_{x_0}.

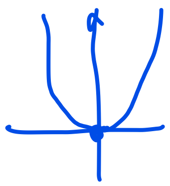
Consider the possible cases.

Case 1: n is odd. Then the factor $(x - x_0)^n$ changes sign in the neighborhood U_{x_0}: if $x < x_0$, then this factor is negative, and if $x > x_0$, then it is positive. So, the entire right-hand side of (2) also changes sign in the neighborhood U_{x_0}. Therefore, the left-hand side of (2), that is, $f(x) - f(x_0)$, also changes sign in the neighborhood U_{x_0}. This means that there is no local extremum at the point x_0, because for some points x from U_{x_0}, the value of $f(x)$ will be greater than $f(x_0)$, and for some points x, the value of $f(x)$ will be less than $f(x_0)$.

Case 2a: n is even and $f^{(n)}(x_0) > 0$. Then both factors on the right-hand side of (2) are positive for all $x \in \mathring{U}_{x_0}$. Therefore, the difference $f(x) - f(x_0)$ is also positive for all $x \in \mathring{U}_{x_0}$. This means that $f(x) > f(x_0)$ for all $x \in \mathring{U}_{x_0}$, that is, the point x_0 is a strict local minimum point.

Case 2b: n is even and $f^{(n)}(x_0) < 0$. Then the right-hand side of (2) is negative for all $x \in \mathring{U}_{x_0}$. Therefore, the difference $f(x) - f(x_0)$ is also negative for all $x \in \mathring{U}_{x_0}$. This means that $f(x) < f(x_0)$ for all $x \in \mathring{U}_{x_0}$, that is, the point x_0 is a strict local maximum point. □

EXAMPLES.

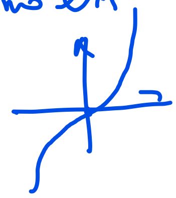


Consider the behavior of the function $f(x) = x^n$ at the point 0 for values n of different parity.

If $n = 2$, then $f'(x) = 2x$, $f''(x) = 2$, therefore $f'(0) = 0$, $f''(0) = 2$. Thus, since 2 is an even number and $f''(0) > 0$, the point 0 is a strict local minimum point for the function x^2.

$n=3$

no extremum



If $n = 3$, then $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, therefore $f'(0) = f''(0) = 0$, $f'''(0) = 6$. Since 3 is an odd number, there is no local extremum at the point 0 for the function x^3.

These results can be generalized as follows: for all even n $(= 2, 4, \dots)$, the function x^n has a strict local minimum at the point 0, and for all odd n $(= 1, 3, \dots)$, the function x^n does not have a local extremum at the point 0.

Convex functions

Definitions of convex functions

`21B/00:00 (16:10)`

DEFINITION 1 OF A CONVEX FUNCTION.

Let the function f be defined on the interval (a, b). A function f is called *convex upwards* (or *concave*) on the interval (a, b) if the graph of the secant drawn through the points $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ lies below the function graph on the interval (x_1, x_2) for any points $x_1, x_2 \in (a, b)$, $x_1 < x_2$.

Similarly, a function f is called *convex downwards* (or just *convex*) on the interval (a, b) if the graph of the secant drawn through the points $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ lies above the function graph in the interval (x_1, x_2) for any points $x_1, x_2 \in (a, b)$, $x_1 < x_2$.

The left-hand part of Fig. 11 shows an example of a function that is convex upwards, and the right-hand part of Fig. 11 shows an example of a function that is convex downwards.

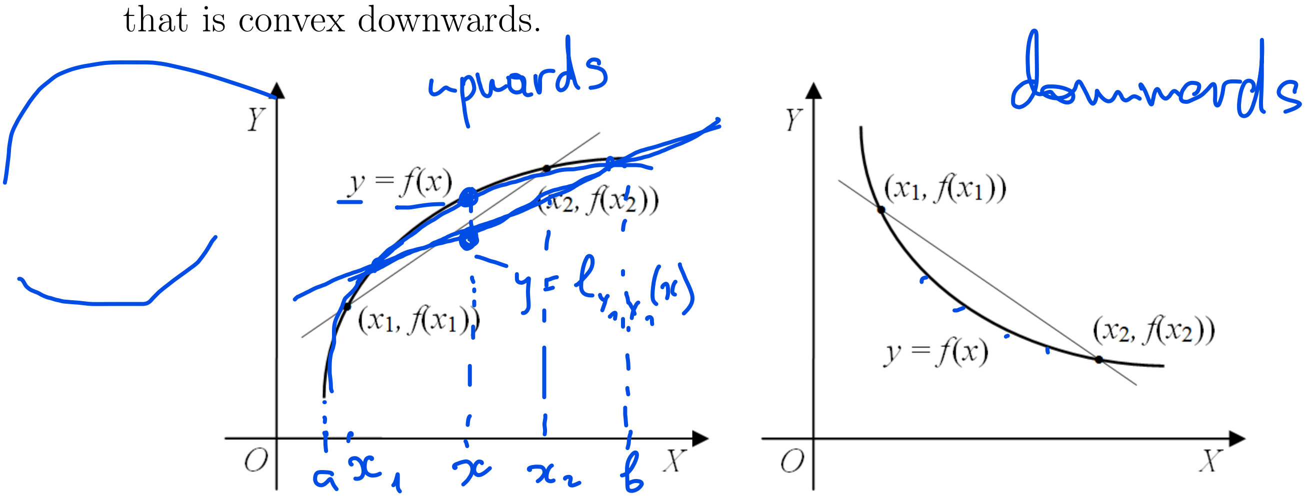


Fig. 11. Convex upwards and convex downwards functions

In order to write down convexity conditions in the form of formulas, we use the equation of a secant passing through the points $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$, $x_1 \neq x_2$. First, we write the equation of the secant in the form that was obtained when studying the geometric sense of the derivative (see Chapter 19):

