11. Absolute and conditional convergence of improper integrals

Cauchy criterion for the convergence of an improper integral

3.9A/00:00 (10:08)

Theorem (Cauchy criterion for the convergence of an improper integral).

Let the function f be defined on the interval $[a, b)$ and there exists the integral $\int_a^c f(x)\,dx$ for any point $c \in (a, b)$. The improper integral $\int_a^b f(x)\,dx$ converges if and only if the following condition is satisfied:

$b = +\infty$

B

$$\forall\, \varepsilon > 0 \quad \exists\, B \in (a, b) \quad \forall\, c', c'', B < c' < c'' < b,$$

$$\left| \int_{c'}^{c''} f(x)\,dx \right| < \varepsilon. \tag{1}$$

Proof.

Let us introduce the auxiliary function $\Phi(c) = \int_a^c f(x)\,dx$. According to the definition of an improper integral, the convergence of the integral $\int_a^b f(x)\,dx$ is equivalent to the existence of the limit of the function $\Phi(c)$ as $c \to b - 0$.

By virtue of the Cauchy criterion for the existence of a function limit, the limit of $\Phi(c)$ as $c \to b - 0$ exists if and only if the following condition is satisfied:

$$\forall\, \varepsilon > 0 \quad \exists\, B \in (a, b) \quad \forall\, c', c'', B < c' < c'' < b,$$

$$|\Phi(c'') - \Phi(c')| < \varepsilon. \tag{2}$$

Let us transform the difference $\Phi(c'') - \Phi(c')$:

$$\Phi(c'') - \Phi(c') = \int_a^{c''} f(x)\,dx - \int_a^{c'} f(x)\,dx =$$

$$= \int_a^{c'} f(x)\,dx + \int_{c'}^{c''} f(x)\,dx - \int_a^{c'} f(x)\,dx = \int_{c'}^{c''} f(x)\,dx.$$

After substituting the found expression for the difference $\Phi(c'') - \Phi(c')$ into condition (2), we obtain condition (1).

Thus, condition (1) is equivalent to the existence of the limit $\lim_{c\to b-0}\Phi(c)$ and the existence of this limit is equivalent to the convergence of the integral $\int_a^b f(x)\,dx$, therefore the condition (1) is necessary and sufficient for the convergence of this integral. $\square$

Absolute convergence of improper integrals

3.9A/10:08 (06:46)

Definition.

Let the function f be defined on the interval $[a,b)$. The improper integral $\int_a^b f(x)\,dx$ is called *absolutely convergent* if the integral $\int_a^b |f(x)|\,dx$ converges.

Theorem (on the convergence of an absolutely convergent integral).

If the improper integral $\int_a^b f(x)\,dx$ absolutely converges, then it converges.

Remark.

The converse is not true: we will show later that a convergent improper integral is not necessarily absolutely convergent. Thus, the property of absolute convergence is stronger than the property of usual convergence.

Proof.

We are given that the integral $\int_a^b |f(x)|\,dx$ converges, and we need to prove that the integral $\int_a^b f(x)\,dx$ converges.

Since the integral $\int_a^b |f(x)|\,dx$ converges, by the necessary condition of the Cauchy criterion for improper integrals, we get

$$\forall\,\varepsilon>0\quad \exists\,B\in(a,b)\quad \forall\,c',c'',B<c'<c''<b,$$
$$\left|\int_{c'}^{c''}|f(x)|\,dx\right|<\varepsilon. \tag{3}$$

Since $c'<c''$, the last inequality in condition (3) can be rewritten without specifying the external absolute value sign on the left-hand side:

$$\int_{c'}^{c''}|f(x)|\,dx<\varepsilon. \tag{4}$$

Recall the property of the integral of the absolute value of a function:

$$\left|\int_{c'}^{c''}f(x)\,dx\right|\le\int_{c'}^{c''}|f(x)|\,dx. \tag{5}$$

Estimates (4) and (5) imply the following estimate:

$$\left|\int_{c'}^{c''}f(x)\,dx\right|<\varepsilon. \tag{6}$$

Thus, we can use (6) as the last inequality in condition (3):

$$\forall\,\varepsilon > 0 \quad \exists\,B \in (a,b) \quad \forall\,c', c'', B < c' < c'' < b, \quad \left|\int_{c'}^{c''} f(x)\,dx\right| < \varepsilon.$$

This means, due to the sufficient condition of the Cauchy criterion for improper integrals, that the integral $\int_a^b f(x)\,dx$ converges. □

Properties of improper integrals of non-negative functions

Criterion for the convergence of improper integrals of non-negative functions

`3.9A/16:54 (10:00)`

In this section, we consider improper integrals of non-negative functions. Since the absolute value of the function is non-negative, all the results obtained in this section can also be used to study the absolute convergence of improper integrals of functions taking both negative and positive values.

