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# HIGHER MATHEMATICS, PART 2

TextBook

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# Chapter 7

# FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

## 7.1. Introduction

Let $x$ be the independent variable, and let $y$ be the dependent variable.

A **differential equation** is an equation, which involves the derivative of a function $y(x)$. The equation may also contain the function itself as well as the independent variable.

The general form of a differential equation of the first order is

$$F(x, y, y') = 0. \tag{1}$$

The solution procedure consists in finding the unknown function $y(x)$, which obeys equation (1) on a given interval.

The general solution of equation (1) is a function $y = \varphi(x, C)$, which is the solution of (1) for any values of a parameter $C$. By setting $C = const$ we obtain a particular solution of equation (1).

Sometimes the solution can be found in the implicit form only. If the equation

$$\Phi(x, y, C) = 0, \tag{2}$$

Cauchy's problem

determines the general solution of (1), then it is called the general integral of the differential equation.

If there given an initial condition $y(x_0) = y_0$ in addition to equation (1), then it is necessary to find the particular solution, which obeys the initial condition.

Here we consider only such classes of first-order differential equations, which can be solved analytically.

$F'(x) = f(x)$

$y$ - antiderivative

## 7.2. Directly Integrable Equations

$\int dy = y + C$

A directly integrable differential equation has the following form:

$$y' = f(x), \tag{3}$$

$y' = f(x)$

where $f(x)$ is a given function.

From this equation follows that the function $y(x)$ is a primitive of $f(x)$ and hence

$$y(x) = \int f(x)dx \cancel{+C}. = F(x) + C \tag{4}$$

$y' = \boxed{\dfrac{dy}{dx}}$

A constant $C$ can be determined from the initial condition, if the one is given.

$\dfrac{dy}{dx} = f(x)$

$\int dy = \int f(x)\,dx$

$y = \int f(x)\,dx$

$y = \int dy = \int f(x)\,dx = F(x) + C$

$\int x\,dx + \int \cos x\,dx = \frac{x^2}{2} + \sin x + C$

$1 = 0 + 0 + C$



**Example**: Find the solution of the equation

$$y'(x) = x + \cos x$$

with the initial condition $y(0) = 1$.

**Solution**: In view of (4) the general solution is

$$y(x) = \int (x + \cos x)dx = \frac{x^2}{2} + \sin x + C.$$

Taking into account the initial condition, we find: $1 = 0 + C$, that is, $C = 1$. Therefore, the function $y(x) = x^2/2 + \sin x + 1$ being the solution of the given equation, satisfies the initial condition.

## 7.3. Separable Equations

A separable differential equation is an equation of the form

$$y' = f(x)g(y), \tag{5}$$

that is, $y'(x)$ equals the product of given functions, $f(x)$ and $g(y)$, each of which is a function of one variable only.

We can not integrate equation (5) directly because the right-hand side contains an unknown function $y(x)$ together with the variable $x$.

To separate the variables we rewrite the equation in the form:

$y = \cancel{\varphi(x)} + C$

$$\frac{dy}{g(y)} = f(x)dx \tag{5a}$$

and then integrate both sides:

$G(y) = F(x) + C$

$$\int \frac{dy}{g(y)} = \int f(x)dx. \tag{6}$$

Thus, the general integral of equation (5) is found.

A differential equations of the form

$$y' = f(ax + by + c) \tag{7}$$

can be reduced to a separable equation by introducing of a new dependent variable $u(x)$ instead of $y$:

$$u = ax + by + c. \tag{8}$$

Next we have to derive the equation for the variable $u(x)$. By differentiating (8), we obtain $u' = a + by'$, which implies the equation

$$u' = a + b\,f(u) \qquad 1$$

being the separable equation.