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

To write down the convexity condition, it is convenient to transform this equation, leaving only the variable y on the left-hand side and moving all other terms to the right-hand side:

$$y = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1).$$

Let us transform the resulting right-hand side by presenting it in a more symmetrical form. First, we reduce both terms to a common denominator:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1) =$$
$$= \frac{\big(f(x_2) - f(x_1)\big)(x - x_1) + f(x_1)(x_2 - x_1)}{x_2 - x_1}.$$

Then we transform the numerator (for brevity, we will not rewrite the denominator):

$$\big(f(x_2) - f(x_1)\big)(x - x_1) + f(x_1)(x_2 - x_1) =$$
$$= xf(x_2) - xf(x_1) - x_1 f(x_2) + x_1 f(x_1) + x_2 f(x_1) - x_1 f(x_1) =$$

$$= xf(x_2) - xf(x_1) - x_1 f(x_2) + x_2 f(x_1) =$$
$$= f(x_1)(x_2 - x) + f(x_2)(x - x_1).$$

Thus, we have obtained the following version of the secant equation:

$$\boxed{y = \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1}.} \tag{3}$$

Denote the right-hand side of equation (3) by $l_{x_1,x_2}(x)$:

$$\underline{l_{x_1,x_2}(x)} \overset{\text{def}}{=} \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1}. \tag{4}$$

Then the secant equation takes the form

$$y = l_{x_1,x_2}(x).$$

$y = f(x)$

Since the equation of the graph of the function f has the form $y = f(x)$, we can now rewrite the definition of convexity in the language of formulas.

Definition 2 of a convex function.

A function f is called *convex upwards* on the interval (a, b) if for any points $x_1, x_2 \in (a, b)$, $x_1 < x_2$, and for any point $x \in (x_1, x_2)$, the inequality $f(x) > l_{x_1,x_2}(x)$ holds.

A function f is called *convex downwards* on the interval (a, b) if for any points $x_1, x_2 \in (a, b)$, $x_1 < x_2$, and for any point $x \in (x_1, x_2)$, the inequality $f(x) < l_{x_1,x_2}(x)$ holds.

Sufficient condition for convexity

`21B/16:10 (15:42)`

$f'(x) > 0$

Theorem (a sufficient condition for convexity).

Let a function f be differentiable up to the second order on the interval (a, b). Then if $f''(x) > 0$ for any $x \in (a, b)$, then f is convex downwards on the interval (a, b), and if $f''(x) < 0$ for any $x \in (a, b)$, then f is convex upwards on the interval (a, b).

$f'(x) < 0$

Remark.

We have previously established that the positive or negative first derivative means an increase or, accordingly, a decrease of the function. Thus, the increase and decrease of the function are associated with the properties of the first derivative, and its convexity is associated with the properties of the second derivative.

Proof.

By virtue of definition 2, to prove the theorem, it suffices to study the difference $l_{x_1,x_2}(x) - f(x)$ and show that for all $x, x_1, x_2 \in (a, b)$ satisfying the double inequality $x_1 < x < x_2$, this difference is positive in the case of a positive second derivative and negative in the case of a negative second derivative.

Let us arbitrarily choose the points $x, x_1, x_2 \in (a, b)$ that satisfy the double inequality $x_1 < x < x_2$, and transform the difference $l_{x_1,x_2}(x) - f(x)$, given the formula (4):

$$l_{(}x_1, x_2)(x) - f(x) = \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1} - f(x).$$

Reduce to a common denominator:

$$\frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1} - f(x) =$$
$$= \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1) - f(x)(x_2 - x_1)}{x_2 - x_1}.$$

From now on, we will only transform the numerator, without writing down the denominator.

Let us represent the difference $(x_2 - x_1)$ in the form $(x_2 - x + x - x_1)$ and rearrange the terms:

$$f(x_1)(x_2 - x) + f(x_2)(x - x_1) - f(x)(x_2 - x + x - x_1) =$$
$$= f(x_1)(x_2 - x) + f(x_2)(x - x_1) - f(x)(x_2 - x) - f(x)(x - x_1) =$$
$$= \big(f(x_1) - f(x)\big)(x_2 - x) + \big(f(x_2) - f(x)\big)(x - x_1) =$$
$$= \big(f(x_2) - f(x)\big)(x - x_1) - \big(f(x) - f(x_1)\big)(x_2 - x).$$

At the last stage, we transformed the expression so that the points were in the same order in the differences of functions and in the differences of points themselves.

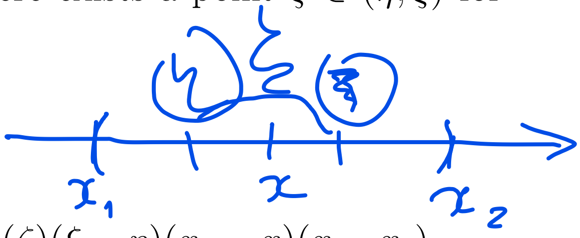
Now for the differences $f(x_2) - f(x)$ and $f(x) - f(x_1)$ we apply Lagrange's theorem, all conditions of which are satisfied. By virtue of this theorem, there exist points $\xi \in (x, x_2)$ and $\eta \in (x_1, x)$ for which the following relations hold:

$$f(x_2) - f(x) = f'(\xi)(x_2 - x),$$
$$f(x) - f(x_1) = f'(\eta)(x - x_1).$$

We continue the transformation of the numerator using the obtained relations:

$$\big(f(x_2) - f(x)\big)(x - x_1) - \big(f(x) - f(x_1)\big)(x_2 - x) =$$
$$= f'(\xi)(x_2 - x)(x - x_1) - f'(\eta)(x - x_1)(x_2 - x) =$$
$$= \big(f'(\xi) - f'(\eta)\big)(x_2 - x)(x - x_1).$$

