5. Definite integral and Darboux sums

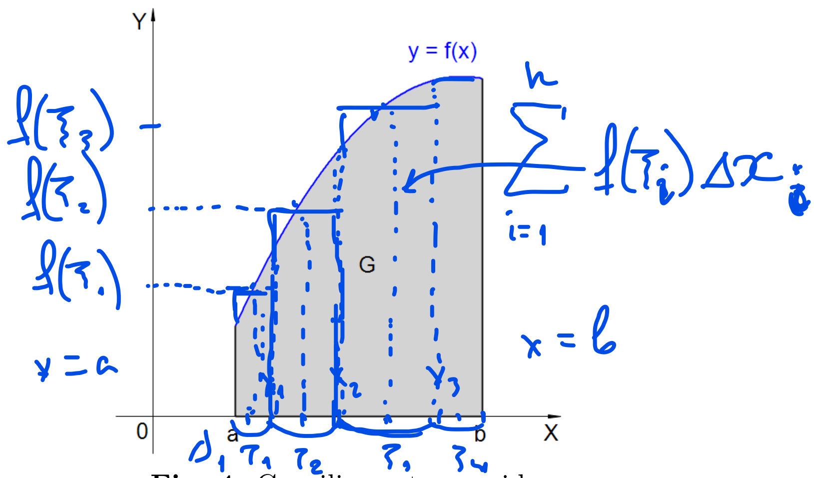
Definite integral

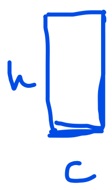
The problem of finding the area of a curvilinear trapezoid

`2.4A/00:00 (07:00)`

The basic concepts related to a definite integral can be considered by the example of the geometric problem of finding the area of a curvilinear trapezoid.

Let a function $f(x)$ be defined on the segment (closed interval) $[a, b]$ and taking positive values on this segment: $f(x) > 0$, $x \in [a, b]$. It is required to find the area of the figure G bounded by the OX axis, the vertical lines $x = a$ and $x = b$, and the graph of the function $y = f(x)$. Such a figure is called a *curvilinear trapezoid* with the base $[a, b]$ (Fig. 4).





$S = c \cdot d$

$\xi_i \in \Delta_i$

Fig. 4. Curvilinear trapezoid

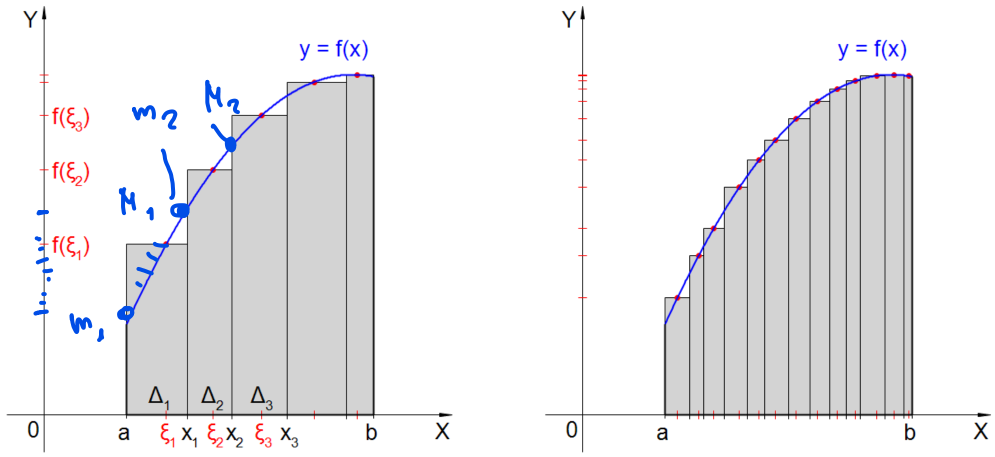
How to find the approximate area of a curvilinear trapezoid?

