6. Classes of integrable functions. Properties of a definite integral

Classes of integrable functions

The simplest example of an integrable function: a constant function

`2.5B/00:00 (02:05)`

Consider the constant function $f(x) = c$ and show that it is integrable on any segment $[a, b]$.

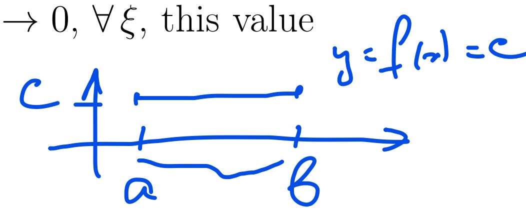
To do this, we calculate the integral sum for the function f on this segment:

$\sigma_T(f, \xi)$

$$\sigma_T(\xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = c\sum_{i=1}^{n} \Delta x_i = c(b-a).$$

$\longrightarrow c(b-a)$

Thus, for any partition T and any sample ξ, the integral sum takes the same value, therefore, when passing to the limit as $l(T) \to 0$, $\forall\,\xi$, this value will not change.



We have proved that

$$\int_a^b c\,dx = c(b-a).$$

Oscillation of a function and its use in integrability criterion

`2.5B/02:05 (04:12)`

We noted earlier that the condition for integrability criterion in terms of Darboux sums can be written as follows:

$$\lim_{l(T)\to 0} (S_T^+ - S_T^-) = 0.$$

Using the definition of Darboux sums, we can transform an expression under the limit sign:

$$S_T^+ - S_T^- = \sum_{i=1}^{n} M_i\Delta x_i - \sum_{i=1}^{n} m_i\Delta x_i = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i.$$

Under the sum sign, the expression $M_i - m_i$ arises, which determines the maximum difference of the values of the function f on the segment Δ_i. This characteristic is called the *oscillation* of the function f on the segment Δ_i and is denoted by $\omega_i(f)$:

$$\omega_i(f) \stackrel{\text{def}}{=} M_i - m_i.$$

$\Delta_i = [x_{i-1}, x_i]$

Thus, the condition from the criterion of integrability of the function can be represented as follows:

$$\lim_{l(T)\to 0} \sum_{i=1}^{n} \omega_i(f)\Delta x_i = 0.$$

$\sup_{\Delta_i} f(x)$ $\inf_{\Delta_i} f(x)$

REMARK.

It can be proved that the following formula holds for the oscillation of a function:

$$\omega_i(f) = \sup_{x',x''\in\Delta_i} |f(x') - f(x'')|. = M_i - m_i \tag{1}$$

We will use this formula to prove the integrability of the product of functions.

Integrability of continuous functions `2.5B/06:17 (13:33)`

THEOREM (INTEGRABILITY THEOREM FOR CONTINUOUS FUNCTIONS).

If the function is continuous on a segment, then it is integrable on this segment.

REMARK.

Continuity is not a necessary condition for integrability. An integrable function may have points of discontinuity.

PROOF.

Let the function f be continuous on $[a, b]$. Let us prove that the conditions of the integrability criterion are satisfied for it.

Condition 1 of the criterion (boundedness of a function on $[a, b]$) follows from the first Weierstrass theorem, which states that any function continuous on a segment is bounded on this segment.

To prove condition 2 of the criterion, we use Cantor's theorem, which states that a function continuous on an segment is uniformly continuous on this segment.

Let us write the definition of uniform continuity for the function f on the segment $[a, b]$ in the following form:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x', x'' \in [a, b], |x' - x''| < \delta,$$
$$|f(x') - f(x'')| < \frac{\varepsilon}{b-a}. \tag{2}$$

We choose the value $\varepsilon > 0$, select the value $\delta > 0$ from (2) and show that condition 2 of the integrability criterion will be satisfied for the given value δ, i. e., that, for all partitions T such that $l(T) < \delta$, the estimate $S_T^+ - S_T^- < \varepsilon$ holds.

So, let us choose some partition T satisfying the condition $l(T) < \delta$.

Let $x', x'' \in \Delta_i$, where Δ_i is some segment defined by the partition T, $i = 1, \ldots, n$. Obviously, $|x' - x''| \leq \Delta x_i$. Considering that the mesh of the partition $l(T)$ is the maximum length of the segments Δ_i and, by the condition, $l(T) < \delta$, we obtain the following chain of inequalities:

$$|x' - x''| \leq \Delta x_i \leq l(T) < \delta.$$

Therefore, if $x', x'' \in \Delta_i$, then the inequality $|x' - x''| < \delta$ holds for these points. Then, by the condition of uniform continuity (2), $|f(x') - f(x'')| < \frac{\varepsilon}{b-a}$.

Since the points x' and x'' can be arbitrarily selected on the segment Δ_i, we choose them so that the maximum value of the function f on the segment Δ_i is reached at the point x', and the minimum value of the function f on this segment is reached at the point x''. Such points exist by virtue of the second Weierstrass theorem, which states that a function continuous on the segment takes its maximum and minimum value:

$$f(x') = \max_{x \in \Delta_i} f(x) = M_i, \quad f(x'') = \min_{x \in \Delta_i} f(x) = m_i.$$

Since the estimate $|x' - x''| < \delta$ is also valid for these points, which means that the estimate $|f(x') - f(x'')| < \frac{\varepsilon}{b-a}$ holds, we obtain

$$|M_i - m_i| < \frac{\varepsilon}{b-a}.$$

In this estimate it is not necessary to use the absolute value sign, since the difference $M_i - m_i$ is always non-negative.

