7. Integral with a variable upper limit. Newton–Leibniz formula

Integral with a variable upper limit

Definition of an integral with a variable upper limit

2.7A/12:56 (04:01)

DEFINITION.

Let the function f be integrable on the segment $[a, b]$. Then, by the integrability theorem on the embedded segment, it is integrable on the segment $[a, x]$ for any $x \in [a, b]$. Therefore, for any $x \in [a, b]$, there exists an integral $\int_a^x f(t)\,dt$. Denote this integral by $F(x)$:

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t)\,dt.$$

$x \in [a, b]$ f – integrable → F – continuous

$\int_a^a = 0$

The function $F(x)$ is called an *integral with a variable upper limit*. Obviously, $F(a) = 0$ as an integral over a segment of zero length.

Theorem on the continuity of an integral with a variable upper limit

2.7A/16:57 (16:48)

THEOREM 1 (ON THE CONTINUITY OF AN INTEGRAL WITH A VARIABLE UPPER LIMIT).

For any function f integrable on the segment $[a, b]$, its integral with a variable upper limit F is a continuous function on this segment.

PROOF.

We choose an arbitrary point $x_0 \in [a, b]$ and prove that the function $F(x)$ is continuous at this point. For definiteness, we assume that $x_0 \in (a, b)$.

We want to prove that the limit of the function $F(x)$ as $x \to x_0$ is equal to the value of the function at the point x_0:

$$\lim_{\Delta x \to 0} \bigl(F(x_0 + \Delta x) - F(x_0)\bigr) = 0.$$

We assume that $x_0 + \Delta x \in [a, b]$; the increment Δx can be both positive and negative.

Consider the difference $|F(x_0+\Delta x)-F(x_0)|$ and transform it using the definition of an integral with a variable upper limit and the additivity theorem for the integral with respect to the integration segment:

$$\underline{|F(x_0+\Delta x)-F(x_0)|} = \left|\int_a^{x_0+\Delta x} f(t)\,dt - \int_a^{x_0} f(t)\,dt\right| =$$

$$= \left|\cancel{\int_a^{x_0} f(t)\,dt} + \int_{x_0}^{x_0+\Delta x} f(t)\,dt - \cancel{\int_a^{x_0} f(t)\,dt}\right| = \underline{\left|\int_{x_0}^{x_0+\Delta x} f(t)\,dt\right|}.$$

If $\Delta x>0$, then the right-hand side of the resulting equality can be estimated using the property of the integral of the absolute value of a function:

$$\left|\int_{x_0}^{x_0+\Delta x} f(t)\,dt\right| \le \int_{x_0}^{x_0+\Delta x} |f(t)|\,dt.$$

A similar estimate can be obtained for the case $\Delta x<0$; in this case, we must use the integral $\int_{x_0+\Delta x}^{x_0} |f(t)|\,dt$ on the right-hand side of the estimate.

If we do not impose additional conditions on Δx, then we can write the following version of the estimate, which is valid for both positive and negative values of Δx:

$$\left|\int_{x_0}^{x_0+\Delta x} f(t)\,dt\right| \le \left|\int_{x_0}^{x_0+\Delta x} \underline{|f(t)|\,dt}\right|.$$

Since the function f is integrable, it is bounded:

$$\exists\, \underline{C}>0 \quad \forall\, x\in[a,b] \quad \underline{|f(x)|\le C}.$$

If we assume that $\Delta x>0$, then from the estimate $|f(x)|\le C$, using the theorem on the comparison of integrals, we obtain the following estimate:

$$\underline{\int_{x_0}^{x_0+\Delta x} |f(t)|\,dt} \le \int_{x_0}^{x_0+\Delta x} C\,dt = C\Delta x\ .$$

If we do not impose additional conditions on Δx, then we have a similar estimate containing the absolute value of the integral and the absolute value of Δx:

$$\underline{\left|\int_{x_0}^{x_0+\Delta x} |f(t)|\,dt\right| \le C|\Delta x|}.$$

Indeed, in the case $\Delta x<0$ we get

$$\left|\int_{x_0}^{x_0+\Delta x} |f(t)|\,dt\right| = \int_{x_0+\Delta x}^{x_0} |f(t)|\,dt \le C(-\Delta x) = C|\Delta x|.$$

So, we started with the expression $|F(x_0+\Delta x)-F(x_0)|$ and, as a result, evaluated it from above with the expression $C|\Delta x|$:

$$0 \le |F(x_0 + \Delta x) - F(x_0)| \le C|\Delta x|.$$

If Δx approaches 0, then the right-hand side of the resulting estimate also approaches 0; therefore, by virtue of the theorem on passing to the limit in inequalities, the left-hand side also approaches 0. We have proved that the function F is continuous at an arbitrary point $x_0 \in (a, b)$.