THEOREM (CRITERION FOR THE CONVERGENCE OF IMPROPER INTEGRALS OF NON-NEGATIVE FUNCTIONS).

Let a function f be defined on $[a,b)$ and $f(x) \geq 0$ for any value $x \in [a,b)$. Suppose that, for any $c \in [a,b)$, there exists an integral $\int_a^c f(x)\,dx$. Then the improper integral $\int_a^b f(x)\,dx$ converges if and only if the set of values of all integrals $\int_a^c f(x)\,dx$ is bounded from above:

$$\exists\,M > 0 \quad \forall\,c \in [a,b) \quad \int_a^c f(x)\,dx \leq M. \tag{7}$$

PROOF.

We introduce an auxiliary function $\Phi(c) = \int_a^c f(x)\,dx$.

The inequality $f(x) \geq 0$, which holds, by condition, for all $x \in [a,b)$, implies the inequality $\int_{c'}^{c''} f(x)\,dx \geq 0$ for any $c', c'' \in [a,b)$ such that $c' < c''$. Therefore, for $c' < c''$, we have

$$\Phi(c'') = \int_a^{c''} f(x)\,dx = \int_a^{c'} f(x)\,dx + \int_{c'}^{c''} f(x)\,dx =$$

$$= \Phi(c') + \int_{c'}^{c''} f(x)\,dx \geq \Phi(c').$$

We obtain that, for all $c' < c''$, the estimate $\Phi(c') \leq \Phi(c'')$ is true. This means that the function $\Phi(c)$ is non-decreasing on the interval $[a,b)$.

1. Sufficiency. Given: condition (7) is satisfied. Prove: the integral $\int_a^b f(x)\,dx$ converges.

Condition (7) means that the function $\Phi(c)$ is bounded from above on the interval $[a,b)$. Thus, the function $\Phi(c)$ is non-decreasing and bounded from above on the interval $[a,b)$, therefore, by virtue of the theorem on the limit of a monotonous and upper-bounded function, there exists a limit of the function $\Phi(c)$ as $c \to b-0$ (equal to $\sup_{c\in[a,b)} \Phi(c)$). It remains to note that the existence of this limit is equivalent to the convergence of the improper integral $\int_a^b f(x)\,dx$. The sufficiency is proven.

2. Necessity. Given: the integral $\int_a^b f(x)\,dx$ converges. Prove: condition (7) is satisfied.

As noted above, if the integral $\int_a^b f(x)\,dx$ converges, then there exists a limit $\lim_{c\to b-0}\Phi(c) = M$. Since the function $\Phi(c)$ is non-decreasing, we obtain that, for all $c', c'' \in [a,b)$ such that $c' < c''$, the inequality holds:

$$\Phi(c') \le \Phi(c'').$$

In this inequality, we pass to the limit as $c'' \to b-0$. By virtue of the theorem on passing to the limit in the inequalities, the following inequality holds for any $c' \in [a,b)$:

$$\Phi(c') \le M.$$

Substituting the definition of the function Φ into this inequality, we obtain the inequality from condition (7). □

The comparison test

3.9A/26:54 (10:35)

Theorem (the comparison test for improper integrals of non-negative functions).

Let the functions f and g be defined on the interval $[a,b)$ and the double inequality $0 \le f(x) \le g(x)$ holds for any $x \in [a,b)$. Suppose that, for any $c \in [a,b)$, there exist integrals $\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$. Then the following two statements are true.

1. If the improper integral $\int_a^b g(x)\,dx$ converges, then the integral $\int_a^b f(x)\,dx$ also converges.

2. If the improper integral $\int_a^b f(x)\,dx$ diverges, then the integral $\int_a^b g(x)\,dx$ also diverges.

Proof.