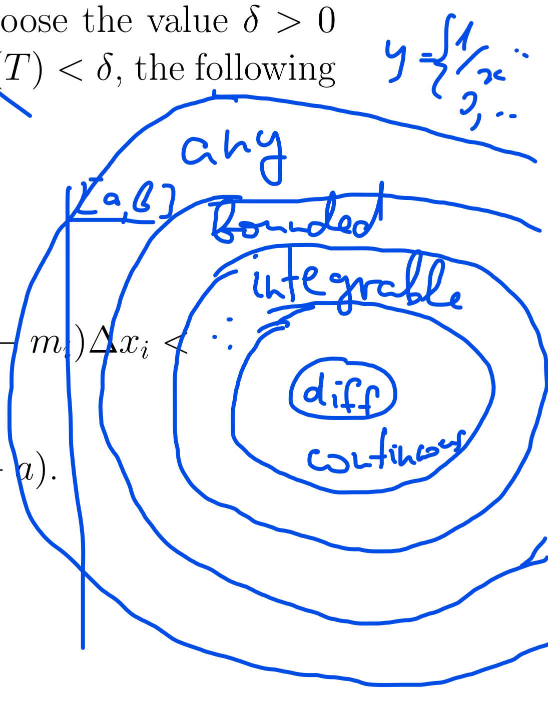
So, we have proved that if for a given $\varepsilon > 0$, we choose the value $\delta > 0$ from condition (2), then, for any partition T for which $l(T) < \delta$, the following relation holds:

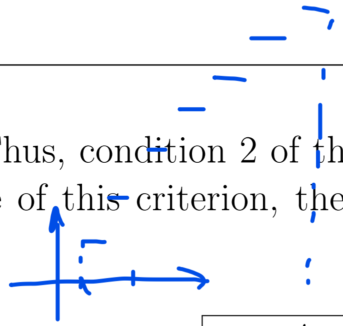
$$M_i - m_i < \frac{\varepsilon}{b-a}, \quad i = 1, \ldots, n.$$

Then, for the difference $S_T^+ - S_T^-$, we get

$$S_T^+ - S_T^- = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i <$$

$$< \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \Delta x_i = \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b-a}(b-a).$$





We got the estimate $S_T^+ - S_T^- < \varepsilon$. Thus, condition 2 of the integrability criterion is also satisfied, and, by virtue of this criterion, the function f is integrable on the segment $[a, b]$. $\square$

Integrability of monotone functions

`2.5B/19:50 (10:18)`

THEOREM (INTEGRABILITY THEOREM FOR MONOTONE FUNCTIONS).

If the function is monotone on the segment, then it is integrable on this segment.

REMARK.

This fact does not follow from the previous theorem, since a monotone function can have a finite or even infinite number of discontinuity points (of the first kind).

PROOF.

Let the function f be monotone on the segment $[a, b]$. For definiteness, we assume that f is non-decreasing on $[a, b]$. Let us prove its integrability using the integrability criterion in terms of Darboux sums.

First, we prove the validity of condition 1 of the criterion, i. e., let us prove the boundedness of the function f.

Since the function f is non-decreasing, we have

$$\forall x \in [a, b] \quad f(a) \leq f(x) \leq f(b).$$

– f is bounded

The resulting double inequality means that the function f is bounded on $[a, b]$.

Now we prove the validity of condition 2 of the criterion. This condition can be represented as

$$\lim_{l(T)\to 0} (S_T^+ - S_T^-) = 0.$$

Choose some partition T. Since the function f is non-decreasing, we have for any segment Δ_i, $i = 1, \ldots, n$,

$$m_i = \min_{x\in\Delta_i} f(x) = f(x_{i-1}), \quad M_i = \max_{x\in\Delta_i} f(x) = f(x_i).$$

Then the difference $S_T^+ - S_T^-$ can be transformed as follows: $\leq$

$$S_T^+ - S_T^- = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\Delta x_i.$$

By the definition of the mesh of the partition, we get $\Delta x_i \leq l(T)$. Since all factors are non-negative, the following estimate holds:

$$\sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big)\Delta x_i \le \sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big)l(T) =$$

$$= l(T)\sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big). \quad = l(T)\cdot\big(f(x_n)-f(x_0)\big)$$

We write out the terms of the last sum in the reverse order and reduce similar terms:

$$\sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big) = \overbrace{\big(f(x_n)-\cancel{f(x_{n-1})}\big)}^{i=n} + \overbrace{\big(\cancel{f(x_{n-1})}-\cancel{f(x_{n-2})}\big)}^{i=n-1} +$$
$$+ \underbrace{\big(\cancel{f(x_{n-2})}-\cancel{f(x_{n-3})}\big)}_{i=n-2} + \cdots + \underbrace{\big(\cancel{f(x_2)}-\cancel{f(x_1)}\big)}_{i=2} + \underbrace{\big(\cancel{f(x_1)}-f(x_0)\big)}_{i=1} =$$
$$= f(x_n)-f(x_0).$$

Thus, we obtained the following double inequality (in which we took into account that $x_0=a$, $x_n=b$):

$$0 \le S_T^+ - S_T^- \le \underbrace{\big(f(b)-f(a)\big)l(T)}_{\to 0}. \to 0$$

$$0 \le S_T^+ - S_T^- \le \otimes$$

If we pass to the limit in the resulting double inequality as $l(T)\to 0$, then the left-hand and right-hand sides of the inequality will be 0; therefore, by virtue of the theorem on passing to the limit in inequalities, the difference $S_T^+ - S_T^-$ will also be 0.