Let us divide the segment $[a, b]$ into smaller segments (not necessarily of equal length) with endpoints $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. For brevity, we denote the obtained segments as follows: $\Delta_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$. We also introduce the notation for the length of the segment Δ_i: $\Delta x_i = x_i - x_{i-1}$, $i = 1, \ldots, n$.

Choose a point ξ_i on each of the segments Δ_i: $\xi_i \in \Delta_i$, $i = 1, \ldots, n$.

Provided that the function f has sufficiently "good" properties, we can assume that the area of the curvilinear trapezoid with the base Δ_i will be close to the area of the rectangle with the same base Δ_i and a height equal to the value of the function f at the point ξ_i. The area of this rectangle is $f(\xi_i)\Delta x_i$.

Summing up the areas of all such rectangles, we get the approximate value of the area of the initial curvilinear trapezoid: $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ (see the left-hand part of Fig. 5).



$\Delta x_i \to 0$

$n \to \infty$

Fig. 5. Curvilinear trapezoid approximation by a set of rectangles

As the number of points x_i increases, the resulting union of the rectangles will be even closer to the initial curvilinear trapezoid (see the right-hand part of Fig. 5).

$\int_a^b f(x)dx$

If the expression $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ has a limit as the number of points x_i unlimitedly increases (and, accordingly, as the length of all segments Δ_i unlimitedly decreases), then it is natural to consider this limit as the area of the initial curvilinear trapezoid.

It is this limit that is called the *definite integral* of the function f over the segment $[a, b]$.

Definition of a definite integral

`2.4A/07:00 (14:01)`

Definition.

Let the function f be defined on the segment $[a, b]$. The *partition* T of the segment $[a, b]$ is the set of points x_i, $i = 0, \ldots, n$, which has the following property:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

For the segments $[x_{i-1}, x_i]$ with endpoints at adjacent points of the partition T, as well as for their lengths $x_i - x_{i-1}$, we will use the notation introduced above:

$$\Delta_i \overset{\text{def}}{=} [x_{i-1}, x_i], \quad \Delta x_i \overset{\text{def}}{=} x_i - x_{i-1}, \quad i = 1, \ldots, n.$$

Obviously, $\Delta x_i > 0$.

The *mesh* of the partition T (notation $l(T)$) is the maximum of the lengths of the segments Δ_i:

$$l(T) \overset{\text{def}}{=} \max_{i=1,\ldots,n} \Delta x_i.$$

A *sample* ξ constructed on the basis of a partition T is an arbitrary set of points $\xi_i \in \Delta_i$, $i = 1, \ldots, n$.

The *integral sum* $\sigma_T(f, \xi)$ of the function f by the partition T and the sample ξ is the following expression:

? $I = \lim\limits_{x \to 0} f(x)$ $\quad 0 < |x| < \delta$

$x \neq 0$

$$\sigma_T(f, \xi) \overset{\text{def}}{=} \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \ 0 < |x| < \delta : |f(x) - I| < \varepsilon$

A function f is called *Riemann integrable* on the segment $[a, b]$ if there exists a number I such that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall T, l(T) < \delta, \quad \forall \xi \quad |\sigma_T(f, \xi) - I| < \varepsilon. \tag{1}$$

Briefly, condition (1) can be written using the limit notation:

$$I = \lim_{l(T) \to 0, \forall \xi} \sigma_T(f, \xi).$$

The number I is called the *Riemann integral*, or the *definite integral*, of the function f over the segment $[a, b]$, and it is denoted as follows: $\int_a^b f(x)\,dx$.

So, the Riemann integral of the function f is the limit of the integral sums $\sigma_T(f, \xi)$ as $l(T) \to 0$, $\forall \xi$, if this limit exists:

$$\int_a^b f(x)dx \overset{\text{def}}{=} \lim_{l(T) \to 0, \forall \xi} \sigma_T(f, \xi).$$

In what follows, Riemann integrability and the Riemann integral will be called simply integrability and integral, respectively.

