12. Numerical series

Numerical series: definition and examples

Definition of a numerical series

3.10A/11:35 (05:39)

Recall how the finite sum of terms is written using the summation symbol $\sum$:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

If the symbol ∞ is indicated in the notation of the sum instead of the finite number n, then this notation can be considered as a formal notation of the sum of an infinite number of terms (such a construction is called a *formal sum*):

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_k + \dots$$

The expression $\sum_{k=1}^{\infty} a_k$ is called a *numerical series*, and the value a_k is called a *common term of the series*. Thus, the series of numbers $\sum_{k=1}^{\infty} a_k$ is the formal sum of all elements of the sequence $\{a_k\}$ (the elements are taken in ascending order of their indices).

Under additional conditions, a specific numerical value (called the *sum of a series*) can be associated with a numerical series. Consider the finite sum

$$S_n = \sum_{k=1}^{n} a_k.$$

This sum is called the *partial sum of the series* $\sum_{k=1}^{\infty} a_k$; it exists for any number $n \in \mathbb{R}$. Thus, we get a sequence of partial sums $\{S_n\}$.

If there exists a finite limit S of the sequence $\{S_n\}$ as $n \to \infty$, then the numerical series $\sum_{k=1}^{\infty} a_k$ is called *convergent* and the limit S is called the *sum* of this numerical series. If the series converges, then its notation $\sum_{k=1}^{\infty} a_k$ usually means the value of its sum, i. e., the limit S (just as the notation of an improper integral means the limit value of usual proper integrals):

$$\sum_{k=1}^{\infty} a_k \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

$$\int_1^{+\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{c \to +\infty} \int_1^{c} f(x)\,dx$$

If the sequence of partial sums $\{S_n\}$ has no limit or has an infinite limit, then the series $\sum_{k=1}^{\infty} a_k$ is called *divergent*; in this case, the sum of the series is not defined (as well as the value of the divergent improper integral).

We emphasize that, in any case, the notation $\sum_{k=1}^{\infty} a_k$ can be considered as a formal sum of an infinite number of terms, regardless of whether this formal notation corresponds to some numerical value or not.

As a summation parameter, the symbols i and j are often used along with the symbol k.

The initial value of the summation parameter does not have to be 1. Series with a summation parameter starting with 0 are often considered. Obviously, if the series $\sum_{k=1}^{\infty} a_k$ converges, then the series $\sum_{k=n_0}^{\infty} a_k$ also converges for any $n_0 \in \mathbb{N}$.

Example of a numerical series: the sum of the elements of a geometric progression `3.10A/17:14 (10:21)`

Let $q \neq 0$ be an arbitrary real number. Consider a series with the common term q^k:

$$\sum_{k=0}^{\infty} q^k = 1 + q + q^2 + \cdots + q^k + \dots$$

$2^{n+1} \to +\infty$ $\quad 1+q$

$q^0 = 1$

$|q|<1$

This series is the formal sum of all terms of the geometric progression with 1 as the first term and q as the ratio.

Recall the formula for the sum of the initial terms of such a geometric progression (provided that $q \neq 1$):

$\{S_n\}$

$$S_n = \sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}.$$

$q^{n+1} \to 0$

$n = 0 \quad S_0 = 1 = \frac{1-q^1}{1-q} = 1$

$n = 1 \quad \frac{1-q^2}{1-q} = \frac{(1-q)(1+q)}{\cancel{1-q}} = 1+q$

In this case, S_n denotes the sum of $(n + 1)$ initial terms of the geometric progression. It is clear that if $q = 1$, then $S_n = n + 1$.

If $|q| < 1$, then $\lim_{n\to\infty} S_n = \frac{1}{1-q}$. If $|q| \geq 1$, then the limit of the sequence $\{S_n\}$ as $n \to \infty$ is either infinite or (for $q = -1$) does not exist (since, for $q = -1$, the sequence $\{q^n\}$ has the form $\{1, -1, 1, -1, \dots\}$ and therefore the sequence $\{S_n\}$ is equal to $\{1, 0, 1, 0, \dots\}$).

So, if $|q| \geq 1$, then the series $\sum_{k=0}^{\infty} q^k$ diverges, and if $|q| < 1$, then the series $\sum_{k=0}^{\infty} q^k$ converges and its sum is $\frac{1}{1-q}$:

$$\boxed{\sum_{k=0}^{\infty} q^k = \frac{1}{1 - q}, \qquad |q| < 1, q \neq 0.}$$

This formula is called the *formula of the sum of an infinitely decreasing geometric progression.*

Cauchy criterion for the convergence of a numerical series and a necessary condition for its convergence

$|a_n - A| < \varepsilon$

$\left(\lim_{m\to\infty} a_m = 0\right) \Leftarrow \left(\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m > N\ |a_{m+1}| < \varepsilon\right)$

Cauchy criterion for the convergence of a numerical series

`3.10A/27:35 (07:57)`

Theorem (Cauchy criterion for the convergence of a numerical series).