So, we have proved that condition 2 of the integrability criterion also holds. By virtue of this criterion, the function f is integrable on the segment $[a,b]$. □

$\int_a^b f(x)\,dx$

Integral properties associated with integrands

Linearity of a definite integral

`2.5B/30:08 (09:46)`

THEOREM 1 (ON LINEARITY OF A DEFINITE INTEGRAL WITH RESPECT TO THE INTEGRAND).

Let the functions f and g be integrable on the segment $[a,b]$, $\alpha,\beta\in\mathbb{R}$. Then the function $\alpha f+\beta g$ is also integrable on $[a,b]$ and the following equality holds:

$$\int_a^b\big(\alpha f(x)+\beta g(x)\big)\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx. \tag{3}$$

PROOF.

Let us prove this fact using the definition of a definite integral. We write down the integral sum for the function $\alpha f+\beta g$ and transform it:

$\int_a^b f\,dx + \beta \int_a^b g(x)\,dx$

$$\sigma_T(\alpha f + \beta g, \xi) = \sum_{i=1}^{n} \big(\alpha f(\xi_i) + \beta g(\xi_i)\big)\Delta x_i =$$

$$= \alpha \sum_{i=1}^{n} f(\xi_i)\Delta x_i + \beta \sum_{i=1}^{n} g(\xi_i)\Delta x_i = \alpha\sigma_T(f,\xi) + \beta\sigma_T(g,\xi).$$

We have obtained the following relation, which is valid for any partition T and any sample ξ:

$$\sigma_T(\alpha f + \beta g, \xi) = \alpha\sigma_T(f,\xi) + \beta\sigma_T(g,\xi). \tag{4}$$

Since, by condition, the functions f and g are integrable on $[a,b]$, the limits $\lim_{l(T)\to 0, \forall\,\xi} \sigma_T(f,\xi)$ and $\lim_{l(T)\to 0, \forall\,\xi} \sigma_T(g,\xi)$ exist and are equal to $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$, respectively.

Then the limit on the right-hand side of equality (4), as $l(T) \to 0$, $\forall\,\xi$, exists and equals $\alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$. Therefore, for the left-hand side of equality (4), there also exists a limit with the same value. Thus, we simultaneously proved the integrability of the function $\alpha f + \beta g$ and the validity of formula (3). □

Integrability of the product

`2.6A/00:00 (16:44)`

Theorem 2 (on integrability of the product of integrable functions).

Let the functions f and g be integrable on the segment $[a,b]$. Then the function fg is also integrable on $[a,b]$.

Remark.

In this case, we can only establish the fact of integrability, since there is no formula expressing the integral of the product of functions in terms of the integrals of the factors.

Proof.

Let us use the integrability criterion in terms of the oscillation of a function, which can be formulated as follows: the function f is integrable if and only if it is bounded and $\sum_{i=1}^{n} \omega_i(f)\Delta x_i \to 0$ as $l(T) \to 0$. To find the oscillation of the function, we apply the formula (1).

First, we note that if the functions f and g are integrable, then they are bounded on $[a,b]$ due to the necessary integrability condition:

$$\exists\, C > 0 \quad \forall\, x \in [a,b] \quad |f(x)| \le C, \quad |g(x)| \le C. \tag{5}$$

Therefore, the product fg is also bounded.

Taking into account (5), we transform the absolute value of the difference $f(x')g(x') - f(x'')g(x'')$ in such a way that it allows us to estimate the oscillation of the product fg through the oscillations of the factors f and g:

$$\begin{aligned} &|f(x')g(x') - f(x'')g(x'')| = \\ &\quad = |f(x')g(x') - f(x'')g(x') + f(x'')g(x') - f(x'')g(x'')| \leq \\ &\quad \leq |g(x')||f(x') - f(x'')| + |f(x'')||g(x') - g(x'')| \leq \\ &\quad \leq C\big(|f(x') - f(x'')| + |g(x') - g(x'')|\big). \end{aligned} \tag{6}$$

We assume that $x', x'' \in \Delta_i$, $i = 1, \dots, n$. Then, by virtue of (1), we obtain

$$|f(x') - f(x'')| \leq \sup_{x', x'' \in \Delta_i} |f(x') - f(x'')| = \omega_i(f).$$

Similarly,

$$|g(x') - g(x'')| \leq \omega_i(g).$$

Given the estimates obtained, relation (6) can be written in the form

$$\forall\, x', x'' \in \Delta_i \quad |f(x')g(x') - f(x'')g(x'')| \leq C\big(\omega_i(f) + \omega_i(g)\big).$$

We have obtained an upper bound for the set of differences of the form $|f(x')g(x') - f(x'')g(x'')|$ when $x', x'' \in \Delta_i$. Therefore, this set is bounded from above and we have the following estimate for its least upper bound:

$$\sup_{x', x'' \in \Delta_i} |f(x')g(x') - f(x'')g(x'')| \leq C\big(\omega_i(f) + \omega_i(g)\big).$$