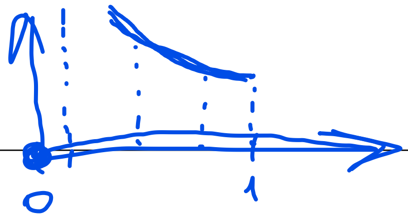
Remarks.

1. Although the limit of integral sums used in the definition of the integral I differs from the usual limit of a function at a point, it is easy to prove, using condition (1), that this limit satisfies both the theorem on arithmetic properties of the limit and the theorem on passing to the limit in the inequalities. We will use these theorems in the next chapter to prove the properties of a definite integral.

2. One can extend the class of integrable functions by giving another definition of integrability. This version of integrability is called *Lebesgue integrability*. Lebesgue integrability is not considered in this book.

$1000000 = M$: $\quad |f(x)| < M \quad x = \dfrac{1}{1000000\ldots}$

$f(x) > M$



$f(x) = \begin{cases} 1/x, & x \in (0,1] \\ 0, & x = 0 \end{cases}$ is not bounded on $[0,1]$

$x \to 0$, $f(x) \to +\infty$

A necessary condition for integrability

2.4A/21:01 (11:50)

Theorem (a necessary condition for integrability).

If the function is integrable on a segment, then it is bounded on this segment.

(f is integrable on $[a,b]$) $\Rightarrow$ (f is bounded on $[a,b]$)

Proof.

Let the function f be integrable on the segment $[a, b]$. This means that there exists a number I for which condition (1) is satisfied. In this condition, we choose the value of ε, setting it equal to 1. Then there exists a value $\delta > 0$ such that the following estimate holds for any partition T of the segment $[a, b]$ satisfying the additional condition $l(T) < \delta$ and any sample ξ constructed on the basis the partition T:

$$|\sigma_T(f, \xi) - I| < 1. \tag{2}$$

We select some partition T satisfying the condition $l(T) < \delta$.

Let us prove the statement of the theorem by contradiction: suppose that the function f is not bounded on $[a, b]$. This means that it is unbounded on at least one segment Δ_i associated with the previously selected partition T. For definiteness, we assume that such a segment is the segment Δ_1.

From the integral sum $\sigma_T(f, \xi)$, we extract the term associated with this segment:

$$\sigma_T(f, \xi) = f(\xi_1)\Delta x_1 + \sum_{i=2}^{n} f(\xi_i)\Delta x_i. \tag{3}$$

Let us fix all the elements of the sample ξ except for the first one, i. e., let us fix the values $\xi_2, \xi_3, \ldots, \xi_n$. In this case, the second term on the right-hand side of equality (3) will be uniquely determined. Denote the value of this term by A:

$$A = \sum_{i=2}^{n} f(\xi_i)\Delta x_i.$$

Then inequality (2) can be transformed as follows:

$$I - 1 < \sigma_T(f, \xi) < I + 1,$$

$$I - 1 < f(\xi_1)\Delta x_1 + A < I + 1.$$

In the resulting relation, we move A from the middle part to the left-hand and right-hand part, after which we divide all parts of the double inequality by Δx_1 (this can be done since $\Delta x_1 > 0$):

$$\frac{I - 1 - A}{\Delta x_1} < f(\xi_1) < \frac{I + 1 - A}{\Delta x_1}. \tag{4}$$

In this double inequality, all values are fixed except for the point ξ_1, which can vary within the segment Δ_1. Thus, we have obtained that the double inequality (4) holds for all $\xi_1 \in \Delta_1$, which implies that the function f is bounded on the segment Δ_1. But this contradicts our assumption that the function f is unbounded on this segment. The obtained contradiction proves the theorem. □

REMARKS.

1. Taking into account this theorem, we will consider only bounded functions hereinafter, not always noting this condition.

2. The converse of the proved theorem is false: if the function is bounded, then this does not follow that it is integrable. We give a corresponding example at the end of this chapter.