1. Let the integral $\int_a^b g(x)\,dx$ converge. Then, by the necessary condition of the previous criterion, we obtain

$$\exists\, M > 0 \quad \forall\, c \in [a, b) \quad \int_a^c g(x)\, dx \le M. \tag{8}$$

Since $f(x) \le g(x)$ for all $x \in [a, b)$, a similar inequality holds for the proper integrals:

$$\int_a^c f(x)\, dx \le \int_a^c g(x)\, dx. \tag{9}$$

Combining estimates (8) and (9), we obtain

$$\int_a^c f(x)\, dx \le \int_a^c g(x)\, dx \le M.$$

Thus, condition (8) is also satisfied for the integral $\int_a^c f(x)\, dx$. Therefore, by virtue of a sufficient part of the previous criterion, the integral $\int_a^b f(x)\, dx$ converges.

2. Let the integral $\int_a^b f(x)\, dx$ diverge.

If we assume that the integral $\int_a^b g(x)\, dx$ converges, then by the already proved statement 1, the integral $\int_a^b f(x)\, dx$ should also converge. But this contradicts the condition. Therefore, the assumption made is false and the integral $\int_a^b g(x)\, dx$ diverges. □

Remark.

Obviously, the theorem remains valid if the functions f and g satisfy the double inequality $0 \le f(x) \le Cg(x)$ with some constant $C > 0$ on the interval $[a, b)$.

Corollary of the comparison test

3.9A/37:29 (04:25)

$\lim\limits_{x \to b-0} \frac{f(x)}{g(x)} = 1$

Corollary.

Let the functions f and g be defined on the interval $[a, b)$ and be non-negative on this interval. Let $\underline{f(x) \sim g(x)}$ as $x \to b - 0$. Then the integrals $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ either both converge or both diverge.

Proof[1].

~~The equivalence of the functions f and g as $x \to b-0$ means that in some left-hand neighborhood U_b^- of the point b, the relation $f(x) = \alpha(x)g(x)$ holds, where $\alpha(x) \to 1$ as $x \to b - 0$.~~

~~Since $\lim_{x \to b-0} \alpha(x) = 1$, we can choose a neighborhood $V_b^- \subset U_b^-$, in which the double inequality holds for the function $\alpha(x)$:~~

$$1 - \frac{1}{2} < \alpha(x) < 1 + \frac{1}{2}.$$

[1] There is no proof of the corollary in video lectures.

We multiply all parts of this inequality by $g(x)$ and take into account that the equality $f(x) = \alpha(x)g(x)$ is true in the neighborhood V_b^-:

$$\Big(1 - \frac{1}{2}\Big)g(x) < \alpha(x)g(x) < \Big(1 + \frac{1}{2}\Big)g(x),$$

$$\frac{1}{2}\,g(x) < f(x) < \frac{3}{2}\,g(x). \tag{10}$$

Choosing some value $B \in V_b^-$, we obtain that inequality (10) holds for all $x \in [B, b)$.

Suppose that the integral $\int_B^b f(x)\,dx$ converges. Then, taking into account statement 1 of the previous theorem, the remark, and the estimate $g(x) < 2f(x)$, which follows from the left-hand side of (10), we obtain that the integral $\int_B^b g(x)\,dx$ also converges. If we assume that the integral $\int_B^b g(x)\,dx$ converges, then from the right-hand side of (10) (i. e., $f(x) < \frac{3}{2}\,g(x)$), it follows that the integral $\int_B^b f(x)\,dx$ also converges.

On the other hand, if we assume that the integral $\int_B^b g(x)\,dx$ diverges, then, taking into account statement 2 of the previous theorem, the remark, and the estimate $g(x) < 2f(x)$, which follows from the left-hand side of (10), we obtain that the integral $\int_B^b f(x)\,dx$ also diverges, and if we assume that the integral $\int_B^b f(x)\,dx$ diverges, then it follows from the right-hand side of (10) that the integral $\int_B^b g(x)\,dx$ also diverges.