The expression on the left-hand side of the last inequality is, by virtue of (1), an oscillation of the function fg. Thus, the resulting inequality takes the form

$$\omega_i(fg) \leq C\big(\omega_i(f) + \omega_i(g)\big).$$

So, we have estimated the oscillation of the product fg through the oscillations of the factors. It remains to multiply both sides by Δx_i and summarize these inequalities by $i = 1, \dots, n$:

$$\sum_{i=1}^{n} \omega_i(fg)\Delta x_i \leq C\Big(\sum_{i=1}^{n} \omega_i(f)\Delta x_i + \sum_{i=1}^{n} \omega_i(g)\Delta x_i\Big).$$

Since, by condition, the functions f and g are integrable on $[a, b]$, we obtain, by the necessary condition of the integrability criterion in terms of the oscillation of the function, that each term on the right-hand side of the inequality approaches 0 as $l(T) \to 0$.

Consequently, the quantity indicated on the left side of the inequality also approaches 0 by virtue of the theorem on passing to the limit in inequalities. Therefore, by virtue of the sufficient condition for the integrability criterion, the function fg is integrable on $[a, b]$. □

Properties associated with integration segments

Integrability on a nested segment

2.6A/16:44 (07:03)

Theorem 3 (on integrability on a nested segment).

If the function f is integrable on the segment $[a, b]$, then it is integrable on any segment $[c, d] \subset [a, b]$.

Proof.

It is enough for us to prove, by virtue of the integrability criterion in terms of the oscillation of the function, that

$$\sum_{T} \omega_i(f)\Delta x_i \to 0, \quad l(T) \to 0. \tag{7}$$

Here, T denotes the partition of the segment $[c, d]$. To make the notation more clear, we used the partition T, according to which the segments Δ_i are constructed, as the summation parameter.

For any partition T, we can add to it new points in such a way as to obtain a partition of the original segment $[a, b]$ as a result. We will denote the resulting partition of the segment $[a, b]$ by T' and we will use the index k to indicate the segments obtained for this partition: Δ_k (such a notation allows us to distinguish these segments from the segments connected with the partition T and marked with the index i). We require that the mesh of the constructed partition T' coincides with $l(T)$: $l(T') = l(T)$. This can be satisfied by choosing new points so that neighboring points are located at a distance not exceeding $l(T)$.

If we consider all possible partitions T' constructed on the basis of partitions T and pass to the limit as $l(T')$ approaches 0, then the mesh of partitions T will also approach 0.

Since, by condition, the function f is integrable on $[a, b]$, we obtain, by virtue of the necessary part of the integrability criterion in terms of the oscillation of the function, that

$$\sum_{T'} \omega_k(f)\Delta x_k \to 0, \quad l(T') \to 0. \tag{8}$$

Note that the integrability criterion assumes that the indicated limit relation is valid for all possible partitions of the interval $[a, b]$. But if this relation is valid for all partitions, then it remains valid for a part of these partitions, namely, a part that is constructed on the basis of partitions T of segment $[c, d]$ as described above.

Since the sum $\sum_{T'} \omega_k(f)\Delta x_k$ contains all terms from the sum $\sum_T \omega_i(f)\Delta x_i$, as well as some additional non-negative terms, corresponding to the segments Δ_k not lying on $[c, d]$, the estimate holds:

$$\sum_T \omega_i(f)\Delta x_i \le \sum_{T'} \omega_k(f)\Delta x_k. \tag{9}$$

It follows from (8) and (9) that $\sum_T \omega_i(f)\Delta x_i \to 0$ as $l(T') \to 0$. Since, by construction, $l(T') = l(T)$, we obtain that relation (7) also holds. $\square$

$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$

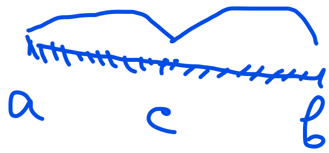
The first theorem on the additivity of a definite integral with respect to the integration segment

2.6A/23:47 (06:27)

Theorem 4 (the first theorem on the additivity of a definite integral with respect to the integration segment).

Let the function f be integrable on $[a, b]$, $c \in (a, b)$ (note that, by virtue of Theorem 3, this function is integrable on the segments $[a, c]$ and $[c, b]$). Then the following equality holds:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{10}$$



Proof.

Let T' be some partition of the segment $[a, c]$, T'' be some partition of the segment $[c, b]$. Then $T = T' \cup T''$ is a partition of the segment $[a, b]$. The partition T necessarily contains the point c and, in addition, we have $l(T) \to 0$ as $l(T') \to 0$ and $l(T'') \to 0$.

Let ξ' and ξ'' be the samples corresponding to the partitions T' and T''. By ξ we denote the sample, which is the union of ξ' and ξ''; this sample corresponds to the partition T.

Then, for the integral sums corresponding to the constructed partitions and samples, the following equality holds:

$$\sigma_T(f, \xi) = \sigma_{T'}(f, \xi') + \sigma_{T''}(f, \xi'').$$

We pass to the limit as $l(T') \to 0$, $\forall\, \xi'$, and $l(T'') \to 0$, $\forall\, \xi''$. By virtue of the already proved integrability of the function f on $[a, c]$ and $[c, b]$, we obtain that the right-hand side of the equality approaches $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

On the other hand, since the function f is integrable on $[a,b]$, we get that the limit of integral sums exists (and is equal to $\int_a^b f(x)\,dx$) for any partitions whose mesh approaches 0 and for any samples related to these partitions. But then the same will be true for the part of the possible partitions T that are constructed on the basis of the partitions T', T'' such that $l(T') \to 0$, $\forall\,\xi'$, and $l(T'') \to 0$, $\forall\,\xi''$.