Darboux sums and Darboux integrals

Definition of Darboux sums

2.4A/32:51 (05:52)

DEFINITION.

Let the function f be defined and bounded on the segment $[a, b]$.

We choose some partition T of this segment and introduce the following notation:

$$M_i = \sup_{x\in\Delta_i} f(x), \quad m_i = \inf_{x\in\Delta_i} f(x), \quad i = 1, \dots, n.$$

$m_i \le M_i$

Since the function f is bounded on $[a, b]$, the values M_i and m_i exist for all $i = 1, \dots, n$.

The *upper Darboux sum* $S_T^+(f)$ and the *lower Darboux sum* $S_T^-(f)$ are defined as follows:

$\xi_i \in \Delta_i$

$$S_T^+(f) \overset{\text{def}}{=} \sum_{i=1}^{n} M_i \Delta x_i, \quad S_T^-(f) \overset{\text{def}}{=} \sum_{i=1}^{n} m_i \Delta x_i.$$

$\sigma_T(f, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$

If it is clear which function f is associated with Darboux sums, then the short notation S_T^+ and S_T^- can be used for them.

REMARK.

The main difference between Darboux sums and integral sums is that the notion of sample ξ is not used in the definition of Darboux sums: Darboux sums depend only on the function f itself and the partition T of the initial segment.

The simplest properties of Darboux sums

`2.4A/38:43 (01:33), 2.4B/00:00 (12:05)`

In the formulations of all properties, it is assumed that the function f is defined and bounded on the segment $[a, b]$.

1. Let T be some partition of the segment $[a, b]$, ξ be an arbitrary sample associated with this partition. Then the following double inequality holds:

$$S_T^-(f) \le \sigma_T(f, \xi) \le S_T^+(f). \qquad \xi_i \in \Delta_i \qquad (5)$$

PROOF.

From the definition of the supremum M_i and the infimum m_i, it follows

$$\sum_{i=1}^{n} \qquad \forall \xi_i \quad m_i \le f(\xi_i) \le M_i. \qquad (6)$$

Δx_i

Multiply all parts of the double inequality (6) by $\Delta x_i > 0$ and take a sum of all inequalities for $i = 1, \ldots, n$:

$$\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(\xi_i) \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i.$$

Given the definitions of the integral sum and Darboux sums, we obtain (5). □

2. For a fixed partition T of the segment $[a, b]$, the following relations hold:

$$\boxed{S_T^+(f)} = \sup_{\xi} \sigma_T(f, \xi), \qquad S_T^-(f) = \inf_{\xi} \sigma_T(f, \xi).$$

PROOF.

Let us prove this property for the upper Darboux sum. Given the definition of supremum, it is necessary to prove two statements:

1) $\forall \xi \quad \sigma_T(f, \xi) \le S_T^+(f)$,
2) $\forall \varepsilon > 0 \quad \exists \xi' \quad \sigma_T(f, \xi') > S_T^+(f) - \varepsilon$.

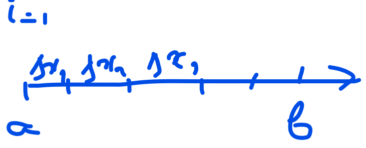
Statement 1 has already been proved (see property 1). Let us prove statement 2. From the definition of the supremum M_i, it follows

$$\forall \varepsilon > 0 \quad \exists \xi_i' \in \Delta_i \quad f(\xi_i') > M_i - \frac{\varepsilon}{b-a}. \qquad \cdot \Delta x_i \qquad \sum_{i=1}^{n} \qquad (7)$$

Multiply both sides of inequality (7) by $\Delta x_i > 0$ and take a sum of all inequalities for $i = 1, \ldots, n$:

$$\underbrace{\sum_{i=1}^{n} f(\xi_i') \Delta x_i}_{\sigma_T(f,\xi')} > \underbrace{\sum_{i=1}^{n} M_i \Delta x_i}_{S_T^+(f)} - \underbrace{\sum_{i=1}^{n} \frac{\varepsilon}{b-a} \Delta x_i}_{= \frac{\varepsilon}{b-a}(b-a)}. \qquad \varepsilon$$

$$\sum_{i=1}^{n} \Delta x_i = b - a$$



Given the definitions of the integral sum and the upper Darboux sum, as well as the fact that $\sum_{i=1}^{n} \frac{\varepsilon}{b-a} \Delta x_i = \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$, we obtain statement 2.