Apply Lagrange's theorem again, now for the difference of the derivatives $f'(\xi) - f'(\eta)$. By virtue of this theorem, there exists a point $\zeta \in (\eta, \xi)$ for which the following relation holds:



$$f'(\xi) - f'(\eta) = f''(\zeta)(\xi - \eta).$$

We finally get:

$$\big(f'(\xi) - f'(\eta)\big)(x_2 - x)(x - x_1) = f''(\zeta)(\xi - \eta)(x_2 - x)(x - x_1).$$

Thus, the difference $l_{x_1,x_2}(x) - f(x)$ can be represented as follows:

$$l_{x_1,x_2}(x) - f(x) = \frac{f''(\zeta)(\xi - \eta)(x_2 - x)(x - x_1)}{x_2 - x_1}. \tag{5}$$

Note that for the points included in the resulting expression, the following estimates hold: $x_1 < \eta < x < \xi < x_2$. Thus, all the differences of the points included in the right-hand side of equality (5) are positive. Therefore, the sign of the difference $l_{x_1,x_2}(x) - f(x)$ coincides with the sign of the second derivative $f''(\zeta)$ at the point $\zeta \in (\eta, \xi) \subset (a, b)$.

So, if $f''(x) > 0$ for all $x \in (a, b)$, then the difference $l_{x_1,x_2}(x) - f(x)$ is positive and, therefore, the function is convex downwards on the interval (a, b), and if $f''(x) < 0$ for all $x \in (a, b)$, then the difference $l_{x_1,x_2}(x) - f(x)$ is negative and, therefore, the function is convex upwards on the interval (a, b). □

Inflection points of a function

Definition of an inflection point

22A/00:00 (07:37)

In studying the properties of functions associated with increasing and decreasing, we introduced the notion of a local extremum point, that is, a point located between the intervals of increasing and decreasing of a function.

Similarly, we can introduce a special notion for a point located between the intervals at which the function is convex downwards and convex upwards.

Definition.

Let the function f be defined in some neighborhood of the point a and be continuous at this point. The point a is called the *inflection point* of the function f if there exist intervals (b, a) and (a, c) such that on one of them the function f is convex downwards and on the other the function f is convex upwards (Fig. 12).

necessary cond. for extrema

for inflection point $f''(x_0)=0$

max $f'(x_0)=0$

min

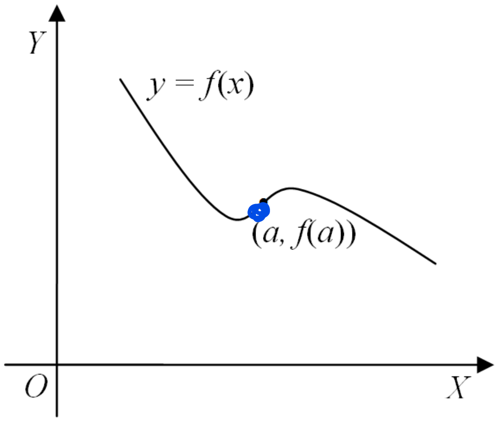


Fig. 12. Inflection point of a function

A necessary condition for the existence of an inflection point

22A/07:37 (10:20)

Theorem (a necessary condition for the existence of an inflection point).

Let a be the inflection point of the function f and let the function f be twice differentiable in some neighborhood of the point a and its second derivative be continuous at the point a. Then $f''(a) = 0$.

Proof.

Let us prove the theorem by contradiction: suppose that $f''(a) \neq 0$. For definiteness, we can assume that $f''(a) > 0$.

By condition of the theorem, the function f'' is continuous at the point a, and by our assumption $f''(a) > 0$. Then, by virtue of the theorem on the simplest properties of continuous functions, it can be stated that there exists a neighborhood U_a of the point a in which the function f'' preserves the sign, that is, in our case, $f''(x) > 0$ for $x \in U_a$.

But if the second derivative of the function is positive in some neighborhood U_a, then this means, by virtue of the previous theorem on a sufficient condition for convexity, that the function f is convex downwards in this neighborhood, which contradicts the condition that a is an inflection point. Note that if we considered the case of $f''(a) < 0$, then, by similar reasoning, we would obtain that the function f is convex upwards in some neighborhood of the point a. Therefore, our assumption is false and $f''(a) = 0$. □

Points at which the second derivative is continuous and vanishes are called *points suspected for inflection*. However, a point suspected for inflection is not necessarily an inflection point.

For example, for the functions $f_1(x) = x^3$ and $f_2(x) = x^4$, the point 0 is a point suspected for inflection, since $f_1''(0) = 6x|_{x=0} = 0$, $f_2''(0) = 12x^2|_{x=0} = 0$. However, the point 0 is an inflection point for the function x^3 and it is not an inflection point for the function x^4 (these facts will be rigorously proved later, by means of sufficient conditions for the existence of an inflection point).

Thus, the formulated necessary condition for the existence of an inflection point is not a sufficient condition.

The first sufficient condition for the existence of an inflection point

22A/17:57 (07:04)

In the sufficient condition considered in this section (as in the first sufficient condition for the existence of a local extremum), the differential properties of the function are analyzed not at the inflection point itself, but in its neighborhood.

Theorem (first sufficient condition for the existence of an inflection point).

$f''(x)>0$

Let the function f be continuous at the point a and twice differentiable in some punctured neighborhood $\mathring{U}_a$ of the point a. If the second derivative of the function f has different signs in the left-hand and the right-hand neighborhoods (U_a^- and U_a^+) of the point a, then the point a is an inflection point, and if the second derivative has the same signs, then the point a is not an inflection point.