Passing to the limit in both sides of the previous equality, we obtain the proved relation (10). □

Remark.

The converse statement, which we accept without proof, is also true: if the function is integrable on the segments $[a,c]$ and $[c,b]$, then it is integrable on the segment $[a,b]$ and equality (10) holds. This fact implies that any function that has a finite number of discontinuities of the first kind on the segment $[a,b]$ is integrable on this segment.

The second theorem on the additivity of a definite integral with respect to the integration segment 2.6A/30:14 (11:39)

$[a,b]$, $a<b$ $b-a$

Definition.

We assume that the integral of any function defined at a over a segment of zero length $[a,a]$ is 0:

$[a,a]$

$$\int_a^a f(x)\,dx \overset{\text{def}}{=} 0.$$

In addition, we define the integral from b to a for $a<b$ as follows:

$a<b$

$$\int_b^a f(x)\,dx \overset{\text{def}}{=} -\int_a^b f(x)\,dx.$$

$a-b<0$

This is a quite natural definition, which follows from the initial definition of a definite integral if we allow the situation $x_{i-1} > x_i$ (for which $\Delta x_i < 0$).

So, we can say that if we swap the limits of integration, then the sign of the integral changes to the opposite.

Theorem 5 (the second theorem on the additivity of a definite integral with respect to the integration segment).

Let the function f be integrable on $[a,b]$, $c_1, c_2, c_3 \in [a,b]$. Then the equality holds:

$$\int_{c_1}^{c_3} f(x)\,dx = \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx. \tag{11}$$

c_2 c_3 c_1

Proof.

Let us prove equality (11) for one of the cases of the location of the points c_1, c_2, c_3 that is different from the case $c_1 < c_2 < c_3$, which is already considered in Theorem 4.

Let, for example, $c_2 < c_1 < c_3$. By virtue of Theorem 4, we have

$$\int_{c_2}^{c_3} f(x)\,dx = \int_{c_2}^{c_1} f(x)\,dx + \int_{c_1}^{c_3} f(x)\,dx.$$

In the obtained relation, we transform the integrals so that their limits correspond to the limits indicated in (11). In this case, we only need to transform the integral from c_2 to c_1, changing its sign:

$$+\int_{c_2}^{c_3} f(x)\,dx = -\int_{c_1}^{c_2} f(x)\,dx + \int_{c_1}^{c_3} f(x)\,dx.$$

If we transfer the integral preceded by a minus sign to another part of the equality and swap the left-hand and right-hand sides of this equality, then we obtain (11).

Any other arrangement of points c_1, c_2, c_3 can be analyzed in a similar way. For example, for the case $c_3 < c_2 < c_1$, we have

$$\int_{c_3}^{c_1} f(x)\,dx = \int_{c_3}^{c_2} f(x)\,dx + \int_{c_2}^{c_1} f(x)\,dx,$$

$$-\int_{c_1}^{c_3} f(x)\,dx = -\int_{c_2}^{c_3} f(x)\,dx - \int_{c_1}^{c_2} f(x)\,dx,$$

Multiplying the resulting equality by -1, we obtain (11). It is even easier to analyze situations in which some points coincide. □

Estimates for integrals

Simple estimates of integrals

2.6A/41:53 (01:17), 2.6B/00:00 (06:47)

Theorem 6 (on the non-negativity of the integral of a non-negative function).

If the function f is integrable on $[a, b]$ and $\forall\, x \in [a, b]\ f(x) \geq 0$, then

$$\int_a^b f(x)\,dx \geq 0. \tag{12}$$

Proof.

Consider the integral sum for some partition T and a sample ξ:

$$\sigma_T(f,\xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i. \quad \geq 0$$

Since $\Delta x_i > 0$ and, by condition, $f(\xi_i) \geq 0$, all terms of this sum are non-negative, therefore the integral sum itself is non-negative too:

$$\sigma_T(f,\xi) \geq 0. \quad \Rightarrow \quad \int_a^b f(x)\,dx \geq 0$$

When passing to the limit as $l(T) \to 0$, $\forall\, \xi$, the sign of the non-strict inequality is preserved, therefore estimate (12) holds. □

Theorem 7 (on the comparison of integrals).

If the functions f and g are integrable on $[a,b]$ and $\forall\, x \in [a,b]$ $f(x) \leq g(x)$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx. \quad f(x) \leq g(x) \tag{13}$$

Proof.

We use the previously proved Theorems 1 and 6. Let us introduce the auxiliary function $h(x) = g(x) - f(x)$. Obviously, this function is non-negative. In addition, by virtue of Theorem 1, this function is integrable; moreover,

$$\int_a^b h(x)\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx. \quad \geq 0$$

According to Theorem 6, the left-hand side of the resulting equality is non-negative:

$$\int_a^b h(x)\,dx \geq 0.$$

Therefore, the right-hand side is also non-negative, therefore estimate (13) holds. □

Corollary.