So, we have proved that the improper integrals $\int_B^b f(x)\,dx$ and $\int_B^b g(x)\,dx$ either both converge or both diverge. Taking into account the theorem on the additivity of an improper integral with respect to the integration interval, we obtain that the same statement holds for the initial integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$. □

Examples of using the comparison test 3.9B/00:00 (08:23)

1. Consider the integral $\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$.

For the absolute value of the integrand, the following estimate holds:

$$\left|\frac{\sin x}{x^2}\right| \le \frac{1}{x^2}.$$

Earlier, we proved that the integral $\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges. Therefore, the integral $\int_1^{+\infty}\left|\frac{\sin x}{x^2}\right|dx$ also converges by the comparison test. And this, in turn, means that the initial integral converges absolutely.

In a similar way, one can prove that absolute convergence holds for the integral $\int_1^{+\infty} \frac{\sin x}{x^\alpha}\,dx$ for any $\alpha > 1$.

2. Consider the integral $\int_2^{+\infty} \frac{1}{\ln x}\, dx$.

For any $x > 1$, the double estimate $0 < \ln x < x$ is valid. It follows that $\frac{1}{x} < \frac{1}{\ln x}$. Earlier, we proved that the integral $\int_1^{+\infty} \frac{1}{x}\, dx$ diverges. Obviously, the integral $\int_2^{+\infty} \frac{1}{x}\, dx$ also diverges. Then the initial integral $\int_2^{+\infty} \frac{1}{\ln x}\, dx$ also diverges by the comparison test.

3. Consider the integral $\int_1^{+\infty} \frac{1}{x^\alpha + \sin x}\, dx$, $\alpha > 1$.

Let us show that the integrand is equivalent to the function $\frac{1}{x^\alpha}$ as $x \to +\infty$:

$$\lim_{x\to+\infty} \frac{\frac{1}{x^\alpha+\sin x}}{\frac{1}{x^\alpha}} = \lim_{x\to+\infty} \frac{x^\alpha}{x^\alpha + \sin x} = \lim_{x\to+\infty} \frac{1}{1 + \frac{\sin x}{x^\alpha}} = 1.$$

So, we have proved that $\frac{1}{x^\alpha+\sin x} \sim \frac{1}{x^\alpha}$, $x \to +\infty$.

Since the integral $\int_1^{+\infty} \frac{1}{x^\alpha}\, dx$ converges for $\alpha > 1$, we obtain from the corollary of the comparison test that the initial integral also converges.

Conditional convergence of improper integrals

3.9B/08:23 (15:56)

Definition.

The improper integral $\int_a^b f(x)\, dx$ is called *conditionally convergent* if this integral converges and the integral $\int_a^b |f(x)|\, dx$ diverges. In other words, the integral converges conditionally if it converges, but it is not absolutely convergent.

It is clear that conditional convergence may hold only for integrals whose integrands change sign.

Example.

Consider the integral $\int_1^{+\infty} \frac{\sin x}{x}\, dx$ and show that it converges conditionally.

We begin by proving the convergence of this integral and consider the proper integral with integration limits from 1 to c, where $c > 1$. We use the integration formula by parts, setting $u = \frac{1}{x}$, $dv = \sin x\, dx$ (in this case, $v = -\cos x$, $du = -\frac{1}{x^2}\, dx$):

$$\int_1^c \frac{\sin x}{x}\, dx = -\frac{\cos x}{x}\Big|_1^c - \int_1^c \frac{\cos x}{x^2}\, dx. \tag{11}$$

The integral $\int_1^c \frac{\cos x}{x^2}\, dx$ converges (moreover, it absolutely converges). This can be proved by the comparison test (see example 1 from the previous section). Thus, the second term on the right-hand side of (11) has a finite limit as $c \to +\infty$. Let us transform the first term:

$$-\frac{\cos x}{x}\bigg|_1^c = -\frac{\cos c}{c} + \frac{\cos 1}{1}.$$

The limit of this expression as $c \to +\infty$ also exists and is equal to $\cos 1$.

Since the right-hand side of equality (11) has a finite limit as $c \to +\infty$, we conclude that the left-hand side also has a finite limit. We have proved that the integral $\int_1^{+\infty} \frac{\sin x}{x}\, dx$ converges.