U^- U^+

$f''(x)<0$

Proof.

This statement immediately follows from the theorem on a sufficient condition for convexity. If, for example, the function f'' is positive in the left-hand neighborhood of the point a and negative in the right-hand neighborhood, then this means that the function f is convex downwards in the left-hand neighborhood and it is convex upwards in the right-hand neighborhood, therefore, the point a is an inflection point. A similar statement is also true if the function f'' is negative in the left-hand neighborhood and positive in the right-hand neighborhood. If the second derivative has the same signs in the left-hand and right-hand neighborhood of the point a, then the function f has the same convexity type to the left and right of the point a, therefore, the point a is not an inflection point. □

Using this sufficient condition, one can easily prove that the point 0 is an inflection point for the function x^3, but it is not an inflection point for the function x^4. Indeed, $(x^3)'' = 6x$ and, therefore, the second derivative takes values of different signs to the left and right of the point 0, while $(x^4)'' = 12x^2$ takes positive values both to the left and to the right of the point 0.

The second sufficient condition for the existence of an inflection point

22A/25:01 (12:39)

In the sufficient condition considered in this section, as in the necessary condition, the differential properties of the function are analyzed at the inflection point itself. However, unlike the necessary condition, it is required to analyze the derivative of both the second and the third order.

Theorem (second sufficient condition for the existence of an inflection point).

Let the function f be three times differentiable in some neighborhood of the point a and its third derivative be continuous at the point a. Let $f''(a) = 0$ and $f'''(a) \neq 0$. Then a is the inflection point of the function f.

Proof.

For definiteness, suppose that $f'''(a) > 0$.

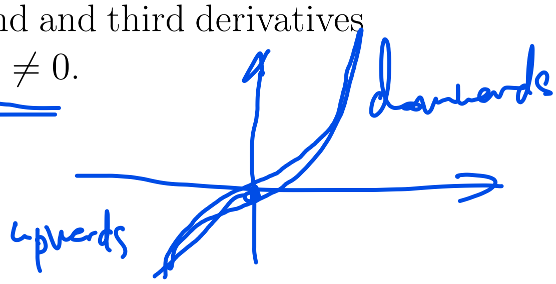
Since the function f''' is continuous at the point a and $f'''(a) > 0$, we obtain, by the theorem on the simplest properties of continuous functions, that there exists a neighborhood U_a of the point a, in which the function f''' preserves the sign, that is, in our case, $f'''(x) > 0$ for $x \in U_a$. For the function f'', this means, by virtue of the second corollary of Lagrange's theorem, that it increases on the set U_a.

Since, by condition, $f''(a) = 0$, and, in addition, the function f'' increases in a neighborhood U_a, we obtain that the function f'' takes only negative values in the left-hand neighborhood U_a^-, and it takes only positive values in the right-hand neighborhood U_a^+: $f''(x) < 0$ for $x \in U_a^-$, $f''(x) > 0$ for $x \in U_a^+$. Thus, the conditions of the theorem on the first sufficient condition for the existence of an inflection point are satisfied, and the point a is an inflection point. The situation $f'''(a) < 0$ is analyzed in the same way. $\square$

Now we can prove that the point 0 is the inflection point of the function $f(x) = x^3$ simply by calculating the values of the second and third derivatives at this point: $f''(0) = 6x|_{x=0} = 0$, $f'''(0) = 6|_{x=0} = 6 \neq 0$.

$f''(x) = 6x < 0, \; x < 0$

$f''(x) = 6x > 0, \; x > 0$



Location of the graph of a function relative to a tangent line

The theorem on the location of a tangent line in the domain of convexity of a function `22A/37:40 (02:04), 22B/00:00 (10:37)`

The convexity properties of a function can also be studied by analyzing the location of a tangent line relative to the graph of a function.

It is natural to assume that if a tangent at any point x_0 of the interval (a, b) lies above the graph of the function, then the function will be convex upwards on (a, b) (see the left-hand part of Fig. 13), and if a tangent at any point $x_0 \in (a, b)$ will lie below the graph of the function, the function will be convex downwards on (a, b) (see the right-hand part of Fig. 13).

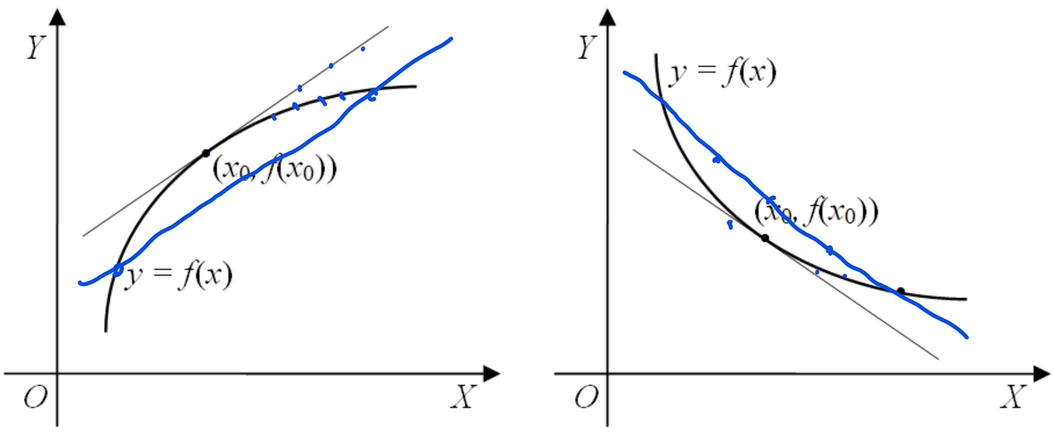


Fig. 13. Convexity property and location of tangent line

Theorem (on the location of a tangent in the domain of convexity of a function).