The case when x_0 coincides with one of the endpoints of the initial segment is considered similarly, taking into account the fact that in this case the limit at the endpoints of the segment should be understood as one-sided limit (and it suffices to consider the positive increment Δx for the point a and negative increment for the point b). □

Theorem on the differentiability of an integral with a variable upper limit and a continuous integrand

`2.7A/33:45 (13:39), 2.7B/00:00 (04:04)`

Theorem 2 (on the differentiability of an integral with a variable upper limit and a continuous integrand).

If the function f is integrable on the segment $[a, b]$ and continuous at the point $x_0 \in (a, b)$, then its integral with a variable upper limit F is a differentiable function at the point x_0 and the formula holds:

$$F'(x_0) = f(x_0).$$

Remarks.

1. It can be proved that the integral with a variable upper limit and an integrand continuous on $[a, b]$ is a differentiable function also at the endpoints of the segment $[a, b]$ if, in this case, we consider the one-sided derivative, that is, one-sided limit of the ratio of the increment of the function to the increment of the argument. However, we will not need this fact.

2. Theorems 1 and 2 indicate that the integration operation "improves" the properties of functions: if the original function is integrable, then its integral with a variable upper limit is a continuous function and if the original function is continuous, then its integral with a variable upper limit is a differentiable function.

Proof.

We need to prove that there exists a limit $\lim_{\Delta x \to 0} \frac{F(x_0+\Delta x)-F(x_0)}{\Delta x}$ and the limit value is $f(x_0)$. In other words, we need to prove that the following equality holds:

$$\lim_{\Delta x \to 0} \left(\frac{F(x_0 + \Delta x) - F(x_0)}{\Delta x} - f(x_0) \right) = 0 .$$

Let us write down what the last equality means in the language ε–δ:

$$\forall\,\varepsilon > 0 \quad \exists\,\delta > 0 \quad \forall\,\Delta x, |\Delta x| < \delta,$$

$$\Big|\frac{F(x_0+\Delta x) - F(x_0)}{\Delta x} - f(x_0)\Big| < \varepsilon. \tag{1}$$

We select some value of $\varepsilon > 0$. By condition, the function f is continuous at the point x_0. This means that the following condition is true for the selected ε:

$$\exists\,\delta > 0 \quad \forall\,\Delta x, |\Delta x| < \delta, \quad |f(x_0+\Delta x) - f(x_0)| < \frac{\varepsilon}{2}. \tag{2}$$

Let us show that the value δ from condition (2) also ensures that condition (1) is satisfied, i. e., that the estimate $|f(x_0+\Delta x) - f(x_0)| < \frac{\varepsilon}{2}$ implies the validity of the estimate $\Big|\frac{F(x_0+\Delta x)-F(x_0)}{\Delta x} - f(x_0)\Big| < \varepsilon$.

Transform the difference $\Big|\frac{F(x_0+\Delta x)-F(x_0)}{\Delta x} - f(x_0)\Big|$ by taking out the factor $\frac{1}{\Delta x}$ and then use the definition of an integral with a variable upper limit:

$$\Big|\frac{1}{\Delta x}\Big(\int_a^{x_0+\Delta x} f(t)\,dt - \int_a^{x_0} f(t)\,dt - f(x_0)\Delta x\Big)\Big|. \tag{3}$$

In the proof of Theorem 1, we have already established that the difference $\int_a^{x_0+\Delta x} f(t)\,dt - \int_a^{x_0} f(t)\,dt$ is an integral from x_0 to $x_0+\Delta x$. Further, the factor Δx in the last term $f(x_0)\Delta x$ of expression (3) can be represented as the integral $\int_{x_0}^{x_0+\Delta x} dt$. Thus, expression (3) takes the form

$$\frac{1}{|\Delta x|}\Big|\int_{x_0}^{x_0+\Delta x} f(t)\,dt - f(x_0)\int_{x_0}^{x_0+\Delta x} dt\Big|.$$