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if the following condition is satisfied:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N \quad \forall p \in \mathbb{N} \quad \left| \sum_{k=m+1}^{m+p} a_k \right| < \varepsilon. \tag{1}$$

$p=1$ $\quad |a_{m+1}| < \varepsilon$

Proof.

Let $S_n = \sum_{k=1}^{n} a_k$ be a partial sum of the initial series. A series converges if and only if the sequence of partial sums $\{S_n\}$ is convergent.

For the sequence $\{S_n\}$, we write the condition from the Cauchy criterion for the convergence of a sequence:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m_1, m_2 > N \quad |S_{m_2} - S_{m_1}| < \varepsilon.$$

If we put $m_1 = m$, $m_2 = m + p$ for some $p \in \mathbb{N}$, then the last condition can be rewritten in the following form:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N \quad \forall p \in \mathbb{N} \quad |S_{m+p} - S_m| < \varepsilon. \tag{2}$$

Let us transform the difference $S_{m+p} - S_m$ taking into account the formula for partial sums:

$$S_{m+p} - S_m = \sum_{k=1}^{m+p} a_k - \sum_{k=1}^{m} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{m+p} a_k - \sum_{k=1}^{m} a_k = \sum_{k=m+1}^{m+p} a_k.$$

Substituting the obtained expression for the difference $S_{m+p} - S_m$ into condition (2), we obtain condition (1).

So, we have shown that condition (1) is necessary and sufficient for the convergence of the sequence $\{S_n\}$, and the convergence of this sequence takes place if and only if the initial series converges. $\square$

A necessary condition for the convergence of a numerical series

`3.10A/35:32 (04:58)`

Corollary (a necessary condition for the convergence of a numerical series).

If the series $\sum_{k=1}^{\infty} a_k$ converges, then its common term a_k approaches zero:

$\sum \underbrace{(1+\overset{\to 0}{q^n})}_{\to 1}$, $|q|<1$

diverges!!

$\sum_{k=1}^{\infty} \frac{1}{k}$

diverges!

$$\lim_{k\to\infty} a_k = 0.$$

REMARKS.

1. This condition means that if the common term of a series does not approach 0, then the series is not convergent. Thus, it makes it easy to prove the divergence of many series. However, it should be emphasized that this condition is not a sufficient condition for convergence: from the fact that the common term of a series approaches 0, it does not follow that the series converges (we will give the corresponding examples later).

2. A similar condition for improper integrals over a semi-infinite interval, generally speaking, does not hold. There exist conditionally convergent improper integrals of the form $\int_a^{+\infty} f(x)\,dx$ for which the integrand $f(x)$ does not approach zero as $x \to +\infty$. An example of such an integral is $\int_1^{+\infty} \sin e^x\,dx$. It is easy to prove the convergence of this integral by changing the variable $t = e^x$, since, as a result of this changing, the integral will take the form $\int_e^{+\infty} \frac{\sin t}{t}\,dt$. At the same time, if the function $f(x)$ is non-negative and non-increasing on the interval $[a, +\infty)$, then the convergence of the integral $\int_a^{+\infty} f(x)\,dx$ implies that $\lim_{x\to+\infty} f(x) = 0$ (this fact follows from the integral convergence test considered in the next chapter).

PROOF.

Since the initial series converges, condition (1) of the Cauchy criterion for the convergence of a numerical series is fulfilled for it. We put $p = 1$ in this condition (this can be done, since it is allowed to take any $p \in \mathbb{N}$ in condition (1)):

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N \quad \left| \sum_{k=m+1}^{m+1} a_k \right| < \varepsilon. \tag{3}$$

Since $\sum_{k=m+1}^{m+1} a_k = a_{m+1}$, the last inequality takes the form $|a_{m+1}| < \varepsilon$.

Thus, condition (3) coincides with the definition (in the language ε–N) of a convergent sequence $\{a_k\}$ in the case when its limit is 0. □

Absolutely convergent numerical series and arithmetic properties of convergent numerical series

Absolutely convergent numerical series

`3.10A/40:30 (01:44), 3.10B/00:00 (03:09)`

DEFINITION.