If the function f is integrable on $[a,b]$ and $\forall\, x \in [a,b]$ $m \leq f(x) \leq M$ for some $m, M \in \mathbb{R}$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a). \tag{14}$$

Proof.

Earlier, we established that the constant function $f(x) = c$ is integrable on any interval and

$$\int_a^b c\,dx = c(b-a).$$

We apply Theorem 7 to the double inequality $m \le f(x) \le M$:

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx.$$

Given the formula for the integral of the constant, we obtain relation (14). □

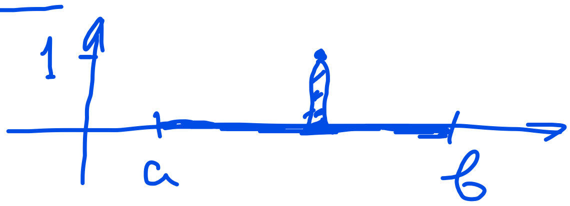
Integral of a positive continuous function `2.6B/06:47 (11:25)`

Theorem 8 (on the integral of a positive continuous function).

$\forall x \in [a, b] \quad f(x) \geqslant 0$

Let the function f be integrable and non-negative on $[a, b]$. Also suppose that the function f is continuous at the point $c \in [a, b]$ and, moreover, $f(c) > 0$. Then

$$\int_a^b f(x)\,dx > 0.$$



Proof.

We use the simplest property of a continuous function: if the function is continuous at the point c and takes a positive value at it, then there exists a neighborhood of this point at which the function remains positive.

If we denote $f(c) = D > 0$, then it can be argued that there exists a neighborhood U_c^δ such that the estimate $f(x) \ge \frac{D}{2}$ holds for any point $x \in U_c^\delta$.

We assume that the neighborhood U_c^δ lies inside the segment $[a, b]$, and also that the estimate $f(x) \ge \frac{D}{2}$ is satisfied at the boundary of the neighborhood U_c^δ (otherwise, it's enough to simply reduce the neighborhood). Then the integral from a to b can be represented as the sum of three integrals:

$$\int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^b f(x)\,dx.$$

The first and third integrals on the right-hand side are non-negative by virtue of Theorem 6. Let us turn to the second integral. Since $\forall x \in [c-\delta, c+\delta]$ $f(x) \ge \frac{D}{2}$, applying the corollary of Theorem 7, we obtain

$$\int_{c-\delta}^{c+\delta} f(x)\,dx \ge \frac{D}{2} \cdot \big(c + \delta - (c - \delta)\big) = D\delta > 0.$$

Thus, the second integral is positive. Therefore, the sum of the three integrals is also positive. □

Corollary.

If the function f is continuous on $[a,b]$ and $\forall\, x \in [a,b]$ $f(x) < M$, then

$$\int_a^b f(x)\,dx < M(b-a).$$

$$M\cdot(b-a) = \int_a^b M\,dx > \int_a^b f(x)\,dx$$

Proof.

Consider the function $h(x) = M - f(x)$. This function is continuous and positive on $[a,b]$. Therefore, by the previous theorem, we obtain $\int_a^b h(x)\,dx > 0$. To get the required estimate, it remains to use the linearity of the integral and the formula for the integral of a constant function. □

Properties of the integral of the absolute value of a function

`2.6B/18:12 (16:20)`

Theorem 9 (on the integral of the absolute value of a function).

1

If the function f is integrable on $[a,b]$, then its absolute value $|f|$ is also integrable on $[a,b]$ and the estimate holds:

$$|a+b| \le |a| + |b|$$

2

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx. \tag{15}$$

$$|a+b+c| \le |a|+|b|+|c|$$

$$\left|\sum a_i\right| \le \sum |a_i|$$

Proof.

First, we prove the integrability of the function $|f|$. Let us use the lower bound for the difference $|t'-t''|$:

$$|t'-t''| \ge \big||t'| - |t''|\big|. \tag{16}$$

We choose the partition T of the segment $[a,b]$, choose some segment Δ_i defined by this partition, and write the estimate (16) for $f(x')$ and $f(x'')$ when $x', x'' \in \Delta_i$, swapping the left-hand and right-hand sides of the estimate:

$$\big||f(x')| - |f(x'')|\big| \le |f(x') - f(x'')|.$$

We will argue in the same way as in the proof of the integrability of the product (see Theorem 2). First, it is obvious that the right-hand side of the resulting inequality is bounded from above by the value $\sup_{x',x''\in\Delta_i} |f(x') - f(x'')|$, which is equal to the oscillation of the function f on the segment Δ_i. Therefore,

$$\big||f(x')| - |f(x'')|\big| \le \omega_i(f).$$

Further, since this estimate is valid for all $x', x'' \in \Delta_i$, we find that a similar estimate holds for the least upper boundary of the left-hand side:

$$\sup_{x',x''\in\Delta_i} \big||f(x')| - |f(x'')|\big| \le \omega_i(f).$$

The left-hand side of the last estimate is the oscillation of the function $|f|$:

$$\sum \boxed{\omega_i(|f|) \le \omega_i(f).} \cdot \Delta x_i$$

So, we have proved that the oscillation of the absolute value of a function does not exceed the oscillation of the function itself. It remains for us to multiply both sides of the resulting estimate by Δx_i and summarize the resulting inequalities for i from 1 to n:

(necessary part of criteria)

$$0 \le \sum_{i=1}^{n} \omega_i(|f|)\Delta x_i \le \sum_{i=1}^{n} \omega_i(f)\Delta x_i. \qquad l(T)\to 0$$

(suff. part) if $|f|$ is integrable

This estimate is valid for an arbitrary partition T. Passing to the limit as $l(T) \to 0$ and taking into account that, by condition, the function f is integrable on $[a,b]$, we obtain, by virtue of the integrability criterion, that the right-hand side of the inequality approaches 0. Then, by virtue of the theorem on passing to the limit in inequalities, the left-hand side also approaches 0; therefore, due to the same integrability criterion, the function $|f|$ is also integrable on $[a,b]$. The first part of the theorem is proved.