Since the obtained integrals have the same integration limits, we can write the last expression as a single integral of the difference of functions:

$$\frac{1}{|\Delta x|}\Big|\int_{x_0}^{x_0+\Delta x} \big(f(t) - f(x_0)\big)\,dt\Big|. \longrightarrow 0 \qquad \Delta x \to 0$$

This expression can be estimated from above by an expression containing the integral of the absolute value of the difference of functions:

$$\frac{1}{|\Delta x|}\Big|\int_{x_0}^{x_0+\Delta x} \big(f(t) - f(x_0)\big)\,dt\Big| \le \frac{1}{|\Delta x|}\Big|\int_{x_0}^{x_0+\Delta x} |f(t) - f(x_0)|\,dt\Big|. \tag{4}$$

We did not remove the absolute value sign for the integral, since the value of Δx can be either positive or negative.

Now let us turn to the estimate $|f(x_0+\Delta x) - f(x_0)| < \frac{\varepsilon}{2}$ from (2). The points x_0 and $x_0+\Delta x$ appearing in this estimate are the limits of the integral on the right-hand side of (4). Any point t located between x_0 and $x_0 + \Delta x$ can be represented as $x_0 + \delta'$, where $|\delta'| < |\Delta x|$. Since it is assumed in condition (2) that $|\Delta x| < \delta$, we see that the same estimate holds for $|\delta'|$: $|\delta'| < \delta$. This means that, for the point $t = x_0 + \delta'$, the estimate $|f(t) - f(x_0)| < \frac{\varepsilon}{2}$ is also valid.

Thus, the integrand on the right-hand side of (4) is estimated by $\frac{\varepsilon}{2}$ for all points t:

$$|f(t) - f(x_0)| < \frac{\varepsilon}{2}.$$

In this estimate, the "$<$" sign can be replaced with the "$\le$" sign. Using the theorem on the comparison of integrals, we obtain

$$\frac{1}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x} |f(t) - f(x_0)|\,dt\right| \le \frac{1}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x} \frac{\varepsilon}{2}\,dt\right| =$$

$$= \frac{1}{|\Delta x|}\cdot\frac{\varepsilon}{2}\left|\int_{x_0}^{x_0+\Delta x} dt\right| = \frac{1}{|\Delta x|}\cdot\frac{\varepsilon}{2}\,|\Delta x| = \frac{\varepsilon}{2} < \varepsilon.$$

So, we have proved that, for any values of Δx satisfying the condition $|\Delta x| < \delta$, the estimate holds:

$$\left|\frac{F(x_0 + \Delta x) - F(x_0)}{\Delta x} - f(x_0)\right| < \varepsilon.$$

This means that condition (1) is satisfied. Therefore, the function $F(x)$ has a derivative at the point x_0 and this derivative is equal to $f(x_0)$. □

Newton–Leibniz formula

f $\quad F = \int_a^x f(t)\,dt$

T.1.: f-integr $\Rightarrow$ F-continuous

T.2.: f-contin. $\Rightarrow$ F-differentiable

Theorems on antiderivatives for continuous functions

2.7B/04:04 (06:23)

Theorem 3 (on the existence of an antiderivative for a continuous function).

Any function f continuous on $[a, b]$ has an antiderivative on (a, b), which is an integral with a variable upper limit: $F(x) = \int_a^x f(t)\,dt$.

Proof.

Since f is continuous on $[a, b]$, it follows from Theorem 2 that its integral with a variable upper limit F is a differentiable function on (a, b) and, for any point $x \in (a, b)$, the equality $F'(x) = f(x)$ is true. We have obtained that $F(x)$ satisfies the definition of the antiderivative of the function $f(x)$ for $x \in (a, b)$. □

Corollary.

If f is a continuous function on $[a, b]$ and $\Phi(x)$ is its antiderivative on (a, b), then this antiderivative can be represented in the following form, where C is some constant:

$$\Phi(x) = \int_a^x f(t)\,dt + C. \tag{5}$$

Proof.