The property for the lower Darboux sum is proved similarly, using the definition of the infimum. □

Darboux sum property related to refinement of a partition

2.4B/12:05 (15:11)

Before stating the next property, we introduce the concept of refinement of a partition.

Definition.

The partition T_2 is called the *refinement* of the partition T_1 if any element of the partition T_1 belongs to the partition T_2, i. e., $T_1 \subset T_2$. In other words, the refinement T_2 of the partition T_1 contains all points of the partition T_1 and possibly some other points of the original segment.

3. If the partition T_2 is a refinement of the partition T_1, then the following chain of inequalities holds:

$T_1 \subset T_2$

$$S_{T_1}^- \le S_{T_2}^- \le S_{T_2}^+ \le S_{T_1}^+. \tag{8}$$

Proof.

The middle inequality in (8) immediately follows from property 1. Let us prove the right-hand inequality: $S_{T_2}^+ \le S_{T_1}^+$.

It is enough for us to consider the case when the refinement T_2 of the partition T_1 differs from T_1 by only one additional point. The case when there are several additional points can be reduced to the case with one point if we add these points to the partition sequentiallyand apply the proved estimate to the resulting refinements.

So, we assume that the refinement T_2 contains one additional point x': $T_1 = \{x_i, i = 0, \dots, n\}$, $T_2 = T_1 \cup \{x'\}$. For definiteness, we also assume that $x' \in \Delta_1$, i. e., $x_0 < x' < x_1$. We also introduce the following notation:

$$\Delta_1' = [x_0, x'], \quad \Delta x_1' = x' - x_0, \quad M_1' = \sup_{x \in \Delta_1'} f(x),$$

$$\Delta_1'' = [x', x_1], \quad \Delta x_1'' = x_1 - x', \quad M_1'' = \sup_{x \in \Delta_1''} f(x).$$

We need to prove that

$$S_{T_1}^+ - S_{T_2}^+ \ge 0. \tag{9}$$

The indicated Darboux sums contain the coinciding terms $M_i\Delta x_i$ for $i = 2, \dots, n$. After reducing these coinciding terms, the difference $S^+_{T_1} - S^+_{T_2}$ takes the following form:

$$S^+_{T_1} - S^+_{T_2} = M_1\Delta x_1 - (M'_1\Delta x'_1 + M''_1\Delta x''_1). \tag{10}$$

Since $\sup A \le \sup B$ for $A \subset B$ and in our case $\Delta'_1 \subset \Delta_1$ and $\Delta''_1 \subset \Delta_1$, we get

$$M'_1 \le M_1, \quad M''_1 \le M_1.$$

Therefore, the right-hand side of equality (10) can be estimated as follows:

$$M_1\Delta x_1 - (M'_1\Delta x'_1 + M''_1\Delta x''_1) \ge M_1\Delta x_1 - (M_1\Delta x'_1 + M_1\Delta x''_1).$$

The right-hand side of the last inequality is 0, since $\Delta x_1 = \Delta x'_1 + \Delta x''_1$. We proved the validity of inequality (9) and thereby the validity of the right-hand inequality in (8).

The left-hand inequality in (8) is proved similarly, by taking into account the following property of the infimum: $\inf A \ge \inf B$ for $A \subset B$. □

Darboux sums associated with different partitions `2.4B/27:16 (05:14)`

4. If T' and T'' are some partitions of the segment $[a, b]$, then the estimate holds:

$$S^-_{T'} \le S^+_{T''}. \tag{11}$$

Thus, any lower Darboux sum of the function f is less than or equal to any of its upper Darboux sums.