It remains for us to show that the integral $\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx$ diverges. We can move the absolute value sign to the numerator, since the denominator of the integrand is positive:

$$\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx = \int_1^{+\infty} \frac{|\sin x|}{x}\, dx.$$

The function $|\sin x|$ can be estimated from below by the function $\sin^2 x$: $|\sin x| \geq \sin^2 x$ for any $x \in \mathbb{R}$. Therefore, for any $x \geq 1$, the estimate holds:

$$\frac{\sin^2 x}{x} \leq \frac{|\sin x|}{x}.$$

If we prove that the integral $\int_1^{+\infty} \frac{\sin^2 x}{x}\, dx$ diverges, then the integral $\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx$ will diverge as well.

So, it remains for us to prove the divergence of the integral $\int_1^{+\infty} \frac{\sin^2 x}{x}\, dx$. We consider the proper integral with integration limits from 1 to c, where $c > 1$, and transform it using the formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$:

$$\int_1^c \frac{\sin^2 x}{x}\, dx = \frac{1}{2}\int_1^c \frac{1 - \cos 2x}{x}\, dx = \frac{1}{2}\int_1^c \frac{1}{x}\, dx - \frac{1}{2}\int_1^c \frac{\cos 2x}{x}\, dx.$$

The second integral on the right-hand side of the last equality converges to a finite limit as $c \to +\infty$. This can be proved using the same method of integration by parts, which we previously applied in the study of the integral $\int_1^c \frac{\sin x}{x}\, dx$.

The first integral on the right-hand side equals $\ln |c|$ and therefore it approaches $+\infty$ as $c \to +\infty$. Thus, the limit of the right-hand side is $+\infty$, so the limit of the left-hand side is also $+\infty$. We have proved that the improper integral $\int_1^{+\infty} \frac{\sin^2 x}{x}\, dx$ diverges.

Therefore, the integral $\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx$ also diverges by the comparison test. The conditional convergence of the initial integral $\int_1^{+\infty} \frac{\sin x}{x}\, dx$ is proved.

Dirichlet's test for conditional convergence of an improper integral

Formulation of Dirichlet's test

3.9B/24:19 (06:36)

THEOREM (DIRICHLET'S TEST FOR CONDITIONAL CONVERGENCE OF AN IMPROPER INTEGRAL).

Let the functions f and g be defined on the interval $[a, b)$ and satisfy the following conditions:

1) the function f is continuous on $[a, b)$, and the integral $\int_a^c f(x)\,dx$ is uniformly bounded for all $c \in (a, b)$, i.e.,

$$\exists\, M > 0 \quad \forall\, c \in (a, b) \quad \left|\int_a^c f(x)\,dx\right| \le M;$$

2) the function g is continuously differentiable on $[a, b)$, and $g(c)$ monotonously approaches 0 as $c \to b - 0$ (the monotonicity condition means that $g'(c)$ preserves the sign for all $c \in (a, b)$).

Then the improper integral $\int_a^b f(x)g(x)\,dx$ converges (generally speaking, conditionally).

Proof of Dirichlet's test

3.9B/30:55 (13:14)

We introduce an auxiliary function $\Phi(c) = \int_a^c f(x)\,dx$. By condition 1, this function is uniformly bounded on (a, b):

$$\exists\, M > 0 \quad \forall\, c \in (a, b) \quad |\Phi(c)| \le M. \tag{12}$$

In addition, the function $\Phi(c)$ is differentiable on (a, b) as an integral with a variable upper limit and the continuous integrand f, and the equality $\Phi'(c) = f(c)$ holds.

Therefore, the proper integral $\int_a^c f(x)g(x)\,dx$ can be represented in the following form:

$$\int_a^c f(x)g(x)\,dx = \int_a^c \Phi'(x)g(x)\,dx.$$

The resulting integral can be transformed by the integration formula by parts, setting $u = g(x)$, $dv = \Phi'(x)\,dx$, whence $v = \Phi(x)$:

$$\int_a^c \Phi'(x)g(x)\,dx = \Phi(x)g(x)\Big|_a^c - \int_a^c \Phi(x)g'(x)\,dx =$$

$$= \Phi(c)g(c) - \Phi(a)g(a) - \int_a^c \Phi(x)g'(x)\,dx. \tag{13}$$

Since the function $\Phi(c)$ is bounded (see (12)) and the function $g(c)$ approaches 0 as $c \to b-0$ by condition 2, we obtain that the first term $\Phi(c)g(c)$ of the right-hand side approaches 0 as $c \to b-0$. The second term $\Phi(a)g(a)$ does not depend on the parameter c.