By Theorem 3, we obtain that the integral with a variable upper limit $\int_a^x f(t)\,dt$ is an antiderivative of the function f. The theorem on antiderivatives of a given function states that any two antiderivatives of the function f are different by some constant term C. □

Newton–Leibniz formula

2.7B/10:27 (05:55)

Theorem 4 (the fundamental theorem of calculus).

If the function f is continuous on $[a, b]$, $\Phi(x)$ is a continuous function on $[a, b]$, and Φ is the antiderivative of the function f on (a, b) (a function Φ with the indicated properties exists by virtue of Theorem 3), then

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a). \tag{6}$$

Formula (6) is called the *Newton–Leibniz formula*.

Remarks.

1. The antiderivative (and the indefinite integral) is defined by means of the differentiation operation, but the definite integral is defined by means of the limit of integral sums and therefore its definition is not related with the differentiation operation. Nevertheless, there is a relation between the operations of differentiation (that is, finding the derivative) and integration (that is, finding the definite integral), which is established by the Newton–Leibniz formula. That is why Theorem 4 is called the *fundamental theorem of calculus*.

2. The Newton–Leibniz formula (6) allows us to reduce the problem of finding a definite integral to the problem of finding the antiderivative of an integrand over a given interval.

3. Formula (6) remains valid for the case $a \geq b$.

4. Formula (6) is often written in the following form:

$$\int_a^b f(x)\,dx = \Phi(x)\Big|_a^b.$$

Proof.

By the corollary of Theorem 3, there exists a constant $C \in \mathbb{R}$ such that the antiderivative $\Phi(x)$ of the function $f(x)$ is representable in the form (5). Given this form, we find the values of the antiderivative $\Phi(x)$ at the endpoints of the segment $[a, b]$:

$$\Phi(a) = \overbrace{\int_a^a f(t)\,dt}^{=0} + C = C, \quad \Phi(b) = \int_a^b f(t)\,dt + \overset{\Phi(a)}{\cancel{C}}.$$

The difference $\Phi(b) - \Phi(a)$ is $\int_a^b f(t)\,dt + C - C = \int_a^b f(t)\,dt$. Thus, the Newton–Leibniz formula is proved, since the value of the integral does not depend on the choice of a letter for the integration parameter (in this case, x or t). □

Additional techniques for calculating definite integrals

Change of variables in a definite integral

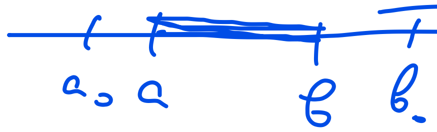
`2.7B/16:22 (11:37)`

Theorem 5 (on the change of variables in a definite integral).

Let the function $f(x)$ be continuous on $[a_0, b_0]$, the function $\varphi(t)$ act from (α_0, β_0) to (a_0, b_0) and be continuously differentiable on (α_0, β_0) (this means that the derivative $\varphi'(t)$ is defined and continuous on (α_0, β_0)). Let, in addition, $\alpha, \beta \in (\alpha_0, \beta_0)$ and $\varphi(\alpha) = a$, $\varphi(\beta) = b$ (moreover, $a, b \in (a_0, b_0)$ due to the properties of the function $\varphi(t)$).

$dx = \varphi'(t)\,dt$

Then



$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\big(\varphi(t)\big)\varphi'(t)\,dt. \tag{7}$$

$x = \varphi(t)$

Remark.

When using Theorem 5 to transform the integral $\int_a^b f(x)\,dx$, the function $\varphi(t)$ arises when we change the previous integration parameter x by the new parameter t: $x = \varphi(t)$. In this case, the differentials will be related as follows: $dx = \varphi'(t)\,dt$. This is similar to the relation used to change of variables in an indefinite integral. The only difference from the case of changing variables in an indefinite integral is that in the case of a definite integral, it is also necessary to change the integration limits using the relations $a = \varphi(\alpha)$, $b = \varphi(\beta)$.

Proof.

First, we note that the integrals on the left-hand and right-hand side of (7) exist, since their integrands are continuous over the entire integration segment.

Since the function $f(x)$ is continuous on $[a_0, b_0]$, it has an antiderivative on (a, b) by virtue of Theorem 3. Denote this antiderivative by $\Phi(x)$.