Suppose that on the interval (a, b), the function f has a second derivative that preserves the sign: either $f''(x) > 0$ for all $x \in (a, b)$, or $f''(x) < 0$ for all $x \in (a, b)$. Then for any point $x_0 \in (a, b)$, there exists a punctured neighborhood $\mathring{U}_{x_0}$ such that the tangent at the point x_0 lies on one side of the function graph in this neighborhood.

Proof.

Let us write the required statement in the form of a formula. To do this, we use the equation of the tangent line to the graph of the function $y = f(x)$ at the point x_0:

$$y - f(x_0) = f'(x_0)(x - x_0).$$

This equation can be written as $y = L_{x_0}(x)$, if we denote

$$L_{x_0}(x) \stackrel{\text{def}}{=} f(x_0) + f'(x_0)(x - x_0). \tag{6}$$

For definiteness, suppose that $f''(x) > 0$ for all $x \in (a, b)$, and prove that there exists a punctured neighborhood $\overset{\circ}{U}_{x_0} \subset (a, b)$ such that the estimate $f(x) - L_{x_0}(x) > 0$ holds for all $x \in \overset{\circ}{U}_{x_0}$. This estimate means that in the case of a function convex downwards on (a, b), the graph of the function lies above the graph of the tangent in $\overset{\circ}{U}_{x_0}$.

Let us choose a neighborhood $\overset{\circ}{U}_{x_0} \subset (a, b)$ and write the expansion of the function f by Taylor's formula at the point x_0 with the remainder term in the Lagrange form:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2, \quad x \in \overset{\circ}{U}_{x_0}. \tag{7}$$

Here ξ is some point lying between x_0 and x. If $x \in \overset{\circ}{U}_{x_0}$, then the point ξ will also lie in this neighborhood.

The first two terms on the right-hand side of (7) coincide with $L_{x_0}(x)$ (see (6)). Replace them with $L_{x_0}(x)$ and move them to the left-hand side. As a result, relation (7) takes the form

$$f(x) - L_{x_0}(x) = \frac{f''(\xi)}{2}(x - x_0)^2. \tag{8}$$

The factor $(x - x_0)^2$ is positive for all $x \in \overset{\circ}{U}_{x_0}$, the second derivative $f''(\xi)$ is also positive, since, by our assumption, $f''(x) > 0$ for all $x \in \overset{\circ}{U}_{x_0}$ and the point ξ belongs to $\overset{\circ}{U}_{x_0}$. Thus, the right-hand side of (8) is greater than zero for $x \in \overset{\circ}{U}_{x_0}$. Therefore, the left-hand side is also greater than zero: $f(x) - L_{x_0}(x) > 0$ for $x \in \overset{\circ}{U}_{x_0}$.

If we assume that $f''(x) < 0$ for all $x \in (a, b)$, then we can prove in the same way that $f(x) - L_{x_0}(x) < 0$ for $x \in \overset{\circ}{U}_{x_0}$, which means that in the case of a function convex upwards in (a, b), the graph of the function lies below the tangent graph in $\overset{\circ}{U}_{x_0}$. □

The theorem on the location of the tangent at an inflection point

22B/10:37 (09:46)

If there is a tangent at the inflection point, then to the left and to the right of the inflection point this tangent will lie on different sides of the function graph (Fig. 14). We prove this statement under the same assumptions as for the second sufficient condition for the existence of an inflection point.

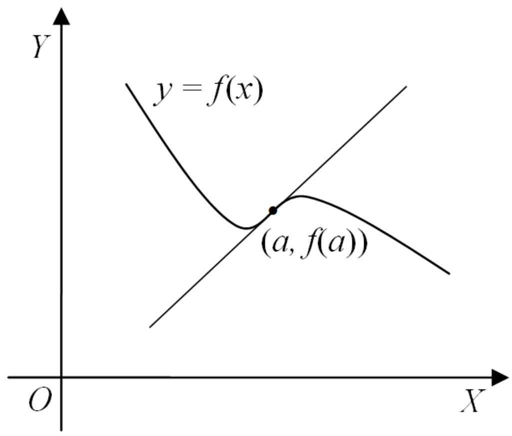


Fig. 14. Tangent line at an inflection point

Theorem (on the location of the tangent at an inflection point).

Let the function f be three times differentiable in some neighborhood of the point a and its third derivative is continuous at the point a. Let $f''(a) = 0$ and $f'''(a) \neq 0$. Then there exists a neighborhood U_a such that the tangent at the point a lies on different sides of the graph of the function on the left-hand and right-hand sides of this neighborhood.

Proof.

As in the proof of the previous theorem, we will use the equation of the tangent of the form $y = L_a(x)$, where the expression $L_a(x)$ is determined by formula (6). For definiteness, suppose that $f'''(a) > 0$ and show that in this case there exists a neighborhood U_a such that $f(x) - L_a(x) < 0$ for $x \in U_a^-$ and $f(x) - L_a(x) > 0$ for $x \in U_a^+$. This means that the tangent lies on different sides of the function graph in this neighborhood.

Since $f'''(a) > 0$ and, moreover, by condition, the third derivative is continuous at the point a, we obtain, by the theorem on the simplest properties of continuous functions, that there exists a neighborhood U_a of the point a, in which the function f''' preserves the sign, that is, in our case, $f'''(x) > 0$ for $x \in U_a$.

Let us write the expansion of the function f by Taylor's formula at the point a with the remainder term in the Lagrange form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(\xi)}{3!}(x-a)^3. \qquad (9)$$

Here ξ is some point lying between a and x. In particular, if $x \in U_a$, then the point ξ will also lie in this neighborhood.