Now let us turn to the proof of estimate (15). We choose an arbitrary partition T of the segment $[a,b]$ and a sample ξ, consider the absolute value of the integral sum for the function f, and transform it using a generalization of the triangle inequality $|t'+t''| \le |t'| + |t''|$ for the case of n terms:

$$\int_a^b f(x)\,dx \qquad \int_a^b |f(x)|\,dx$$

$$|\sigma_T(f,\xi)| = \left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i\right| \le \sum_{i=1}^{n} |f(\xi_i)|\Delta x_i.$$

On the right-hand side, we get the integral sum for the function $|f|$ over the same partition T and the sample ξ. Therefore,

$$|\sigma_T(f,\xi)| \le \sigma_T(|f|,\xi).$$

f integrable $\quad |f|$ integrable

Since we have already proved that the function $|f|$ is integrable, the limits of the integral sums as $l(T) \to 0$, $\forall\,\xi$, exist both on the left-hand side and on the right-hand side. These limits are equal to the integrals of the corresponding functions and the same estimate holds for them. $\square$

Remark.

The integrability of the absolute value of a function does not imply the integrability of the function itself. To prove this statement, it suffices to give an example. Consider the following function (which can be obtained from the Dirichlet function by stretching and shifting along the OY axis):

$[0,1]$

$|f(x)| = 1$ on $[0,1]$

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ -1, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function, like the Dirichlet function, is not integrable on any segment of positive length, because for any segment $[a, b]$, its upper Darboux integral is $(b - a)$, and it differs from the lower Darboux integral equal to $-(b - a)$. At the same time, the absolute value of this function is a constant: $|f(x)| = 1$, and the constant is integrable on any interval.

Mean value theorems for definite integrals

The first mean value theorem

`2.6B/34:32 (10:39)`

THEOREM 10 (THE FIRST MEAN VALUE THEOREM).

Suppose that the functions f and g are integrable on $[a, b]$ and the following conditions are satisfied for them:

1) for the function f, a double estimate holds: $m \le f(x) \le M$, $x \in [a, b]$;

2) the function g preserves the sign on $[a, b]$, i. e., either $g(x) \ge 0$ for $x \in [a, b]$ or $g(x) \le 0$ for $x \in [a, b]$.

$+$

$-$

Then there exists a value $\mu \in [m, M]$ such that the following equality holds:

$$\int_a^b f(x)g(x)\,dx = \mu \int_a^b g(x)\,dx. \tag{17}$$

$\int_a^b g(x)\,dx \geq 0$

PROOF.

First, we consider the case when $g(x) \ge 0$ for $x \in [a, b]$.

We multiply all the terms of the estimate from condition 1 by $g(x)$. The signs of inequality will not change, since, by our assumption, the function g is non-negative:

$$mg(x) \le f(x)g(x) \le Mg(x).$$

By virtue of Theorem 2, each of the obtained products is an integrable function. We integrate all the terms of the double inequality from a to b. By virtue of Theorem 7, the signs of inequality will not change. In addition, the constants m and M can be taken out of the signs of the integrals:

$$m \int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M \int_a^b g(x)\,dx.$$

Thus, we obtain the integral $\int_a^b g(x)\,dx$ on the left-hand and right-hand sides of the resulting double inequality.

If $\int_a^b g(x)\,dx = 0$, then the last double inequality takes the form $0 \le \int_a^b f(x)g(x)\,dx \le 0$, which implies that $\int_a^b f(x)g(x)\,dx = 0$. In this case, equality (17) is satisfied and any value from the interval $[m, M]$ can be taken as μ.

If $\int_a^b g(x)\,dx \ne 0$, then we can divide all parts of the double inequality by this nonzero value. As a result, we get

$$m \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le M.$$

Denote the obtained quotient of integrals by μ:

$$\mu = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}. \tag{18}$$

Thus, the double inequality $m \le \mu \le M$ holds for μ and, in addition, relation (18) can be transformed to (17) by multiplying both sides of the equality by $\int_a^b g(x)\,dx$.

So, we have proved the theorem for the case $g(x) \ge 0$.