Then we obtain $\int \frac{du}{b\,f(u) + a} = \int dx \quad \Rightarrow \quad \int \frac{du}{b\,f(u) + a} = x + C.$

**Example 1:** Solve the equation

$$y' = e^{2x-3y}.$$

$\frac{dy}{dx} = \underbrace{e^{2x}}_{f(x)} \cdot \underbrace{e^{-3y}}_{g(y)}$

**Solution**: The variables can be easily separated:

$$\int e^{3y}dy = \int e^{2x}dx.$$

By integrating, we obtain a general integral of the given equation:

$$\frac{1}{3}e^{3y} = \frac{1}{2}e^{2x} + C_1.$$

$e^{3y} = \frac{3}{2}e^{2x} + C'$

$\ln e^{3y} = \ln\left(\frac{3}{2}e^{2x} + C'\right)$

$3y$

$\ln e = 1$

By means of simple formula manipulations we can also write the general solution in the explicit form:

$$y = \frac{1}{3}\ln(\frac{3}{2}e^{2x} + C),$$

where the constant $3C_1$ is denoted by $C$.

**Example 2**: Find the solution of the equation

$$y' = \cos(x + y), \tag{9}$$

which obeys the initial condition $y(0) = \pi/2$.

**Solution**: Let us introduce a new variable:

$$u = x + y.$$

$u' = 1 + y' = 1 + \cos(x + y)$

Then from (9) we obtain the separable equation for $u(x)$

$$u' = 1 + \cos u.$$

$g(u)\ f(x) = 1$

By separating the variables and integrating, we have:

$\int \frac{du}{1+\cos u} = \int 1 \cdot dx$

$$\int \frac{du}{1+\cos u} = x + C.$$

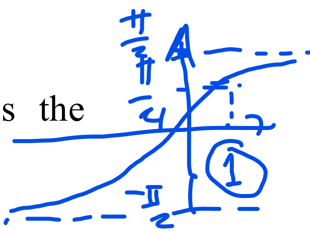
Using the formula $1 + \cos u = 2\cos^2 u/2$ we obtain the algebraic equation

$$\tan(u/2) = x + C,$$

$\frac{u}{2} = \arctan(x + C)$

which implies

$$u = 2\arctan(x + C).$$



Since $y = u - x$, the general solution of the given equation is the following one: $y = 2\arctan(x + C) - x$.

The initial condition yields: $\pi/2 = 2\arctan C$, so that $C = 1$.

$\arctan C = \frac{\pi}{4}$

$\Downarrow$

$C = 1$

Finally we obtain:

$$y = 2\arctan(x + 1) - x.$$

## 7.4. Homogeneous Equations

If some differential equation can be represented in the following form:

$$y' = f(\frac{y}{x}), \tag{10}$$

then it is called a homogeneous equation.

One of the main methods of solving differential equations is based on introducing a new dependent variable $u(x)$ instead of $y$. There is no general rule to make the right choice of $u$ because it depends on the form of the equation. That is why it is necessary to consider different classes of equations separately. One of typical techniques of such a kind is illustrated below by solving an homogeneous equation.

The right-hand side of equation (10) suggests the substitution $u = y/x$.

Then we have to derive the equation for the new dependent variable $u$.

To find the derivative of $y = ux$, we use the rule of differentiation of the product:

$$y' = u'x + u. = f(\frac{y}{x})$$

$u$

From (10) we obtain the equation

$$u'x + u = f(u),$$

which being rewritten in the form $x$ $u$

$$u' = \frac{1}{x}(f(u) - u) \tag{11}$$

is a separable equation. Then the problem of integration is solved just in the same way as above. (See equation (5).)

$$\frac{1}{x}\sqrt{x \cdot y} = \sqrt{\frac{xy}{x^2}} = \sqrt{\frac{y}{x}}$$

**Example**: Solve the equation

$$y' = \frac{y}{x - \sqrt{xy}}. \tag{12}$$

**Solution**: Since

$$\frac{1}{x} : \frac{y}{x - \sqrt{xy}} = \frac{y/x}{1 - \sqrt{y/x}} = f(\frac{y}{x}),$$

$$f(t) = \frac{t}{1 - \sqrt{t}}$$

the given equation is the homogeneous equation.