The first two terms on the right-hand side of (9) coincide with $L_a(x)$ (see (6)). Replace them with $L_a(x)$ and move them to the left-hand side. In addition, we take into account that, by condition, $f''(a) = 0$. As a result, relation (9) takes the form:

$$f(x) - L_a(x) = \frac{f'''(\xi)}{3!}(x-a)^3. \tag{10}$$

We have chosen a neighborhood U_a in such a way that $f'''(x) > 0$ for $x \in U_a$. Since the point ξ also belongs to U_a when $x \in U_a$, we see that the factor $f'''(\xi)$ on the right-hand side of (10) is positive for all $x \in U_a$.

The difference $(x-a)$ on the right-hand side of (10) has an odd power, therefore the expression $(x-a)^3$ will be negative for $x \in U_a^-$ and it will be positive for $x \in U_a^+$. Therefore, the same estimates will be fulfilled for the left-hand side of (10): $f(x) - L_a(x) < 0$ for $x \in U_a^-$ and $f(x) - L_a(x) > 0$ for $x \in U_a^+$.

Similar reasonings allow us to prove that signs alternate for the expression $f(x) - L_a(x)$ in the case of $f'''(a) < 0$. □

Asymptotes[1]

Definition.

Let the function f be defined in a punctured neighborhood of the point x_0 (the neighborhood can be one-sided). The line $x = x_0$ is called a *vertical asymptote* of the graph of the function $y = f(x)$ if at least one of the following conditions is true:

$$\lim_{x\to x_0-0} f(x) = \infty, \qquad \lim_{x\to x_0+0} f(x) = \infty.$$

Let the function f be defined in a neighborhood of $+\infty$. The line $y = kx+b$ is called a *non-vertical asymptote* of the graph of the function $y = f(x)$, as $x \to +\infty$, if

$$\lim_{x\to+\infty}\big(f(x) - (kx+b)\big) = 0. \tag{11}$$

The non-vertical asymptote $y = kx + b$ as $x \to -\infty$ is defined similarly (provided that the function f is defined in a neighborhood of the point $-\infty$):

$$\lim_{x\to-\infty}\big(f(x) - (kx+b)\big) = 0.$$

If $k = 0$, then the non-vertical asymptote is called a *horizontal asymptote*, and if $k \neq 0$, then it is called an *oblique asymptote* or *slant asymptote*.

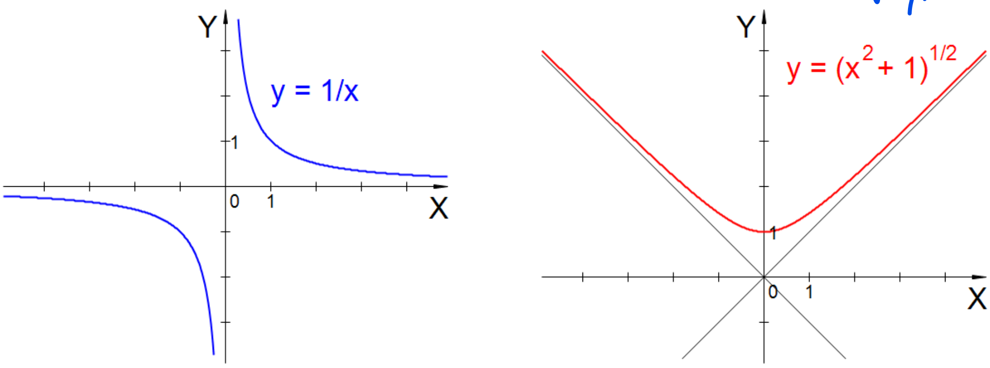
Examples.

The graph of the function $y = \frac{1}{x}$ has a vertical asymptote $x = 0$ and a horizontal asymptote $y = 0$ (see the left-hand part of Fig. 15). The graph

[1] This section is missing in video lectures.

of the function $y = \sqrt{x^2+1}$ has two oblique asymptotes: $y = x$ and $y = -x$ (see the right-hand part of Fig. 15). Note that both graphs are hyperbole branches.

$\sqrt{x^2+1}$



$k=1$

$k=-1$

Fig. 15. Examples of asymptotes

The graph of the function $y = \tan x$ has an infinite number of vertical asymptotes of the form $y = \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$ (see Fig. 4 in the section "Preliminary information"). The graph of the function $y = \arctan x$ has two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ (see Fig. 7 in Chapter 14), and the graph of the function $y = \operatorname{artanh} x$ has two horizontal asymptotes $y = -1$ and $y = 1$ (see Fig. 8 in Chapter 18).

THEOREM (CRITERION FOR THE EXISTENCE OF A NON-VERTICAL ASYMPTOTE).

For the line $y = kx + b$ to be an asymptote of the graph of the function $y = f(x)$ as $x \to +\infty$, it is necessary and sufficient that there exist finite limits

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k, \qquad \lim_{x\to+\infty} \big(f(x) - kx\big) = b. \tag{12}$$

REMARK.

$y=kx+b$

A similar criterion holds for the case $x \to -\infty$.

PROOF.

1. Necessity. Given: the line $y = kx + b$ is the asymptote of the graph of the function $y = f(x)$ for $x \to +\infty$. Prove: relations (12) holds.

Denote $\alpha(x) = f(x) - (kx + b)$. By the definition of the non-vertical asymptote (see (11)), we obtain that $\alpha(x) \to 0$ as $x \to +\infty$.