PROOF.

Consider the union of two given partitions: $T = T' \cup T''$. The resulting partition T is a refinement of both the partition T' and the partition T''. Therefore, applying property 3, we obtain the following chain of inequalities:

$$S^-_{T'} \le S^-_T \le S^+_T \le S^+_{T''}. \tag{12}$$

In this case, we applied the left-hand inequality from (8) for T' and its refinement T, the middle inequality from (8) for T, and the right-hand inequality from (8) for T'' and its refinement T.

Consequently, the boundary terms of the obtained chain of inequalities (12) satisfy inequality (11). □

Darboux integrals `2.4B/32:30 (07:05)`

5. There exist values $I^-(f) = \sup_T S^-_T(f)$, $I^+(f) = \inf_T S^+_T(f)$ and the following estimate holds for them:

$S_T^- \le I^- \le I^+ \le S_T^+$

$$I^-(f) \le I^+(f). \qquad (13)$$

PROOF.

Consider the previously proved inequality (11), fix the partition T'' in it, and consider the arbitrary partition T of the segment $[a, b]$ as the partition T':

$$S_T^-(f) \le S_{T''}^+(f).$$

This inequality means that the set of all lower Darboux sums over arbitrary partitions T is bounded from above by $S_{T''}^+(f)$. Therefore, the set of all lower Darboux sums for the function f is bounded from above, which means that it has the least upper bound $I^-(f)$.

Since the value $S_{T''}^+(f)$ is the upper bound for the set of all lower Darboux sums and the value $I^-(f)$ is the least upper bound for these sums, we obtain the following inequality:

$$I^-(f) \le S_{T''}^+(f).$$

In the last inequality, we can assume that T'' is an arbitrary partition of the segment $[a, b]$. Therefore, the set of all upper Darboux sums of the function f over an arbitrary partition T'' is bounded from below by the value $I^-(f)$. So, this set has the greatest lower bound $I^+(f)$.

The estimate (13) follows from the fact that the quantity $I^-(f)$ is the lower bound for the set of all upper Darboux sums and the value $I^+(f)$ is the greatest lower bound for these sums.

DEFINITION.

The values $I^-(f)$ and $I^+(f)$ are called the *lower and upper Darboux integrals* for the function f on the segment $[a, b]$, respectively. Thus, by virtue of property 5, any bounded function has the lower and upper Darboux integrals and inequality (13) holds for them.

Integrability criterion in terms of Darboux sums

Formulation of the integrability criterion

`2.5A/00:00 (09:13)`

THEOREM (INTEGRABILITY CRITERION IN TERMS OF DARBOUX SUMS).

The function f is integrable on the segment $[a, b]$ if and only if two conditions are satisfied:

$I^+ = I^-$

1) f is bounded on $[a, b]$,

2) $\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, T, l(T) < \delta, \quad S_T^+(f) - S_T^-(f) < \varepsilon.$

1 2 2

Df: $\forall \varepsilon > 0\ \exists \delta > 0\ \forall T, l(T) < \delta,\ \forall \xi\ \ |\sigma_T(f, \xi) - I| < \varepsilon$

Remark.

Condition 2 of the theorem can be written as follows:

$$\lim_{l(T)\to 0}\left(S_T^+(f) - S_T^-(f)\right) = 0.$$

Proof of necessity 2.5A/09:13 (09:43)

Given: the function f is integrable on $[a, b]$. Prove: conditions 1 and 2 are satisfied.

The validity of condition 1 follows from the necessary condition for integrability. It remains for us to prove the validity of condition 2.