It remains to show that the integral $\int_a^c \Phi(x)g'(x)\,dx$ also has a finite limit as $c \to b-0$, i. e., that the improper integral $\int_a^b \Phi(x)g'(x)\,dx$ converges.

We prove the convergence of the integral $\int_a^b \Phi(x)g'(x)\,dx$ using the Cauchy criterion for the convergence of the improper integral.

However, we first use the Cauchy criterion for the existence of a function limit. By condition 2, the function $g(c)$ has a limit as $c \to b-0$. By virtue of the necessary part of the Cauchy criterion for the existence of a function limit, this means the following:

$$\forall\,\varepsilon>0 \quad \exists\,B\in(a,b) \quad \forall\,c',c'',B<c'<c''<b,$$
$$|g(c'')-g(c')|<\frac{\varepsilon}{M}. \tag{14}$$

In inequality (14), we used the constant M from estimate (12).

According to the Cauchy criterion, to prove the convergence of the integral $\int_a^b \Phi(x)g'(x)\,dx$, it suffices to establish the following fact:

$$\forall\,\varepsilon>0 \quad \exists\,B\in(a,b) \quad \forall\,c',c'',B<c'<c''<b,$$
$$\left|\int_{c'}^{c''}\Phi(x)g'(x)\,dx\right|<\varepsilon. \tag{15}$$

We choose an arbitrary value $\varepsilon>0$, get the value $B\in(a,b)$ from condition (14), and show that estimate (15) holds for this value B. To do this, we transform the integral $\left|\int_{c'}^{c''}\Phi(x)g'(x)\,dx\right|$ using the theorem on the integral of the absolute value of a function and estimate (12):

$$\left|\int_{c'}^{c''}\Phi(x)g'(x)\,dx\right| \le \int_{c'}^{c''}|\Phi(x)|\cdot|g'(x)|\,dx \le M\int_{c'}^{c''}|g'(x)|\,dx. \tag{16}$$

Since, by condition 2, the derivative $g'(x)$ preserves the sign on (a,b), we can move the absolute value sign outside the integral sign in the integral $\int_{c'}^{c''}|g'(x)|\,dx$:

$$\int_{c'}^{c''}|g'(x)|\,dx = \left|\int_{c'}^{c''}g'(x)\,dx\right|.$$

Indeed, if the derivative $g'(x)$ is always positive, then the absolute value can be omitted, and if the derivative $g'(x)$ is always negative, then the minus sign can be taken out of the integral sign, the positive function remains under the integral sign, and the external minus can be removed using the external operation of taking the absolute value.

The integral on the right-hand side of the last equality can be transformed according to the Newton–Leibniz formula:

$$\int_{c'}^{c''} g'(x)\,dx = g(c'') - g(c').$$

Thus, the chain of inequalities (16) can be continued as follows:

$$M\int_{c'}^{c''} |g'(x)|\,dx \le M|g(c'') - g(c')|.$$

Since the values of c' and c'' are chosen so that condition (14) is satisfied, the expression $M|g(c'') - g(c')|$ is estimated by ε. Taking into account that the transformations in the chain of inequalities (16) started with the integral $\left|\int_{c'}^{c''} \Phi(x)g'(x)\,dx\right|$, we finally obtain the estimate

$$\left|\int_{c'}^{c''} \Phi(x)g'(x)\,dx\right| < \varepsilon.$$

Thus, condition (15) is satisfied. Therefore, by virtue of a sufficient part of the Cauchy criterion for the convergence of the improper integral, the integral $\int_a^b \Phi(x)g'(x)\,dx$ converges.

We have proved that all terms on the right-hand side of equality (13) have a finite limit as $c \to b - 0$. This means that the integral $\int_a^c f(x)g(x)\,dx$ also has a finite limit and therefore the initial improper integral $\int_a^b f(x)g(x)\,dx$ converges. $\square$

Integrals with several singularities

3.10A/00:00 (11:35)

Let the function f be defined on the interval (a, b) and either the endpoints of this interval are points at infinity or the function is unbounded in a neighborhood of these endpoints (or a combination of these situations takes place). Then the improper integral $\int_a^b f(x)\,dx$, which has singularities at both endpoints of the integration interval, can be represented as the sum of the improper integrals considered above with unique singularity:

$$\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx. \tag{17}$$

Here d is some point belonging to the interval (a, b). If both integrals on the right-hand side of equality (17) converge, then the initial integral $\int_a^b f(x)\,dx$ is also convergent and its value is equal to the sum of the values of the integrals on the right-hand side. If at least one integral on the right-hand side diverges, then the initial integral is divergent too.