Then the function $f(x)$ can be represented as

$$f(x) = kx + b + \alpha(x). \tag{13}$$

Divide both sides of equality (13) by x:

$$\frac{f(x)}{x} = k + \frac{b}{x} + \frac{\alpha(x)}{x}.$$

Taking into account the property of the function $\alpha(x)$, we obtain that the limit of the right-hand side of the last equality, as $x \to +\infty$, is k, which implies the first of relations (12).

Now we transform equality (13) as follows:

$$f(x) - kx = b + \alpha(x).$$

The right-hand side of this equality, as $x \to +\infty$, is b, whence the second of relations (12) follows. The necessity is proven.

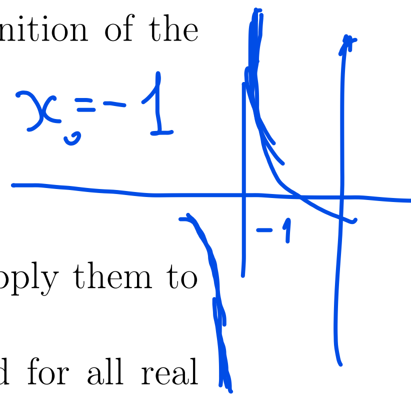
2. Sufficiency. Given: relations (12) holds. Prove: the line $y = kx + b$ is the asymptote of the graph of the function $y = f(x)$ for $x \to +\infty$.

Taking into account the second of relations (12), we obtain:

$$\lim_{x\to+\infty}\big(f(x) - (kx+b)\big) = \lim_{x\to+\infty}\big(f(x) - kx\big) - b = b - b = 0.$$

Thus, for the line $y = kx + b$, condition (11) from the definition of the non-vertical asymptote is satisfied. □

$x_0 = -1$



Example of a function study[2]

To illustrate the described methods of functions study, we apply them to the rational function $f(x) = \frac{x^2-3x-2}{x+1}$ and draw its graph.

1. Vertical asymptotes. The function $f(x)$ is defined for all real arguments, except for the point $x = -1$. At the point $x = -1$, the function has a discontinuity of the second kind, since

$\frac{2}{-0}$ $\frac{2}{+0}$

$$\lim_{x\to-1-1}\frac{x^2-3x-2}{x+1} = -\infty, \qquad \lim_{x\to-1+1}\frac{x^{-}3x-2}{x+1} = +\infty.$$

Therefore, the graph of the function has one vertical asymptote $x = -1$.

2. Non-vertical asymptotes. Let us use the criterion for the existence of non-vertical asymptotes:

$$k_\pm = \lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\frac{x^2-3x-2}{x^2+x} = 1,$$

$$b_\pm = \lim_{x\to\pm\infty}(f(x) - k_\pm x) = \lim_{x\to\pm\infty}\left(\frac{x^2-3x-2}{x+1} - x\right) =$$

$y = x - 4$

$$= \lim_{x\to\pm\infty}\frac{x^2-3x-2-x^2-x}{x+1} = \lim_{x\to\pm\infty}\frac{-4x-2}{x+1} = -4.$$

Thus, the graph of the function $f(x)$ has one non-vertical (more precisely, oblique) asymptote $y = x - 4$.

[2] This section is missing in video lectures.

$f(x) = 0$

3. Intersection points with coordinate axes. Since $f(0) = -2$, the graph intersects the OY axis at a single point $(0, -2)$. To find the intersection points with the OX axis, we solve the quadratic equation $x^2 - 3x - 2 = 0$:

$$x_{1,2} = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}.$$

Using the approximate value of 4.1 for $\sqrt{17}$, we get $x_1 \approx \frac{3-4.1}{2} = -0.55$, $x_2 \approx \frac{3+4.1}{2} = 3.55$. Thus, the graph intersects the OX axis at points whose approximate coordinates are $(-0.6, 0)$, $(3.6, 0)$.

4. Critical points. Find the first derivative of this function:

$$f'(x) = \left(\frac{x^2 - 3x - 2}{x+1}\right)' = \frac{(2x-3)(x+1) - (x^2 - 3x - 2)\cdot 1}{(x+1)^2} =$$

$$= \frac{x^2 + 2x - 1}{(x+1)^2}.$$

$f'(x) = 0$

Now we can find the critical points of the function f, i. e., points at which the derivative f' vanishes. To do this, solve the quadratic equation $x^2 + 2x - 1 = 0$:

$$x^*_{1,2} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}.$$

Using the approximate value of 1.4 for $\sqrt{2}$, we get $x^*_1 \approx -1 - 1.4 = -2.4$, $x^*_2 \approx -1 + 1.4 = 0.4$.

5. Intervals of monotonicity and local extrema. Since in the formula for the derivative $f'(x)$ the denominator $(x+1)^2$ is non-negative, and the numerator $x^2 + 2x - 1$ takes positive values for $x \in (-\infty, x^*_1) \cup (x^*_2, +\infty)$ and negative values for $x \in (x^*_1, x^*_2)$, we obtain, by virtue of the first sufficient condition for the existence of a local extremum, that the point x^*_1 is a local maximum point (the sign of the derivative changes from "+" to "−" in a neighborhood of this point), and the point x^*_2 is a local minimum point (sign the derivative changes from "−" to "+" in a neighborhood of this point).

Let us also calculate the values of the function f at the points of local extrema:

$$f(x^*_1) = \frac{(-1-\sqrt{2})^2 - 3(-1-\sqrt{2}) - 2}{(-1-\sqrt{2}) + 1} =$$

$$= \frac{1 + 2\sqrt{2} + 2 + 3 + 3\sqrt{2} - 2}{-\sqrt{2}} = -\frac{4 + 5\sqrt{2}}{\sqrt{2}} =$$

$$= -2\sqrt{2} - 5 \approx -2 \cdot 1.4 - 5 = -7.8,$$

$$f(x_2^*) = \frac{(-1+\sqrt{2})^2 - 3(-1+\sqrt{2}) - 2}{(-1+\sqrt{2})+1} =$$
$$= \frac{1 - 2\sqrt{2} + 2 + 3 - 3\sqrt{2} - 2}{\sqrt{2}} = \frac{4 - 5\sqrt{2}}{\sqrt{2}} =$$
$$= 2\sqrt{2} - 5 \approx 2 \cdot 1.4 - 5 = -2.2.$$

So the coordinates of the local maximum and local minimum of the function f are approximately equal to $(-2.4, -7.8)$ and $(0.4, -2.2)$.