Now suppose that $g(x) \le 0$ for $x \in [a, b]$. Consider the auxiliary function $\tilde g(x) = -g(x)$. The function $\tilde g(x)$ is non-negative: $\tilde g(x) \ge 0$ for $x \in [a, b]$ and the theorem has already been proved for the case of non-negative functions. Therefore, there exists a value $\mu \in [m, M]$ such that

$$\int_a^b f(x)\tilde g(x)\,dx = \mu \int_a^b \tilde g(x)\,dx.$$

Let's get back to the function $g(x)$:

$$\int_a^b f(x)\bigl(-g(x)\bigr)\,dx = \mu \int_a^b \bigl(-g(x)\bigr)\,dx.$$

To obtain equality (17), it suffices to put the signs "minus" behind the signs of the integrals and multiply both sides of the resulting equality by (-1). Thus, equality (17) is valid for the function $g(x)$ also in the case $g(x) \le 0$. □

The second and the third mean value theorems

2.7A/00:00 (12:56)

THEOREM 11 (THE SECOND MEAN VALUE THEOREM).

Suppose that the functions f and g are defined on $[a, b]$ and the following conditions are satisfied for them:

1) the function f is continuous on $[a, b]$ (this condition immediately implies the integrability of the function f on $[a, b]$);

2) the function g is integrable on $[a, b]$ and preserves the sign on this segment, i. e., either $g(x) \geq 0$ for $x \in [a, b]$ or $g(x) \leq 0$ for $x \in [a, b]$.

Then there exists a point $c \in [a, b]$ such that the following equality holds:

$$\int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx. \tag{19}$$

Proof.

We use the already proved Theorem 10, for which all conditions are satisfied. In particular, since the function f is continuous on a segment, for it, by virtue of the first Weierstrass theorem, there exist numbers $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for $x \in [a, b]$ (note that the boundedness of the function f follows not only from the first Weierstrass theorem, but also from the necessary integrability condition).

$c_1 < c_2$

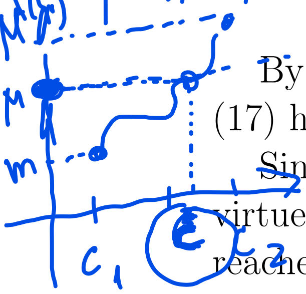
$m \leq \mu \leq M$

As m and M, we can take the values $\inf_{x\in[a,b]} f(x)$ and $\sup_{x\in[a,b]} f(x)$, respectively:

$$m = \inf_{x\in[a,b]} f(x), \quad M = \sup_{x\in[a,b]} f(x).$$

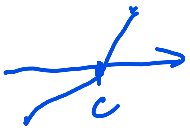
$f(c_1) \leq \mu \leq f(c_2)$

$\int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx$



By virtue of Theorem 10, there exists a value $\mu \in [m, M]$ for which equality (17) holds.

Since the function f is continuous on the segment $[a, b]$, we obtain, by virtue of the second Weierstrass theorem, that the values of m and M are reached at some points, i. e., there exist points $c_1, c_2 \in [a, b]$ for which the equalities $f(c_1) = m$, $f(c_2) = M$ hold.



By virtue of the corollary of the intermediate value theorem, for the function f, there exists a point c lying on a segment with endpoints c_1 and c_2, in which the function f takes the value μ: $f(c) = \mu$. Since $c_1, c_2 \in [a, b]$, we obtain that the point c also belongs to the segment $[a, b]$.

Substituting the value $f(c)$ into (17) instead of μ, we get equality (19). □

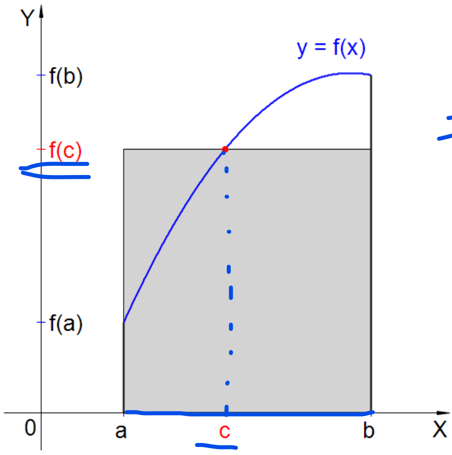
Theorem 12 (the third mean value theorem).

Let the function f be continuous on $[a, b]$. Then there exists a point $c \in [a, b]$ such that the following equality holds:

$$\int_a^b f(x)\, dx = f(c)(b - a). \tag{20}$$

Remark (geometric sense of the third mean value theorem).

Assume that $f(x) > 0$ for $x \in [a, b]$. We noted earlier that the value of a definite integral $\int_a^b f(x)\, dx$ can be interpreted as the area of a curvilinear trapezoid bounded by the graph $y = f(x)$, the segment of the axis OX, and the lines $x = a$ and $x = b$ (this fact will be proved later when we give a rigorous definition of area). Formula (20) means that there exists a point $c \in [a, b]$ for which a rectangle with the base $[a, b]$ and the height $f(c)$ has an area equal to the area of this curvilinear trapezoid (Fig. 6).



$= \int_a^b f(x)\,dx$

Fig. 6. Geometric sense of the third mean value theorem

Proof.

It is enough to use the second mean value theorem (Theorem 11) by putting $g(x) \equiv 1$ in it. Obviously, in this case the function $g(x)$ preserves the sign. Then

$$\int_a^b g(x)\,dx = \int_a^b dx = b - a.$$

Substituting the function $g(x) \equiv 1$ and the found value of the integral of this function into formula (19), we obtain (20). $\square$

$\int_a^b f(x)\,dx = f(c) \int_a^b g(x)\,dx$ — Theorem 11

$g(x) \equiv 1$

$\int_a^b f(x)\,dx = f(c)(b-a)$ — Theorem 12