To solve this problem, we introduce the variable $u = y/x$ instead of $y$ and derive a differential equation for $u(x)$.

$u' = \frac{du}{dx}$

First, $y = ux$, so $y' = u'x + u$. Therefore, by (12)

$$u'x + u = \frac{u}{1 - \sqrt{u}} \quad \Rightarrow \quad u'x = \frac{\sqrt{u^3}}{1 - \sqrt{u}} \quad \Rightarrow$$

$(1-\sqrt{u})$

$$u'x = \frac{u}{1 - \sqrt{u}} - u = \frac{\cancel{u} - \cancel{u} + u\sqrt{u}}{1 - \sqrt{u}} = \frac{\sqrt{u^3}}{1 - \sqrt{u}}$$

$\int u^{d} dx = \frac{u^{d+1}}{d+1}$

$d \neq -1$

$$\frac{1-\sqrt{u}}{\sqrt{u^3}}du = \frac{dx}{x} \quad \Rightarrow \quad \int (u^{-3/2} - \frac{1}{u})du + C = \int \frac{dx}{x} \quad \Rightarrow$$

$$-2/\sqrt{u} - \ln|u| = \ln|x| + C.$$

Replacing $u$ by $y/x$ we obtain the general integral of equation (12):

$-\ln|u| = -\ln|\frac{y}{x}| =$

$= -\ln|y| + \cancel{\ln|x|}$

$$\ln|y| + 2\sqrt{x/y} = C. \tag{13}$$

## 7.5. Linear Equations

A linear differential equation is an equation, which can be represented as

$$y' + P(x)y = Q(x), \tag{14}$$

where $P(x)$ and $Q(x)$ are given functions.

To solve the equation, we introduce a new dependent variable $u(x)$ instead of $y$ by the equality

$$y = u(x)\,v(x), \tag{15}$$

keeping in mind to determine a function $v(x)$ later.

To derive the differential equation for $u(x)$ we find the derivative $y' = u'v + uv'$ and substitute it into original equation (14):

$$u'v + v'u + P(x)\,u\,v = Q(x).$$

$v' = -P(x) \cdot v$

Next we group the terms and take out the common factor:

$$u'v + u(v' + P(x)v) = Q(x). \tag{16}$$

$\frac{dv}{v} = -P(x)dx$

Now we are ready to determine the function $v(x)$. Let $v(x)$ be a function such that

$$v' + P(x)v = 0. \tag{17}$$

By separating the variables, we obtain the solution of equation (17):

$$\int \frac{dv}{v} = -\int P(x)dx \quad \Rightarrow \quad \ln|v| = -\int P(x)dx \quad \Rightarrow$$

$$v = e^{-\int P(x)dx}. \tag{18}$$

A constant of integration is chosen to be equal to zero because it is enough to have one function only, which obeys condition (17).

$u'v = Q(x)$

In view of (18), equation (16) is reduced to the directly integrable equation of the form

$u' = \frac{Q(x)}{v}$

$$u' = Q(x)e^{f(x)}, \tag{19}$$

where $f(x) = \int P(x)dx$ is one of primitives of $P$.

Therefore,

$$u(x) = \int Q(x)e^{f(x)}dx + C. \tag{20}$$

Thus, equation (14) has the following general solution:

$$y(x) = e^{-f(x)}\left(\int Q(x)e^{f(x)}dx + C\right). \tag{21}$$

**Example**: Find the general solution of the equation

$P(x) = \frac{-3}{x}$

$Q(x) = x$

$$y' = 3y/x + x. \tag{22}$$

**Solution**: Let $y = uv$. Then $y' = u'v + uv'$.

$\frac{dv}{dx} = \frac{3v}{x}$

Substituting these expressions into the original equation, we obtain

$$u'v + v'u = 3uv/x + x \qquad \Rightarrow$$

$$u'v + \cancel{u(v' - 3v/x)} = x. \tag{23}$$

$\frac{dv}{v} = \frac{3\,dx}{x}$

Then we find the function $v(x)$ by solving of the equation

$$v' - 3v/x = 0.$$

The variables are easily separated and we have

$$\int \frac{dv}{v} = 3\int \frac{dx}{x} \qquad \Rightarrow \qquad \ln|v| = 3\ln|x| \qquad \Rightarrow \qquad v = x^3.$$