The same can be done if the singularity arises at some internal point of the integration interval. Let the function f be defined on the set $[a,b] \setminus \{d\}$ and be unbounded in a neighborhood of the point d. Then the integral $\int_a^b f(x)\,dx$ must be understood as an improper integral defined by the same relation (17), in which the improper integrals on the right-hand side have unique singularity at the point d.

In this case, given the definition of an improper integral, the value of the integral $\int_a^b f(x)\,dx$ (provided that it converges) will be equal to the sum of the following limits:

$$\int_a^b f(x)\,dx = \lim_{c \to d-0} \int_a^c f(x)\,dx + \lim_{c' \to d+0} \int_{c'}^b f(x)\,dx.$$

The rate at which the point c approaches d from the left and the rate at which the point c' approaches d from the right are not related in any way: these limits must be considered independently of each other.

If we define an improper integral with a singularity at one internal point in this way, then the integral $\int_{-1}^1 \frac{1}{x}\,dx$ (with a singularity at 0) will diverge, since the integral $\int_0^1 \frac{1}{x}\,dx$ diverges:

$$\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to +0} \int_\varepsilon^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to +0} (\ln|x|)\Big|_\varepsilon^1 = \lim_{\varepsilon \to +0} (-\ln|\varepsilon|) = +\infty.$$

Similarly, we can establish that the integral $\int_{-1}^0 \frac{1}{x}\,dx$ also diverges:

$$\int_{-1}^0 \frac{1}{x}\,dx = \lim_{\varepsilon \to -0} \int_{-1}^\varepsilon \frac{1}{x}\,dx = \lim_{\varepsilon \to -0} (\ln|x|)\Big|_{-1}^\varepsilon = \lim_{\varepsilon \to -0} \ln|\varepsilon| = -\infty.$$

However, it is easy to see that we would get a finite limit value if this limit was calculated not for two integrals separately, but simultaneously for the sum of the integrals $\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_\varepsilon^1 \frac{1}{x}\,dx$ as $\varepsilon \to +0$:

$$\lim_{\varepsilon \to +0} \left(\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_\varepsilon^1 \frac{1}{x}\,dx \right) = \lim_{\varepsilon \to +0} (\ln|-\varepsilon| - \ln|\varepsilon|) = \lim_{\varepsilon \to +0} 0 = 0.$$

The main feature here is that the parameter ε approaches a singular point on the left and right at the same rate, which allows us to eliminate two infinitely growing terms.

Such a type of convergence of an improper integral with a singularity at an internal point is called *convergence in the sense of the principal value*. We give a general definition of this type of convergence.

Definition.

Let the function f be defined on the set $[a, b] \setminus \{d\}$ and be unbounded in a neighborhood of the point d. The improper integral $\int_a^b f(x)\,dx$ is said to *converge in the sense of the principal value* (or *in the sense of the Cauchy principal value*) if there exists a finite limit on the sum of the integrals $\int_a^{d-\varepsilon} f(x)\,dx + \int_{d+\varepsilon}^b f(x)\,dx$ as $\varepsilon \to +0$. For this limit, the notation (v. p.) $\int_a^b f(x)\,dx$ is used:

$$\text{(v. p.)} \int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{\varepsilon \to +0} \Bigl(\int_a^{d-\varepsilon} f(x)\,dx + \int_{d+\varepsilon}^b f(x)\,dx \Bigr).$$

Thus, the previously obtained result for the integral of the function $\frac{1}{x}$ on the segment $[-1, 1]$ can be written as follows:

$$\text{(v. p.)} \int_{-1}^1 \frac{1}{x}\,dx = 0.$$

There is an extensive theory related to the convergence of improper integrals in the sense of the Cauchy principal value, but we will not study this type of convergence in this book.