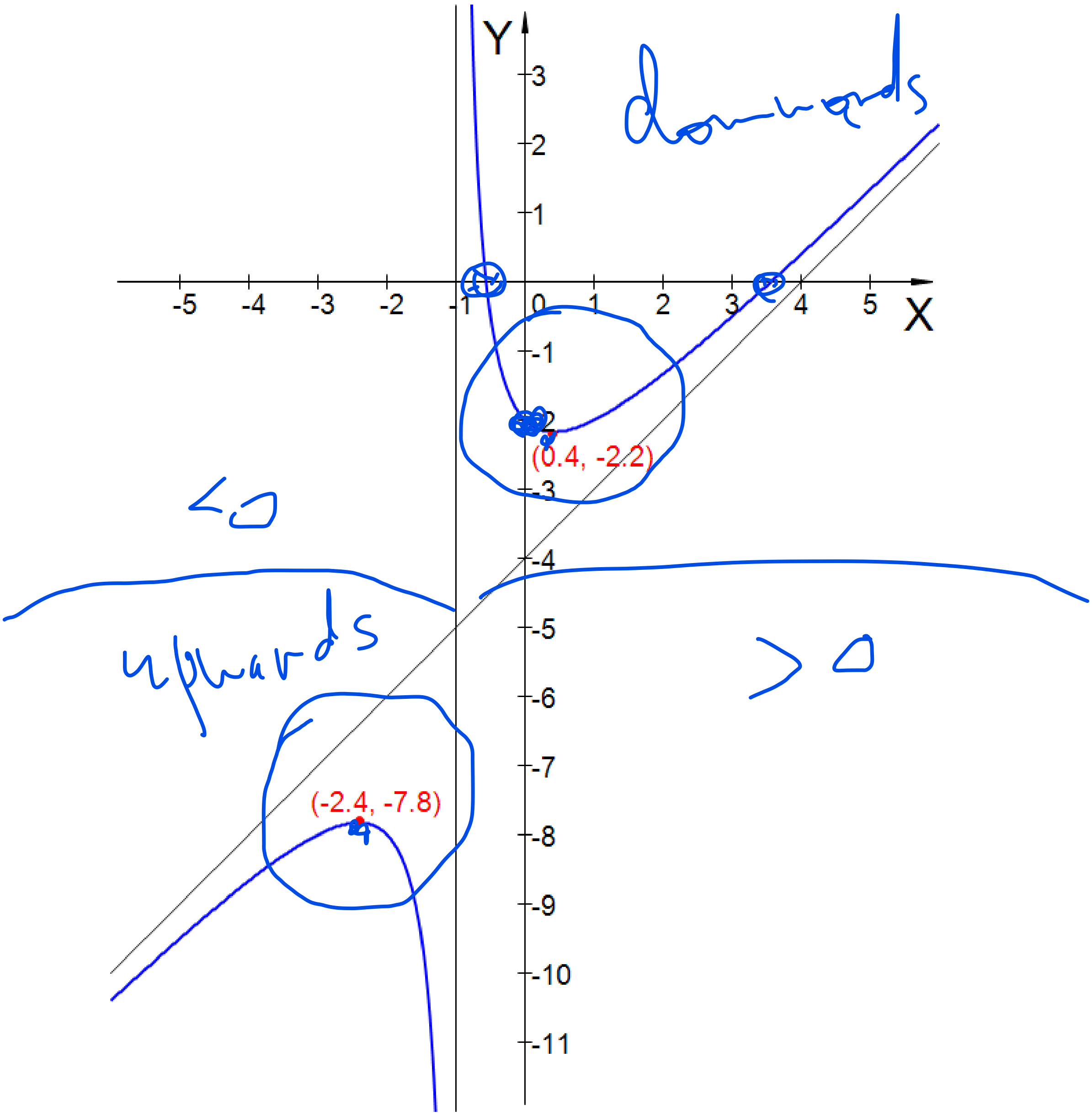


Fig. 16. Graph of the function $f(x) = \frac{x^2-3x-2}{x+1}$

6. Intervals of convexity and inflection points. To find the intervals of convexity and inflection points of the function f, we find its second derivative:

$$
\begin{aligned}
f''(x) &= \big(f'(x)\big)' = \Big(\frac{x^2+2x-1}{(x+1)^2}\Big)' = \\
&= \frac{(2x+2)(x+1)^2-(x^2+2x-1)\cdot 2(x+1)}{(x+1)^4} = \\
&= \frac{2(x+1)^2-2(x^2+2x-1)}{(x+1)^3} = \frac{4}{(x+1)^3}.
\end{aligned}
$$

Thus, $f''(x)<0$ for $x\in(-\infty,-1)$ and $f''(x)>0$ for $x\in(-1,+\infty)$. By virtue of the sufficient condition for convexity, we obtain that the function f is convex upward on the interval $(-\infty,-1)$ and the function is convex downward on the interval $x\in(-1,+\infty)$. The function f does not have inflection points.

The graph of the function f is shown in Fig. 16. The figure also shows the asymptotes $x=-1$, $y=x-4$ and points of the local extremum $(-2.4,-7.8)$, $(0.4,-2.2)$.