Let us differentiate the superposition $\Phi\big(\varphi(t)\big)$, which is defined for $t \in (\alpha_0, \beta_0)$:

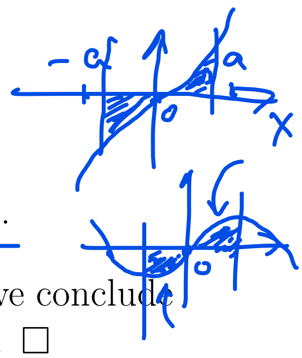
$$\Big(\Phi\big(\varphi(t)\big)\Big)' = \Phi'(x)\big|_{x=\varphi(t)} \cdot \varphi'(t) = f\big(\varphi(t)\big)\varphi'(t).$$

Thus, the superposition $\Phi\big(\varphi(t)\big)$ is the antiderivative for the integrand of the right-hand side of equality (7) on the interval (α_0, β_0).

Now we apply the Newton–Leibniz formula for the integrals indicated on the left-hand side and the right-hand side of (7):

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a),$$

$$\int_\alpha^\beta f\big(\varphi(t)\big)\varphi'(t)\,dt = \Phi\big(\varphi(\beta)\big) - \Phi\big(\varphi(\alpha)\big) = \Phi(b) - \Phi(a).$$



Since the right-hand sides of the obtained equalities coincide, we conclude that the left-hand sides coincide too, i. e., that equality (7) holds. $\square$

Corollaries of the theorem on the change of variables in a definite integral

2.7B/27:59 (09:28)

1. Let the function f be an odd function defined and continuous on the segment $[-a, a]$. Then $\int_{-a}^{a} f(t)\,dt = 0$.

$f(-x) = -f(x)$

Proof.

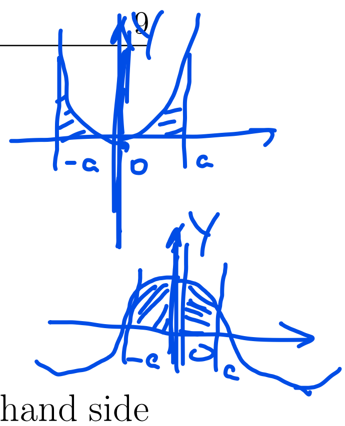
We represent this integral as the sum of the integrals:

$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_0^a f(t)\,dt. \tag{8}$$

In the first integral from the right-hand side of equality (8), we make the variable change $t = -x$. Then $dt = -dx$, the integration limits $-a$ and 0 will change by a and 0, respectively, and this integral will take the form

$$\int_{-a}^{0} f(t)\,dt = \int_a^0 f(-x)\,(-dx).$$

Since the function f is odd, the equality $f(-x) = -f(x)$ holds. Thus,



$$\int_a^0 f(-x)\,(-dx) = \int_a^0 \bigl(-f(x)\bigr)\,(-dx) = \int_a^0 f(x)\,dx.$$

Now change the integration limits:

$$\int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx.$$

Substituting this representation for the first integral in the right-hand side of (8), we obtain

$$-\int_0^a f(x)\,dx + \int_0^a f(t)\,dt = 0. \;\square$$

$\forall x \; f(-x) = f(x)$

2. Let the function f be an even function defined and continuous on the segment $[-a, a]$. Then $\int_{-a}^{a} f(t)\,dt = 2\int_0^a f(t)\,dt$.

Proof.

As in the proof of corollary 1, we represent this integral as the sum of integrals (8) and make the same variable change $t = -x$ in the first integral from the right-hand side of (8):

$$\int_{-a}^0 f(t)\,dt = \int_a^0 f(-x)\,(-dx).$$

In this case, the function is even, i. e., $f(-x) = f(x)$, so further transformations of the integral will be as follows:

$T = 2\pi$

$$\int_a^0 f(-x)\,(-dx) = -\int_a^0 f(x)\,dx = \int_0^a f(x)\,dx.$$

Substituting this representation for the first integral in the right-hand side of (8), we obtain the required expression:

$T \;\; \forall x \;\; f(x+T) = f(x)$

$$\int_0^a f(x)\,dx + \int_0^a f(t)\,dt = 2\int_0^a f(t)\,dt. \;\square$$

3. Let the function f be a continuous periodic function with period T. Then

$$\forall\, a \in \mathbb{R} \qquad \int_a^{a+T} f(t)\,dt = \int_0^T f(t)\,dt. \tag{9}$$

Thus, the integral of a periodic function over any segment whose length is equal to its period T is equal to the integral over the segment $[0, T]$.