The series $\sum_{k=1}^{\infty} a_k$ *absolutely converges* if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Theorem (on the convergence of an absolutely convergent numerical series).

If the series absolutely converges, then it is convergent.

Proof.

Let the series $\sum_{k=1}^{\infty} a_k$ absolutely converge. This means that the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Therefore, by virtue of the necessary part of the Cauchy criterion for the convergence of a numerical series, condition (1) is satisfied:

necessary part

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m > N \quad \forall p \in \mathbb{N} \quad \sum_{k=m+1}^{m+p} |a_k| < \varepsilon.$$

The sum $\sum_{k=m+1}^{m+p} |a_k|$ can be estimated from below using the following absolute value property (which is a generalization of the triangle inequality for the case of the sum of n terms):

$$\left| \sum_{k=m+1}^{m+p} a_k \right| \leq \sum_{k=m+1}^{m+p} |a_k|. \quad < \varepsilon$$

Since the right-hand side of the last inequality is bounded from above by ε, the same estimate holds for the left-hand side of the inequality:

$$\left| \sum_{k=m+1}^{m+p} a_k \right| < \varepsilon.$$

sufficient part

This inequality coincides with condition (1) of the Cauchy criterion for the convergence of the numerical series $\sum_{k=1}^{\infty} a_k$. Therefore, by virtue of a sufficient part of the Cauchy criterion, this series converges. □

Arithmetic properties of convergent numerical series

3.10B/03:09 (08:19)

Theorem (on arithmetic properties of convergent numerical series).

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be convergent series with sums S_a and S_b, respectively. Let $\alpha, \beta \in \mathbb{R}$.

Then the series $\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k)$ also converges and its sum is $\alpha S_a + \beta S_b$.

Thus, for convergent series, the same arithmetic transformations can be used as for finite sums:

$$\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k. \tag{4}$$

In addition, if the initial series converge absolutely, then the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ also converges absolutely.

PROOF.

Let us introduce partial sums:

$$S'_n = \sum_{k=1}^{n} a_k, \quad S''_n = \sum_{k=1}^{n} b_k, \quad S_n = \sum_{k=1}^{n}(\alpha a_k + \beta b_k).$$

$\alpha S_a + \beta S_b$

Obviously, for these finite sums, the equality holds:

$$S_n = \sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k = \alpha S'_n + \beta S''_n.$$

Since, by condition, $\lim_{n\to\infty} S'_n = S_a$, $\lim_{n\to\infty} S''_n = S_b$, we obtain, by arithmetic properties of the limit of a sequence, that the limit S_n as $n \to \infty$ exists and is equal to $\alpha S_a + \beta S_b$. Thus, we simultaneously proved the convergence of the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ and formula (4).

To prove the absolute convergence of the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ in the case when the initial series absolutely converge, we use the Cauchy criterion.

For $\alpha = \beta = 0$, the statement is obvious; therefore, we will assume that $|\alpha| + |\beta| \neq 0$. Let us choose the value $\varepsilon > 0$. For absolutely convergent initial series, by virtue of the Cauchy criterion, the following conditions are satisfied:

$$\exists\, N_1 \in \mathbb{N} \quad \forall m > N_1 \quad \forall p \in \mathbb{N} \qquad \sum_{k=m+1}^{m+p} |a_k| < \frac{\varepsilon}{|\alpha| + |\beta|},$$

$$\exists\, N_2 \in \mathbb{N} \quad \forall m > N_2 \quad \forall p \in \mathbb{N} \qquad \sum_{k=m+1}^{m+p} |b_k| < \frac{\varepsilon}{|\alpha| + |\beta|}.$$

If we put $N = \max\{N_1, N_2\}$, then the following estimate will be true for any $m > N$ and $p \in \mathbb{N}$:

$$\sum_{k=m+1}^{m+p} |\alpha a_k + \beta b_k| \leq \sum_{k=m+1}^{m+p} (|\alpha| \cdot |a_k| + |\beta| \cdot |b_k|) =$$

$$= |\alpha| \sum_{k=m+1}^{m+p} |a_k| + |\beta| \sum_{k=m+1}^{m+p} |b_k| < |\alpha| \cdot \frac{\varepsilon}{|\alpha| + |\beta|} + |\beta| \cdot \frac{\varepsilon}{|\alpha| + |\beta|} = \varepsilon.$$

So, we have proved that, for the series $\sum_{k=1}^{\infty} |\alpha a_k + \beta b_k|$, condition (1) of the Cauchy criterion is fulfilled. Therefore, this series converges, which means that the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ converges absolutely. □