Now we come back to (23), which is reduced to the separable equation

$$u'x^3 = x.$$

$u' = \frac{1}{x^2}$

Therefore,

$$u = \int \frac{dx}{x^2} = -\frac{1}{x} + C.$$

Finally, we obtain

$$y = u\,v = \left(-\frac{1}{x} + C\right)x^3 = -x^2 + Cx^3.$$

## 7.6. The Bernoulli Equations

The Bernoulli Equation is an equation of the form

$$y'(x) + P(x)y = Q(x)y^n, \tag{24}$$

where $n$ is any rational number except $0$ and $1$.

The technique of solving the Bernoulli equations is just the same as for linear equations: A new dependent variable $u(x)$ is introduced by means of the equality

$$y = u(x)v(x). \tag{25}$$

This variable satisfies the equation

$$u'v + u(v' + P(x)v) = Q(x)u^n v^n, \tag{26}$$

0

where the function $v(x)$ is a partial solution of the equation

$$v' + P(x)\,v = 0 \tag{27}$$

and hence,

$$v = e^{-\int P(x)dx}. \tag{28}$$

Therefore, equation (26) is transformed to the form

$$u'v = Q(x)\,u^n v^n$$

and can be rewritten as a separable equation:

$$u^{-n}du = Q(x)\,v^{n-1}dx\,.$$

By integrating, we obtain

$$\frac{1}{-n+1}u^{-n+1} = \int Q(x)\,v^{n-1}dx + C\,. \tag{29}$$

Thus,

$$u(x) = \Big((1-n)\int Q(x)\,v^{n-1}dx + C\Big)^{\frac{1}{1-n}}. \tag{30}$$

The general solution of (24) is $y(x) = u(x)\,v(x)$.

**Example**: Find the general solution of the equation

$$y' + 4xy = 2xe^{-x^2}\sqrt{y}\,. \tag{31}$$

$\sqrt{y} = y^{1/2}$ $\quad \frac{1}{2} = n$

**Solution**: Let $y = uv$. Since the derivative of $y$ is $y' = u'v + uv'$, then (31) can be transformed to the equation with respect to the variable $u(x)$:

$$u'v + v'u + 4xuv = 2xe^{-x^2}\sqrt{uv} \qquad \Rightarrow$$

$$u'v + u(v' + 4xv) = 2xe^{-x^2}\sqrt{uv}\,. \tag{32}$$

To find the function $v(x)$, we solve the equation

$$v' + 4vx = 0\,.$$

This is the separable equation, and its partial solution is

$$v = e^{-2x^2}. \tag{33}$$

$\sqrt{e^{-2x^2}} = e^{-x^2}$

From (32) we have

$$u'\cancel{e^{-2x^2}} = 2x\cancel{e^{-x^2}}\sqrt{u\cancel{e^{-2x^2}}} \quad \Rightarrow \quad u' = 2x\sqrt{u} \quad \Rightarrow$$

$$\int \frac{du}{\sqrt{u}} = \int 2xdx + C \quad \Rightarrow \quad 2\sqrt{u} = x^2 + C \quad \Rightarrow$$

$$u = (x^2 + C)^2/4\,. \tag{34}$$

Therefore, the general solution of the given equation is

$$y(x) = \frac{1}{4}(x^2 + C)^2 e^{-2x^2}\,. \tag{35}$$