Proof.

Using the second theorem on the additivity of a definite integral with respect to the integration segment, we transform the integral $\int_a^{a+T} f(t)\,dt$ as follows:

$$\int_a^{a+T} f(t)\,dt = \cancel{\int_a^0 f(t)\,dt} + \int_0^T f(t)\,dt + \cancel{\int_T^{a+T} f(t)\,dt}. \tag{10}$$

In the last integral of the right-hand side of (10), we make the variable change $x = t - T$. Then $dx = dt$, the integration limits T, $a + T$ change by 0, a, and this integral takes the form

$$\int_T^{a+T} f(t)\,dt = \int_0^a f(x+T)\,dx = -\int_a^0 f(x \cancel{+T})\,dx.$$

Since the function f is periodic with the period T, the equality $f(x+T) = f(x)$ holds. We got that the third integral on the right-hand side of (10) is $-\int_a^0 f(x)\,dx$ and, in combination with the first integral, gives the value 0. Thus, equality (10) turns into equality (9). $\square$

Version of the theorem on the change of variables in a definite integral

2.8A/00:00 (10:45)

The theorem on the change of variables in a definite integral considers the intervals (a_0, b_0) and (α_0, β_0) containing segments with endpoints a, b and α, β, over which the integration is carried out in (7). The purpose of this formulation is to guarantee the existence of the derivative $\varphi'(t)$ at all points of the integration segment.

If we consider the derivatives defined on the segment, assuming that the derivatives are calculated as one-sided limits at the endpoints of the segment, then the condition of the theorem can be simplified by requiring that the function $f(x)$ is continuous on $[a, b]$, the function $\varphi(t)$ acts from $[\alpha, \beta]$ to $[a, b]$ and is continuously differentiable on $[\alpha, \beta]$, and the equalities $\varphi(\alpha) = a$, $\varphi(\beta) = b$ hold.

Integration formula by parts for a definite integral

2.8A/10:45 (04:49)

Theorem (on integration by parts of a definite integral)..
Let the functions u, v be continuously differentiable on the interval (a_0, b_0) and the segment $[a, b]$ be contained in the interval (a_0, b_0). Then the following formula holds:

$$\int_a^b uv'\,dx = uv\big|_a^b - \int_a^b u'v\,dx. \tag{11}$$

Formula (11) is called the *integration formula by parts* for a definite integral. Recall that the expression $uv\big|_a^b$ means the difference $u(b)v(b) - u(a)v(a)$.

Remark.

As in the case of the theorem on changing a variable in a definite integral, if we consider the derivatives defined on the segment, assuming that the derivatives at the endpoints of the segment are calculated as one-sided limits, then the condition of the theorem can be simplified by requiring only that the functions u, v were continuously differentiable on the segment $[a, b]$.

Proof.

By the formula of the derivative of the product, we have

$$\big(u(x)v(x)\big)' = u'(x)v(x) + u(x)v'(x).$$

Let us express the product $u(x)v'(x)$ from the last equality:

$$u(x)v'(x) = \big(u(x)v(x)\big)' - u'(x)v(x).$$

The expressions on the left and on the right are continuous functions and therefore they are integrable. Integrating the left-hand side and the right-hand side of the equality from a to b and using the linearity of a definite integral with respect to the integrand, we obtain

$$\int_a^b u(x)v'(x)\,dx = \int_a^b \big(u(x)v(x)\big)'\,dx - \int_a^b u'(x)v(x)\,dx. \tag{12}$$

Obviously, the function $F(x) = u(x)v(x)$ is the antiderivative for the function $\big(u(x)v(x)\big)'$. Then, according to the Newton–Leibniz formula, we have

$$\int_a^b \big(u(x)v(x)\big)'\,dx = F(b) - F(a) = F(x)\big|_a^b = u(x)v(x)\big|_a^b.$$

Substituting the obtained representation of the integral $\int_a^b \big(u(x)v(x)\big)'\,dx$ into (12), we get equality (11). □