Since the function f is integrable, the following limit exists:

$$\lim_{l(T)\to 0,\forall\,\xi} \sigma_T(f, \xi) = I.$$

We choose some value of $\varepsilon > 0$. Due to the integrability of the function f, we obtain

$$\exists\,\delta > 0 \quad \forall\,T, l(T) < \delta, \quad \forall\,\xi \quad |\sigma_T(f, \xi) - I| < \frac{\varepsilon}{3}. \tag{14}$$

Let us show that the choice of the same value δ ensures the fulfillment of condition 2 of the theorem.

Transform estimate (14) as follows:

$$I - \frac{\varepsilon}{3} < \sigma_T(f, \xi) < I + \frac{\varepsilon}{3}. \tag{15}$$

This double estimate is valid for any sample ξ. Thus, we have lower and upper bounds for the set of integral sums $\sigma_T(f, \xi)$ for a fixed partition T and any sample ξ.

By property 2 of Darboux sums, we have

$$S_T^+(f) = \sup_{\xi} \sigma_T(f, \xi), \qquad S_T^-(f) = \inf_{\xi} \sigma_T(f, \xi).$$

Since the expressions $I - \frac{\varepsilon}{3}$ and $I + \frac{\varepsilon}{3}$ are, by virtue of (15), the lower and upper bounds of the integral sums, respectively, and $S_T^-(f)$ and $S_T^+(f)$ are, by virtue of property 2 of Darboux sums, the greatest lower bound and the least upper bound of the integral sums, we obtain the following chain of inequalities:

$$I - \frac{\varepsilon}{3} \le S_T^-(f) \le S_T^+(f) \le I + \frac{\varepsilon}{3}. \tag{16}$$

Since the distance between the internal terms of the triple inequality (16) cannot exceed the distance between its external terms, the following estimate follows from this triple inequality:

$$S_T^+(f) - S_T^-(f) \le \left(I + \frac{\varepsilon}{3}\right) - \left(I - \frac{\varepsilon}{3}\right). \tag{17}$$

Simplify the right-hand side:

$$\left(I + \frac{\varepsilon}{3}\right) - \left(I - \frac{\varepsilon}{3}\right) = \frac{2\varepsilon}{3} < \varepsilon.$$

Thus, estimate (17) can be rewritten in the form

$$S_T^+(f) - S_T^-(f) < \varepsilon.$$

We obtained an estimate from condition 2 of the theorem. The necessity is proven.

Proof of sufficiency

`2.5A/18:56 (14:25)`

Given: conditions 1 and 2 are satisfied. Prove: the function f is integrable on $[a, b]$.

Condition 1 (i. e., the boundedness of the function f) is required only to guarantee the existence of lower and upper Darboux sums for the function f.

By condition 2, for any $\varepsilon > 0$, there exists a value $\delta > 0$ such that, for all partitions T with mesh $l(T) < \delta$, the estimate holds:

$$S_T^+(f) - S_T^-(f) < \varepsilon. \tag{18}$$

On the other hand, for any partition T, by virtue of property 5 of the Darboux sums, the following triple estimate holds:

$$S_T^-(f) \le I^-(f) \le I^+(f) \le S_T^+(f).$$

This estimate implies the inequality

$$I^+(f) - I^-(f) \le S_T^+(f) - S_T^-(f).$$

Given (18), we obtain

$$I^+(f) - I^-(f) < \varepsilon.$$

The left-hand side of the resulting inequality does not depend on ε, therefore, this inequality can be true for arbitrary $\varepsilon > 0$ only if $I^+(f) - I^-(f) = 0$, i. e., $\boxed{I^+(f) = I^-(f).}$ $= I$

Thus, we have proved that, under condition 2 of the theorem, the lower and upper Darboux integrals coincide. We denote their value by I and show that the value of I is equal to the integral of the function f on the segment $[a, b]$, i. e., that the following condition is true:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, T, l(T) < \delta, \quad \forall\, \xi \quad |\sigma_T(f, \xi) - I| < \varepsilon. \tag{19}$$

We choose the value $\varepsilon > 0$ and select the value $\delta > 0$ from it using condition 2 of the theorem. Then, for any partition T satisfying the condition $l(T) < \delta$, estimate (18) holds.

Using property 1 of the Darboux sums, we obtain

$$\forall\,\xi \quad S_T^-(f) \le \sigma_T(f,\xi) \le S_T^+(f).$$

In addition, by virtue of property 5 of the Darboux sums, we have

$$S_T^-(f) \le I \le S_T^+(f).$$

Thus, the values of $\sigma_T(f,\xi)$ and I are between the values of $S_T^-(f)$ and $S_T^+(f)$. Therefore, the following estimate is true:

$$\forall\,\xi \quad |\sigma_T(f,\xi) - I| \le S_T^+(f) - S_T^-(f).$$

Given that $S_T^+(f) - S_T^-(f) < \varepsilon$, we finally get

$$\forall\,\xi \quad |\sigma_T(f,\xi) - I| < \varepsilon.$$

We proved that condition (19) is satisfied for the function f; therefore, the function f is integrable. $\square$

Corollary of the criterion and an example of a non-integrable function

`2.5A/33:21 (08:48)`

Corollary.

If the function f is integrable on the segment $[a,b]$, then its upper and lower Darboux integrals coincide and, moreover, they are equal to the integral of the function f over the segment $[a,b]$.

Proof.

If the function is integrable, then condition 2 of the theorem is fulfilled for it, which implies both the coincidence of the upper and lower Darboux integrals and their equality to the integral of this function (see the proof of sufficiency). $\square$

$I^+(f) \neq I^-(f) \Rightarrow f$ is not integrable

Remark.

It follows from the corollary that if the upper and lower Darboux integrals are different, then the function is not integrable.

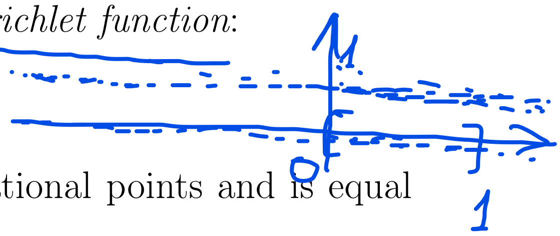
An example of a bounded function that is not integrable.

$x = \frac{m}{n}$

We define the following function, called the *Dirichlet function*:

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

$\sqrt{2}, \pi$



Thus, the Dirichlet function is equal to 1 at rational points and is equal to 0 at irrational points.

$|D(x)| \le 1$

This function is bounded. However, it is not integrable on any segment $[a, b]$ of nonzero length. We show this for the segment $[0, 1]$.

Let us accept without proof the following fact: in any arbitrarily small neighborhood of any rational number, there exists some irrational number and vice versa, in any arbitrarily small neighborhood of any irrational number, there exists some rational number. Therefore, any segment of nonzero length necessarily contains both irrational and rational numbers. This means that, for any partition T of the segment $[0, 1]$, the following relations hold:

$T\ [0,1]$

$$m_i = \inf_{x \in \Delta_i} D(x) = 0, \quad M_i = \sup_{x \in \Delta_i} D(x) = 1.$$

$\forall \Delta_i\ \exists x \in \Delta_i,\ x \in \mathbb{Q}$

$\exists y \in \Delta_i : y \in \mathbb{R} \setminus \mathbb{Q}$

Then, for Darboux sums of the Dirichlet function over any partition T of the segment $[0, 1]$, we have

$$S_T^+(D) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1,$$

$$S_T^-(D) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0.$$

Similar relations hold for Darboux integrals:

$$I^+(D) = \inf_T S_T^+(D) = \inf_T 1 = 1,$$

$$I^-(D) = \sup_T S_T^-(D) = \sup_T 0 = 0.$$

We proved that $I^-(D) \neq I^+(D)$, therefore, the Dirichlet function is not integrable on the interval $[0, 1]$.